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Axiomatization of Boolean Algebras Based on Sheffer Stroke

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Summary. We formalized another axiomatization of Boolean algebras. The classical one is introduced in [9], "the fourth set of postulates" due to Huntington [3] ([2] in Mizar) and the single axiom in terms of disjunction and negation is codified recently in [7]. In this article, we aimed at the description of Boolean algebras using Sheffer stroke according to [6], namely by the following three axioms:

$$(x|x)|(x|x) = x$$

$$x|(y|(y|y)) = x|x$$

$$(x|(y|z))|(x|(y|z)) = ((y|y)|x)|((z|z)|x)$$

($\restriction$ is used instead of $|$ in the translation of our Mizar article). Since Sheffer in his original paper proved its equivalence and Huntington's "first set of postulates", we have also introduced this axiomatization of BAs.

MML Identifier: `SHEFFER1`.

The terminology and notation used here are introduced in the following articles: [8], [9], [5], [1], [4], and [2].

1. Preliminaries

The following two propositions are true:

(1) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of L. Then $(a+b)^{\rm c} = a^{\rm c} * b^{\rm c}$.

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(2) Let L be a join-commutative join-associative Huntington non empty complemented lattice structure and a, b be elements of L. Then $(a * b)^{\mathrm{c}} = a^{\mathrm{c}} + b^{\mathrm{c}}$.

2. Huntington's First Axiomatization of Boolean Algebras

Let I_1 be a non empty lattice structure. We say that I_1 is upper-bounded' if and only if:

(Def. 1) There exists an element c of I_1 such that for every element a of I_1 holds $c \sqcap a = a$ and $a \sqcap c = a$.

Let L be a non empty lattice structure. Let us assume that L is upper-bounded'. The functor $\top'_L$ yields an element of L and is defined by:

(Def. 2) For every element a of L holds $\top'_L \sqcap a = a$ and $a \sqcap \top'_L = a$.

Let I_1 be a non empty lattice structure. We say that I_1 is lower-bounded' if and only if:

(Def. 3) There exists an element c of I_1 such that for every element a of I_1 holds $c \sqcup a = a$ and $a \sqcup c = a$.

Let L be a non empty lattice structure. Let us assume that L is lower-bounded'. The functor $\perp'_L$ yields an element of L and is defined as follows:

(Def. 4) For every element a of L holds $\perp'_L \sqcup a = a$ and $a \sqcup \perp'_L = a$.

Let I_1 be a non empty lattice structure. We say that I_1 is distributive' if and only if:

(Def. 5) For all elements a, b, c of I_1 holds $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$.

Let L be a non empty lattice structure and let a, b be elements of L. We say that a is a complement' of b if and only if:

(Def. 6) $b \sqcup a = \top'_L$ and $a \sqcup b = \top'_L$ and $b \sqcap a = \perp'_L$ and $a \sqcap b = \perp'_L$.

Let I_1 be a non empty lattice structure. We say that I_1 is complemented' if and only if:

(Def. 7) For every element b of I_1 holds there exists an element of I_1 which is a complement' of b.

Let L be a non empty lattice structure and let x be an element of L. Let us assume that L is complemented', distributive, upper-bounded', and meet-commutative. The functor $x^{\mathrm{c}'}$ yields an element of L and is defined as follows:

(Def. 8) $x^{\mathrm{c}'}$ is a complement' of x.

Let us mention that there exists a non empty lattice structure which is Boolean, join-idempotent, upper-bounded', complemented', distributive', lower-bounded', and lattice-like.

Next we state several propositions:

(3) Let L be a complemented' join-commutative meet-commutative distributive upper-bounded' distributive' non empty lattice structure and x be an element of L. Then $x \sqcup x^{c'} = \top'_L$.

(4) Let L be a complemented' join-commutative meet-commutative distributive upper-bounded' distributive' non empty lattice structure and x be an element of L. Then $x \sqcap x^{c'} = \bot'_L$.

(5) Let L be a complemented' join-commutative meet-commutative join-idempotent distributive upper-bounded' distributive' non empty lattice structure and x be an element of L. Then $x \sqcup \top'_L = \top'_L$.

(6) Let L be a complemented' join-commutative meet-commutative join-idempotent distributive upper-bounded' lower-bounded' distributive' non empty lattice structure and x be an element of L. Then $x \sqcap \bot'_L = \bot'_L$.

(7) Let L be a join-commutative meet-absorbing meet-commutative join-absorbing join-idempotent distributive non empty lattice structure and x, y, z be elements of L. Then $(x \sqcup y \sqcup z) \sqcap x = x$.

(8) Let L be a join-commutative meet-absorbing meet-commutative join-absorbing join-idempotent distributive' non empty lattice structure and x, y, z be elements of L. Then $(x \sqcap y \sqcap z) \sqcup x = x$.

