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The Differentiable Functions on Normed Linear Spaces

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Summary. In this article, the basic properties of the differentiable functions on normed linear spaces are described.

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The notation and terminology used in this paper are introduced in the following papers: [20], [23], [4], [24], [6], [5], [19], [3], [10], [1], [18], [7], [21], [22], [11], [8], [9], [25], [13], [15], [16], [17], [12], [14], and [2].

For simplicity, we adopt the following rules: n, k denote natural numbers, x, X, Z denote sets, g, r denote real numbers, S denotes a real normed space, r_1 denotes a sequence of real numbers, s_1, s_2 denote sequences of S, x_0 denotes a point of S, and Y denotes a subset of S.

Next we state several propositions:

(1) For every point x_0 of S and for all neighbourhoods N_1, N_2 of x_0 there exists a neighbourhood N of x_0 such that $N \subseteq N_1$ and $N \subseteq N_2$.

(2) Let X be a subset of S. Suppose X is open. Let r be a point of S. If $r \in X$, then there exists a neighbourhood N of r such that $N \subseteq X$.

(3) Let X be a subset of S. Suppose X is open. Let r be a point of S. If $r \in X$, then there exists g such that $0 < g$ and $\{y; y$ ranges over points of S: $\|y - r\| < g\} \subseteq X$.

(4) Let X be a subset of S. Suppose that for every point r of S such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$. Then X is open.

(5) Let X be a subset of S. Then for every point r of S such that $r \in X$ there exists a neighbourhood N of r such that $N \subseteq X$ if and only if X is open.

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Let S be a zero structure and let f be a sequence of S. We say that f is non-zero if and only if:

(Def. 1) $\operatorname{rng} f \subseteq (\text{the carrier of } S) \setminus \{0_S\}$.

We introduce f is non-zero as a synonym of f is non-zero.

We now state two propositions:

(6) s_1 is non-zero iff for every x such that $x \in \mathbb{N}$ holds $s_1(x) \neq 0_S$.

(7) s_1 is non-zero iff for every n holds $s_1(n) \neq 0_S$.

Let R_1 be a real linear space, let S be a sequence of R_1, and let a be a sequence of real numbers. The functor $a\,S$ yields a sequence of R_1 and is defined as follows:

(Def. 2) For every n holds $(a\,S)(n) = a(n) \cdot S(n)$.

Let R_1 be a real linear space, let z be a point of R_1, and let a be a sequence of real numbers. The functor $a \cdot z$ yields a sequence of R_1 and is defined by:

(Def. 3) For every n holds $(a \cdot z)(n) = a(n) \cdot z$.

Next we state a number of propositions:

(8) For all sequences r_2, r_3 of real numbers holds $(r_2 + r_3)\,s_1 = r_2\,s_1 + r_3\,s_1$.

(9) For every sequence r_1 of real numbers and for all sequences s_2, s_3 of S holds $r_1\,(s_2 + s_3) = r_1\,s_2 + r_1\,s_3$.

(10) For every sequence r_1 of real numbers holds $r \cdot (r_1\,s_1) = r_1\,(r \cdot s_1)$.

(11) For all sequences r_2, r_3 of real numbers holds $(r_2 - r_3)\,s_1 = r_2\,s_1 - r_3\,s_1$.

(12) For every sequence r_1 of real numbers and for all sequences s_2, s_3 of S holds $r_1\,(s_2 - s_3) = r_1\,s_2 - r_1\,s_3$.

(13) If r_1 is convergent and s_1 is convergent, then $r_1\,s_1$ is convergent.

(14) If r_1 is convergent and s_1 is convergent, then $\lim(r_1\,s_1) = \lim r_1 \cdot \lim s_1$.

(15) $(s_1 + s_2) \uparrow k = s_1 \uparrow k + s_2 \uparrow k$.

(16) $(s_1 - s_2) \uparrow k = s_1 \uparrow k - s_2 \uparrow k$.

(17) If s_1 is non-zero, then $s_1 \uparrow k$ is non-zero.

(18) $s_1 \uparrow k$ is a subsequence of s_1.

(19) If s_1 is constant and s_2 is a subsequence of s_1, then s_2 is constant.

(20) If s_1 is constant and s_2 is a subsequence of s_1, then $s_1 = s_2$.

Let us consider S and let I_1 be a sequence of S. We say that I_1 is convergent to 0 if and only if:

(Def. 4) I_1 is non-zero and convergent and $\lim I_1 = 0_S$.

The following propositions are true:

(21) Let X be a real normed space and s_1 be a sequence of X. Suppose s_1 is constant. Then s_1 is convergent and for every natural number k holds $\lim s_1 = s_1(k)$.

(22) For every real number r such that $0 < r$ and for every n holds $s_1(n) = \frac{1}{n+r} \cdot x_0$ holds s_1 is convergent.

(23) For every real number r such that $0 < r$ and for every n holds $s_1(n) = \frac{1}{n+r} \cdot x_0$ holds $\lim s_1 = 0_S$.

