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Short Sheffer Stroke-Based Single Axiom for Boolean Algebras

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Summary. We continue the description of Boolean algebras in terms of the Sheffer stroke as defined in [2]. The single axiomatization for BAs in terms of disjunction and negation was shown in [3]. As was checked automatically with the help of automated theorem prover Otter, single axiom of the form

$$(x|((y|x)|x))|(y|(z|x)) = y \quad \text{(Sh1)}$$

is enough to axiomatize the class of all Boolean algebras ($\upharpoonleft$ is used instead of $|$ in translation of our Mizar article). Many theorems in Section 2 were automatically translated from the Otter proof object.

MML Identifier: SHEFFER2.

The terminology and notation used in this paper are introduced in the following papers: [4], [1], and [2].

1. First Implication

Let L be a non empty Sheffer structure. We say that L satisfies (Sh_1) if and only if:

(Def. 1) For all elements x, y, z of L holds $x\upharpoonleft(y\upharpoonleft x\upharpoonleft x)\upharpoonleft(y\upharpoonleft(z\upharpoonleft x)) = y$.

Let us observe that every non empty Sheffer structure which is trivial satisfies also (Sh_1).

Let us observe that there exists a non empty Sheffer structure which satisfies (Sh_1), $(\mathrm{Sheffer}_1)$, $(\mathrm{Sheffer}_2)$, and $(\mathrm{Sheffer}_3)$.

In the sequel L is a non empty Sheffer structure satisfying (Sh_1).

One can prove the following propositions:

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(1) For all elements x, y, z, u of L holds $(x \upharpoonright (y \upharpoonright z) \upharpoonright (x \upharpoonright (x \upharpoonright (y \upharpoonright z)))) \upharpoonright (z \upharpoonright (x \upharpoonright z \upharpoonright z) \upharpoonright (u \upharpoonright (x \upharpoonright (y \upharpoonright z)))) = z \upharpoonright (x \upharpoonright z \upharpoonright z)$.

(2) For all elements x, y, z of L holds $(x \upharpoonright y \upharpoonright (y \upharpoonright (z \upharpoonright y \upharpoonright y) \upharpoonright (x \upharpoonright y) \upharpoonright (x \upharpoonright y))) \upharpoonright z = y \upharpoonright (z \upharpoonright y \upharpoonright y)$.

(3) For all elements x, y, z of L holds $x \upharpoonright (y \upharpoonright x \upharpoonright x) \upharpoonright (y \upharpoonright (z \upharpoonright (x \upharpoonright z \upharpoonright z))) = y$.

(4) For all elements x, y of L holds $x \upharpoonright (x \upharpoonright (x \upharpoonright x \upharpoonright x) \upharpoonright (y \upharpoonright (x \upharpoonright (x \upharpoonright x \upharpoonright x)))) = x \upharpoonright (x \upharpoonright x \upharpoonright x)$.

(5) For every element x of L holds $x \upharpoonright (x \upharpoonright x \upharpoonright x) = x \upharpoonright x$.

(6) For every element x of L holds $x \upharpoonright (x \upharpoonright x \upharpoonright x) \upharpoonright (x \upharpoonright x) = x$.

(7) For all elements x, y, z of L holds $x \upharpoonright x \upharpoonright (x \upharpoonright (y \upharpoonright x)) = x$.

(8) For all elements x, y of L holds $x \upharpoonright (y \upharpoonright y \upharpoonright x \upharpoonright x) \upharpoonright y = y \upharpoonright y$.

(9) For all elements x, y of L holds $(x \upharpoonright y \upharpoonright (x \upharpoonright y \upharpoonright (x \upharpoonright y) \upharpoonright (x \upharpoonright y))) \upharpoonright (x \upharpoonright y \upharpoonright (x \upharpoonright y)) = y \upharpoonright (x \upharpoonright y \upharpoonright (x \upharpoonright y) \upharpoonright y \upharpoonright y)$.

(10) For all elements x, y of L holds $x \upharpoonright (y \upharpoonright x \upharpoonright (y \upharpoonright x) \upharpoonright x \upharpoonright x) = y \upharpoonright x$.

(11) For all elements x, y of L holds $x \upharpoonright x \upharpoonright (y \upharpoonright x) = x$.

(12) For all elements x, y of L holds $x \upharpoonright (y \upharpoonright (x \upharpoonright x)) = x \upharpoonright x$.