Let G be a non empty $\sqcap$-semi lattice structure. We say that G is meet-idempotent if and only if:

(Def. 9) For every element x of G holds $x \sqcap x = x$.

Next we state a number of propositions:

(9) Every complemented' join-commutative meet-commutative distributive upper-bounded' lower-bounded' distributive' non empty lattice structure is meet-idempotent.

(10) Every complemented' join-commutative meet-commutative distributive upper-bounded' lower-bounded' distributive' non empty lattice structure is join-idempotent.

(11) Every complemented' join-commutative meet-commutative join-idempotent distributive upper-bounded' distributive' non empty lattice structure is meet-absorbing.

(12) Every complemented' join-commutative upper-bounded' meet-commutative join-idempotent distributive distributive' lower-bounded' non empty lattice structure is join-absorbing.

(13) Every complemented' join-commutative meet-commutative upper-bounded' lower-bounded' join-idempotent distributive distributive' non empty lattice structure is upper-bounded.

(14) Every Boolean lattice-like non empty lattice structure is upper-bounded'.

(15) Every complemented' join-commutative meet-commutative upper-bounded' lower-bounded' join-idempotent distributive distributive' non

empty lattice structure is lower-bounded.

(16) Every Boolean lattice-like non empty lattice structure is lower-bounded'.

(17) Every join-commutative meet-commutative meet-absorbing join-absorbing join-idempotent distributive non empty lattice structure is join-associative.

(18) Every join-commutative meet-commutative meet-absorbing join-absorbing join-idempotent distributive' non empty lattice structure is meet-associative.

(19) Let L be a complemented' join-commutative meet-commutative lower-bounded' upper-bounded' join-idempotent distributive distributive' non empty lattice structure. Then $\top_L = \top'_L$.

(20) Let L be a complemented' join-commutative meet-commutative lower-bounded' upper-bounded' join-idempotent distributive distributive' non empty lattice structure. Then $\bot_L = \bot'_L$.

(21) For every Boolean distributive' lattice-like non empty lattice structure L holds $\top_L = \top'_L$.

(22) Let L be a Boolean complemented lower-bounded upper-bounded distributive distributive' lattice-like non empty lattice structure. Then $\bot_L = \bot'_L$.

(23) Let L be a complemented' lower-bounded' upper-bounded' join-commutative meet-commutative join-idempotent distributive distributive' non empty lattice structure and x, y be elements of L. Then x is a complement' of y if and only if x is a complement of y.

(24) Every complemented' join-commutative meet-commutative lower-bounded' upper-bounded' join-idempotent distributive distributive' non empty lattice structure is complemented.

(25) Every Boolean lower-bounded' upper-bounded' distributive' lattice-like non empty lattice structure is complemented'.

(26) Let L be a non empty lattice structure. Then L is a Boolean lattice if and only if L is lower-bounded', upper-bounded', join-commutative, meet-commutative, distributive, distributive', and complemented'.

Let us note that every non empty lattice structure which is Boolean and lattice-like is also lower-bounded', upper-bounded', complemented', join-commutative, meet-commutative, distributive, and distributive' and every non empty lattice structure which is lower-bounded', upper-bounded', complemented', join-commutative, meet-commutative, distributive, and distributive' is also Boolean and lattice-like.

3. Axiomatization Based on Sheffer Stroke

We introduce Sheffer structures which are extensions of 1-sorted structure and are systems

⟨ a carrier, a Sheffer stroke ⟩,

where the carrier is a set and the Sheffer stroke is a binary operation on the carrier.

We consider Sheffer lattice structures as extensions of Sheffer structure and lattice structure as systems

⟨ a carrier, a join operation, a meet operation, a Sheffer stroke ⟩,

where the carrier is a set, the join operation is a binary operation on the carrier, the meet operation is a binary operation on the carrier, and the Sheffer stroke is a binary operation on the carrier.

We consider Sheffer ortholattice structures as extensions of Sheffer structure and ortholattice structure as systems

⟨ a carrier, a join operation, a meet operation, a complement operation, a Sheffer stroke ⟩,

where the carrier is a set, the join operation is a binary operation on the carrier, the meet operation is a binary operation on the carrier, the complement operation is a unary operation on the carrier, and the Sheffer stroke is a binary operation on the carrier.

The Sheffer ortholattice structure TrivShefferOrthoLattStr is defined by:

(Def. 10) $\mathrm{TrivShefferOrthoLattStr} = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_2, \mathrm{op}_1, \mathrm{op}_2 \rangle$.

One can verify the following observations:

* there exists a Sheffer structure which is non empty,
* there exists a Sheffer lattice structure which is non empty, and
* there exists a Sheffer ortholattice structure which is non empty.

Let L be a non empty Sheffer structure and let x, y be elements of L. The functor $x \upharpoonleft y$ yields an element of L and is defined as follows:

(Def. 11) $x \upharpoonleft y =$ (the Sheffer stroke of L)(x, y).