(24) Let a be a convergent to 0 sequence of real numbers and z be a point of S. If $z \neq 0_S$, then $a \cdot z$ is convergent to 0.

(25) For every point r of S holds $r \in Y$ iff $r \in$ the carrier of S iff $Y =$ the carrier of S.

For simplicity, we adopt the following rules: S, T denote non trivial real normed spaces, f, f_1, f_2 denote partial functions from S to T, s_4, s_1 denote sequences of S, and x_0 denotes a point of S.

Let S be a non trivial real normed space. Note that there exists a sequence of S which is convergent to 0.

Let us consider S. Note that there exists a sequence of S which is constant.

In the sequel h is a convergent to 0 sequence of S and c is a constant sequence of S.

Let us consider S, T and let I_1 be a partial function from S to T. We say that I_1 is rest-like if and only if:

(Def. 5) I_1 is total and for every h holds $\|h\|^{-1} (I_1 \cdot h)$ is convergent and $\lim(\|h\|^{-1} (I_1 \cdot h)) = 0_T$.

Let us consider S, T. Observe that there exists a partial function from S to T which is rest-like.

Let us consider S, T. A rest of S, T is a rest-like partial function from S to T.

We now state two propositions:

(26) Let R be a partial function from S to T. Suppose R is total. Then R is rest-like if and only if for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every point z of S such that $z \neq 0_S$ and $\|z\| < d$ holds $\|z\|^{-1} \cdot \|R_z\| < r$.

(27) For every rest R of S, T and for every convergent to 0 sequence s of S holds $R \cdot s$ is convergent and $\lim(R \cdot s) = 0_T$.

In the sequel R, R_2, R_3 are rests of S, T and L is a point of RNormSpaceOfBoundedLinearOperators(S,T).

Next we state several propositions:

(28) $\mathrm{rng}(s_1 \uparrow n) \subseteq \mathrm{rng}\, s_1$.

(29) For every partial function h from S to T and for every sequence s_1 of S such that $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, h$ holds $(h \cdot s_1) \uparrow n = h \cdot (s_1 \uparrow n)$.

(30) Let h_1, h_2 be partial functions from S to T and s_1 be a sequence of S. If h_1 is total and h_2 is total, then $(h_1 + h_2) \cdot s_1 = h_1 \cdot s_1 + h_2 \cdot s_1$ and $(h_1 - h_2) \cdot s_1 = h_1 \cdot s_1 - h_2 \cdot s_1$.

(31) Let h be a partial function from S to T, s_1 be a sequence of S, and r be a real number. If h is total, then $(r\,h) \cdot s_1 = r \cdot (h \cdot s_1)$.

(32) f is continuous in x_0 if and only if the following conditions are satisfied:
  (i) $x_0 \in \operatorname{dom} f$, and
  (ii) for every sequence s_4 of S such that $\operatorname{rng} s_4 \subseteq \operatorname{dom} f$ and s_4 is convergent and $\lim s_4 = x_0$ and for every n holds $s_4(n) \neq x_0$ holds $f \cdot s_4$ is convergent and $f_{x_0} = \lim(f \cdot s_4)$.

(33) For all R_2, R_3 holds $R_2 + R_3$ is a rest of S, T and $R_2 - R_3$ is a rest of S, T.

(34) For all r, R holds $r\,R$ is a rest of S, T.

Let us consider S, T, let f be a partial function from S to T, and let x_0 be a point of S. We say that f is differentiable in x_0 if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and there exist L, R such that for every point x of S such that $x \in N$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x-x_0}$.

Let us consider S, T, let f be a partial function from S to T, and let x_0 be a point of S. Let us assume that f is differentiable in x_0. The functor $f'(x_0)$ yielding a point of RNormSpaceOfBoundedLinearOperators(S, T) is defined by the condition (Def. 7).

(Def. 7) There exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and there exists R such that for every point x of S such that $x \in N$ holds $f_x - f_{x_0} = f'(x_0)(x - x_0) + R_{x-x_0}$.

Let us consider X, let us consider S, T, and let f be a partial function from S to T. We say that f is differentiable on X if and only if:

(Def. 8) $X \subseteq \operatorname{dom} f$ and for every point x of S such that $x \in X$ holds $f{\restriction}X$ is differentiable in x.

Next we state three propositions:

(35) Let f be a partial function from S to T. If f is differentiable on X, then X is a subset of the carrier of S.

(36) Let f be a partial function from S to T and Z be a subset of S. Suppose Z is open. Then f is differentiable on Z if and only if the following conditions are satisfied:
  (i) $Z \subseteq \operatorname{dom} f$, and
  (ii) for every point x of S such that $x \in Z$ holds f is differentiable in x.

(37) Let f be a partial function from S to T and Y be a subset of S. If f is differentiable on Y, then Y is open.