(13) For all elements x, y of L holds $x \upharpoonright y \upharpoonright (x \upharpoonright y) \upharpoonright y = x \upharpoonright y$.

(14) For all elements x, y of L holds $x \upharpoonright (y \upharpoonright x \upharpoonright x) = y \upharpoonright x$.

(15) For all elements x, y, z of L holds $x \upharpoonright y \upharpoonright (x \upharpoonright (z \upharpoonright y)) = x$.

(16) For all elements x, y, z of L holds $x \upharpoonright (y \upharpoonright z) \upharpoonright (x \upharpoonright z) = x$.

(17) For all elements x, y, z of L holds $x \upharpoonright (x \upharpoonright y \upharpoonright (z \upharpoonright y)) = x \upharpoonright y$.

(18) For all elements x, y, z of L holds $(x \upharpoonright (y \upharpoonright z) \upharpoonright z) \upharpoonright x = x \upharpoonright (y \upharpoonright z)$.

(19) For all elements x, y of L holds $x \upharpoonright (y \upharpoonright x \upharpoonright x) = x \upharpoonright y$.

(20) For all elements x, y of L holds $x \upharpoonright y = y \upharpoonright x$.

(21) For all elements x, y of L holds $x \upharpoonright y \upharpoonright (x \upharpoonright x) = x$.

(22) For all elements x, y, z of L holds $x \upharpoonright y \upharpoonright (y \upharpoonright (z \upharpoonright x)) = y$.

(23) For all elements x, y, z of L holds $x \upharpoonright (y \upharpoonright z) \upharpoonright (z \upharpoonright x) = x$.

(24) For all elements x, y, z of L holds $x \upharpoonright y \upharpoonright (y \upharpoonright (x \upharpoonright z)) = y$.

(25) For all elements x, y, z of L holds $x \upharpoonright (y \upharpoonright z) \upharpoonright (y \upharpoonright x) = x$.

(26) For all elements x, y, z of L holds $x \upharpoonright y \upharpoonright (x \upharpoonright z) \upharpoonright z = x \upharpoonright z$.

(27) For all elements x, y, z of L holds $x \upharpoonright (y \upharpoonright (x \upharpoonright (y \upharpoonright z))) = x \upharpoonright (y \upharpoonright z)$.

(28) For all elements x, y, z of L holds $(x \upharpoonright (y \upharpoonright (x \upharpoonright z))) \upharpoonright y = y \upharpoonright (x \upharpoonright z)$.

(29) For all elements x, y, z, u of L holds $(x \upharpoonright (y \upharpoonright z)) \upharpoonright (x \upharpoonright (u \upharpoonright (y \upharpoonright x))) = x \upharpoonright (y \upharpoonright z) \upharpoonright (y \upharpoonright x)$.

(30) For all elements x, y, z of L holds $(x \upharpoonright (y \upharpoonright (x \upharpoonright z))) \upharpoonright y = y \upharpoonright (z \upharpoonright x)$.

(31) For all elements x, y, z, u of L holds $x \upharpoonright (y \upharpoonright z) \upharpoonright (x \upharpoonright (u \upharpoonright (y \upharpoonright x))) = x$.

(32) For all elements x, y of L holds $x \upharpoonright (y \upharpoonright (x \upharpoonright y)) = x \upharpoonright x$.

(33) For all elements x, y, z of L holds $x|(y|z) = x|(z|y)$.

(34) For all elements x, y, z of L holds $x|(y|(x|(z|(y|x)))) = x|x$.

(35) For all elements x, y, z of L holds $(x|(y|z))|(y|x|x) = x|(y|z)|(x|(y|z))$.

(36) For all elements x, y, z of L holds $x|(y|x)|y = y|y$.

(37) For all elements x, y, z of L holds $(x|y)|z = z|(y|x)$.

(38) For all elements x, y, z of L holds $x|(y|(z|(x|y))) = x|(y|y)$.

(39) For all elements x, y, z of L holds $(x|y|y)|(y|(z|x)) = y|(z|x)|(y|(z|x))$.

(40) For all elements x, y, z, u of L holds $(x|y)|(z|u) = u|z|(y|x)$.

(41) For all elements x, y, z of L holds $x|(y|(y|x|z)) = x|(y|y)$.

(42) For all elements x, y of L holds $x|(y|x) = x|(y|y)$.