Let L be a non empty Sheffer ortholattice structure. We say that L is properly defined if and only if the conditions (Def. 12) are satisfied.

(Def. 12)(i) For every element x of L holds $x \upharpoonleft x = x^{\mathrm{c}}$,

(ii) for all elements x, y of L holds $x \sqcup y = x \upharpoonleft x \upharpoonleft (y \upharpoonleft y)$,

(iii) for all elements x, y of L holds $x \sqcap y = x \upharpoonleft y \upharpoonleft (x \upharpoonleft y)$, and

(iv) for all elements x, y of L holds $x \upharpoonleft y = x^{\mathrm{c}} + y^{\mathrm{c}}$.

Let L be a non empty Sheffer structure. We say that L satisfies (Sheffer$_1$) if and only if:

(Def. 13) For every element x of L holds $x \upharpoonleft x \upharpoonleft (x \upharpoonleft x) = x$.

We say that L satisfies (Sheffer$_2$) if and only if:

(Def. 14) For all elements x, y of L holds $x \upharpoonleft (y \upharpoonleft (y \upharpoonleft y)) = x \upharpoonleft x$.

We say that L satisfies (Sheffer$_3$) if and only if:

(Def. 15) For all elements x, y, z of L holds $(x \upharpoonleft (y \upharpoonleft z)) \upharpoonleft (x \upharpoonleft (y \upharpoonleft z)) = y \upharpoonleft y \upharpoonleft x \upharpoonleft (z \upharpoonleft z \upharpoonleft x)$.

Let us note that every non empty Sheffer structure which is trivial satisfies also (Sheffer$_1$), (Sheffer$_2$), and (Sheffer$_3$).

One can verify that every non empty $\sqcup$-semi lattice structure which is trivial is also join-commutative and join-associative and every non empty $\sqcap$-semi lattice structure which is trivial is also meet-commutative and meet-associative.

Let us note that every non empty lattice structure which is trivial is also join-absorbing, meet-absorbing, and Boolean.

One can check the following observations:

* TrivShefferOrthoLattStr is non empty,
* TrivShefferOrthoLattStr is trivial, and
* TrivShefferOrthoLattStr is properly defined and well-complemented.

Let us mention that there exists a non empty Sheffer ortholattice structure which is properly defined, Boolean, well-complemented, and lattice-like and satisfies (Sheffer$_1$), (Sheffer$_2$), and (Sheffer$_3$).

Next we state three propositions:

(27) Every properly defined Boolean well-complemented lattice-like non empty Sheffer ortholattice structure satisfies (Sheffer$_1$).

(28) Every properly defined Boolean well-complemented lattice-like non empty Sheffer ortholattice structure satisfies (Sheffer$_2$).

(29) Every properly defined Boolean well-complemented lattice-like non empty Sheffer ortholattice structure satisfies (Sheffer$_3$).

Let L be a non empty Sheffer structure and let a be an element of L. The functor a^{-1} yielding an element of L is defined as follows:

(Def. 16) $a^{-1} = a \upharpoonleft a$.

One can prove the following propositions:

(30) Let L be a non empty Sheffer ortholattice structure satisfying (Sheffer$_3$) and x, y, z be elements of L. Then $(x \upharpoonleft (y \upharpoonleft z))^{-1} = y^{-1} \upharpoonleft x \upharpoonleft (z^{-1} \upharpoonleft x)$.

(31) For every non empty Sheffer ortholattice structure L satisfying (Sheffer$_1$) and for every element x of L holds $x = (x^{-1})^{-1}$.

(32) Let L be a properly defined non empty Sheffer ortholattice structure satisfying (Sheffer$_1$), (Sheffer$_2$), and (Sheffer$_3$) and x, y be elements of L. Then $x \upharpoonleft y = y \upharpoonleft x$.

(33) Let L be a properly defined non empty Sheffer ortholattice structure satisfying (Sheffer$_1$), (Sheffer$_2$), and (Sheffer$_3$) and x, y be elements of L. Then $x \upharpoonleft (x \upharpoonleft x) = y \upharpoonleft (y \upharpoonleft y)$.

(34) Every properly defined non empty Sheffer ortholattice structure satisfying (Sheffer_1), (Sheffer_2), and (Sheffer_3) is join-commutative.

(35) Every properly defined non empty Sheffer ortholattice structure satisfying (Sheffer_1), (Sheffer_2), and (Sheffer_3) is meet-commutative.

(36) Every properly defined non empty Sheffer ortholattice structure satisfying (Sheffer_1), (Sheffer_2), and (Sheffer_3) is distributive.

(37) Every properly defined non empty Sheffer ortholattice structure satisfying (Sheffer_1), (Sheffer_2), and (Sheffer_3) is distributive'.

(38) Every properly defined non empty Sheffer ortholattice structure satisfying (Sheffer_1), (Sheffer_2), and (Sheffer_3) is a Boolean lattice.

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