Let us consider S, T, let f be a partial function from S to T, and let X be a set. Let us assume that f is differentiable on X. The functor $f'_{\restriction X}$ yielding a partial function from S to RNormSpaceOfBoundedLinearOperators(S,T) is defined by:

(Def. 9) $\mathrm{dom}(f'_{\restriction X}) = X$ and for every point x of S such that $x \in X$ holds $(f'_{\restriction X})_x = f'(x)$.

One can prove the following proposition

(38) Let f be a partial function from S to T and Z be a subset of S. Suppose Z is open and $Z \subseteq \mathrm{dom}\, f$ and there exists a point r of T such that $\mathrm{rng}\, f = \{r\}$. Then f is differentiable on Z and for every point x of S such that $x \in Z$ holds $(f'_{\restriction Z})_x = 0_{\mathrm{RNormSpaceOfBoundedLinearOperators}(S,T)}$.

Let us consider S and let us consider h, n. Observe that $h \uparrow n$ is convergent to 0.

Let us consider S and let us consider c, n. Observe that $c \uparrow n$ is constant.

The following propositions are true:

(39) Let x_0 be a point of S and N be a neighbourhood of x_0. Suppose f is differentiable in x_0 and $N \subseteq \mathrm{dom}\, f$. Let h be a convergent to 0 sequence of S and given c. If $\mathrm{rng}\, c = \{x_0\}$ and $\mathrm{rng}(h+c) \subseteq N$, then $f \cdot (h+c) - f \cdot c$ is convergent and $\lim(f \cdot (h+c) - f \cdot c) = 0_T$.

(40) Let given f_1, f_2, x_0. Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0. Then $f_1 + f_2$ is differentiable in x_0 and $(f_1 + f_2)'(x_0) = {f_1}'(x_0) + {f_2}'(x_0)$.

(41) Let given f_1, f_2, x_0. Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0. Then $f_1 - f_2$ is differentiable in x_0 and $(f_1 - f_2)'(x_0) = {f_1}'(x_0) - {f_2}'(x_0)$.

(42) For all r, f, x_0 such that f is differentiable in x_0 holds $r\,f$ is differentiable in x_0 and $(r\,f)'(x_0) = r \cdot f'(x_0)$.

(43) Let f be a partial function from S to S and Z be a subset of S. Suppose Z is open and $Z \subseteq \mathrm{dom}\, f$ and $f \restriction Z = \mathrm{id}_Z$. Then f is differentiable on Z and for every point x of S such that $x \in Z$ holds $(f'_{\restriction Z})_x = \mathrm{id}_{\text{the carrier of } S}$.

(44) Let Z be a subset of S. Suppose Z is open. Let given f_1, f_2. Suppose $Z \subseteq \mathrm{dom}(f_1 + f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 + f_2$ is differentiable on Z and for every point x of S such that $x \in Z$ holds $((f_1 + f_2)'_{\restriction Z})_x = {f_1}'(x) + {f_2}'(x)$.

(45) Let Z be a subset of S. Suppose Z is open. Let given f_1, f_2. Suppose $Z \subseteq \mathrm{dom}(f_1 - f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 - f_2$ is differentiable on Z and for every point x of S such that $x \in Z$ holds $((f_1 - f_2)'_{\restriction Z})_x = {f_1}'(x) - {f_2}'(x)$.

(46) Let Z be a subset of S. Suppose Z is open. Let given r, f. Suppose $Z \subseteq \mathrm{dom}(r\,f)$ and f is differentiable on Z. Then $r\,f$ is differentiable on Z and for every point x of S such that $x \in Z$ holds $((r\,f)'_{\restriction Z})_x = r \cdot f'(x)$.

(47) Let Z be a subset of S. Suppose Z is open. Suppose $Z \subseteq \mathrm{dom}\, f$ and f

is a constant on Z. Then f is differentiable on Z and for every point x of S such that $x \in Z$ holds $(f'_{\restriction Z})_x = 0_{\mathrm{RNormSpaceOfBoundedLinearOperators}(S,T)}$.

(48) Let f be a partial function from S to S, r be a real number, p be a point of S, and Z be a subset of S. Suppose Z is open. Suppose $Z \subseteq \mathrm{dom}\, f$ and for every point x of S such that $x \in Z$ holds $f_x = r \cdot x + p$. Then f is differentiable on Z and for every point x of S such that $x \in Z$ holds $(f'_{\restriction Z})_x = r \cdot \mathrm{FuncUnit}(S)$.

(49) For every point x_0 of S such that f is differentiable in x_0 holds f is continuous in x_0.

(50) If f is differentiable on X, then f is continuous on X.

(51) For every subset Z of S such that Z is open holds if f is differentiable on X and $Z \subseteq X$, then f is differentiable on Z.

(52) Suppose f is differentiable in x_0. Then there exists a neighbourhood N of x_0 such that
  (i) $N \subseteq \mathrm{dom}\, f$, and
  (ii) there exists R such that $R_{0_S} = 0_T$ and R is continuous in 0_S and for every point x of S such that $x \in N$ holds $f_x - f_{x_0} = f'(x_0)(x - x_0) + R_{x - x_0}$.

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