(43) For all elements x, y of L holds $(x|y)|y = y|(x|x)$.

(44) For all elements x, y, z of L holds $x|(y|y) = x|(x|y)$.

(45) For all elements x, y, z of L holds $(x|(y|y))|(x|(z|y)) = x|(z|y)|(x|(z|y))$.

(46) For all elements x, y, z of L holds $(x|(y|z))|(x|(y|y)) = x|(y|z)|(x|(y|z))$.

(47) For all elements x, y, z of L holds $x|(y|y|(z|(x|(x|y)))) = x|(y|y|(y|y))$.

(48) For all elements x, y, z of L holds $(x|(y|z)|(x|(y|z)))|(y|y) = x|(y|y)$.

(49) For all elements x, y, z of L holds $x|(y|y|(z|(x|(x|y)))) = x|y$.

(50) For all elements x, y, z of L holds $(x|y|(x|y)|(z|(x|y|z)|(x|y)))|(x|x) = z|(x|y|z)|(x|x)$.

(51) For all elements x, y, z of L holds $(x|(y|z|x))|(y|y) = y|z|(y|y)$.

(52) For all elements x, y, z of L holds $x|(y|z|x)|(y|y) = y$.

(53) For all elements x, y, z of L holds $x|(y|(x|z|y)|x) = y|(x|z|y)$.

(54) For all elements x, y, z of L holds $x|(y|(y|(z|x))|x) = y|(x|(y|(x|z))|y)$.

(55) For all elements x, y, z of L holds $x|(y|(y|(z|x))|x) = y|(y|(z|x))$.

(56) For all elements x, y, z, u of L holds $x|(y|(z|(z|(u|(y|x))))) = x|(y|y)$.

(57) For all elements x, y, z of L holds $x|(y|(y|(z|(x|y)))) = x|(y|(x|x))$.

(58) For all elements x, y, z of L holds $x|(y|(y|(z|(x|y)))) = x|x$.

(59) For all elements x, y of L holds $x|(y|(y|y)) = x|x$.

(60) For all elements x, y, z of L holds $x|(y|(z|x)|(y|(z|x))|(z|z)) = x|(y|(z|x))$.

(61) For all elements x, y, z of L holds $x|(y|(z|z)) = x|(y|(z|x))$.

(62) For all elements x, y, z of L holds $x|(y|(z|z|x)) = x|(y|z)$.

(63) For all elements x, y, z of L holds $(x|(y|y))|(x|(z|(y|y|x))) = x|(z|y)|(x|(z|y))$.

(64) For all elements x, y, z of L holds $(x \upharpoonleft (y \upharpoonleft y)) \upharpoonleft (x \upharpoonleft (z \upharpoonleft (x \upharpoonleft (y \upharpoonleft y)))) = x \upharpoonleft (z \upharpoonleft y) \upharpoonleft (x \upharpoonleft (z \upharpoonleft y))$.

(65) For all elements x, y, z of L holds $(x \upharpoonleft (y \upharpoonleft y)) \upharpoonleft (x \upharpoonleft (z \upharpoonleft z)) = x \upharpoonleft (z \upharpoonleft y) \upharpoonleft (x \upharpoonleft (z \upharpoonleft y))$.

(66) For all elements x, y, z of L holds $(x \upharpoonleft x \upharpoonleft y) \upharpoonleft (z \upharpoonleft z \upharpoonleft y) = y \upharpoonleft (x \upharpoonleft z) \upharpoonleft (y \upharpoonleft (x \upharpoonleft z))$.

(67) For every non empty Sheffer structure L such that L satisfies (Sh_1) holds L satisfies $(\mathrm{Sheffer}_1)$.

(68) For every non empty Sheffer structure L such that L satisfies (Sh_1) holds L satisfies $(\mathrm{Sheffer}_2)$.

(69) For every non empty Sheffer structure L such that L satisfies (Sh_1) holds L satisfies $(\mathrm{Sheffer}_3)$.

Let us mention that there exists a non empty Sheffer ortholattice structure which is properly defined, Boolean, well-complemented, lattice-like, and de Morgan and satisfies $(\mathrm{Sheffer}_1)$, $(\mathrm{Sheffer}_2)$, $(\mathrm{Sheffer}_3)$, and (Sh_1).

Let us mention that every non empty Sheffer ortholattice structure which is properly defined satisfies $(\mathrm{Sheffer}_1)$, $(\mathrm{Sheffer}_2)$, and $(\mathrm{Sheffer}_3)$ is also Boolean and lattice-like and every non empty Sheffer ortholattice structure which is Boolean, lattice-like, well-complemented, and properly defined satisfies also $(\mathrm{Sheffer}_1)$, $(\mathrm{Sheffer}_2)$, and $(\mathrm{Sheffer}_3)$.

2. Second Implication

We adopt the following rules: L denotes a non empty Sheffer structure satisfying $(\mathrm{Sheffer}_1)$, $(\mathrm{Sheffer}_2)$, and $(\mathrm{Sheffer}_3)$ and v, q, p, w, z, y, x denote elements of L.

One can prove the following propositions:

(70) For all x, w holds $w \upharpoonleft (x \upharpoonleft x \upharpoonleft x) = w \upharpoonleft w$.

(71) For all p, x holds $x = x \upharpoonleft x \upharpoonleft (p \upharpoonleft (p \upharpoonleft p))$.

(72) For all y, w holds $w \upharpoonleft w \upharpoonleft (w \upharpoonleft (y \upharpoonleft (y \upharpoonleft y))) = w$.

(73) For all q, p, y, w holds $(w \upharpoonleft (y \upharpoonleft (y \upharpoonleft y)) \upharpoonleft p) \upharpoonleft (q \upharpoonleft q \upharpoonleft p) = p \upharpoonleft (w \upharpoonleft q) \upharpoonleft (p \upharpoonleft (w \upharpoonleft q))$.

(74) For all q, p, x holds $(x \upharpoonleft p) \upharpoonleft (q \upharpoonleft q \upharpoonleft p) = p \upharpoonleft (x \upharpoonleft x \upharpoonleft q) \upharpoonleft (p \upharpoonleft (x \upharpoonleft x \upharpoonleft q))$.

(75) For all w, p, y, q holds $(w \upharpoonleft w \upharpoonleft p) \upharpoonleft (q \upharpoonleft (y \upharpoonleft (y \upharpoonleft y)) \upharpoonleft p) = p \upharpoonleft (w \upharpoonleft q) \upharpoonleft (p \upharpoonleft (w \upharpoonleft q))$.

(76) For all p, x holds $x = x \upharpoonleft x \upharpoonleft (p \upharpoonleft p \upharpoonleft p)$.

(77) For all y, w holds $w \upharpoonleft w \upharpoonleft (w \upharpoonleft (y \upharpoonleft y \upharpoonleft y)) = w$.

(78) For all y, w holds $w \upharpoonleft (y \upharpoonleft y \upharpoonleft y) \upharpoonleft (w \upharpoonleft w) = w$.

(79) For all p, y, w holds $w \upharpoonleft (y \upharpoonleft y \upharpoonleft y) \upharpoonleft (p \upharpoonleft (p \upharpoonleft p)) = w$.

(80) For all p, x, y holds $y \upharpoonleft (x \upharpoonleft x) \upharpoonleft (y \upharpoonleft (x \upharpoonleft x)) \upharpoonleft (p \upharpoonleft (p \upharpoonleft p)) = (x \upharpoonleft x) \upharpoonleft y$.

(81) For all x, y holds $y \upharpoonleft (x \upharpoonleft x) = (x \upharpoonleft x) \upharpoonleft y$.

(82) For all y, w holds $w \upharpoonleft y = y \upharpoonleft y \upharpoonleft (y \upharpoonleft y) \upharpoonleft w$.

(83) For all y, w holds $w{\upharpoonleft}y = y{\upharpoonleft}w$.

(84) For all x, p holds $(p{\upharpoonleft}x){\upharpoonleft}(p{\upharpoonleft}(x{\upharpoonleft}x{\upharpoonleft}(x{\upharpoonleft}x))) = x{\upharpoonleft}x{\upharpoonleft}(x{\upharpoonleft}x){\upharpoonleft}p{\upharpoonleft}(x{\upharpoonleft}x{\upharpoonleft}(x{\upharpoonleft}x){\upharpoonleft}p)$.

(85) For all x, p holds $(p{\upharpoonleft}x){\upharpoonleft}(p{\upharpoonleft}x) = x{\upharpoonleft}x{\upharpoonleft}(x{\upharpoonleft}x){\upharpoonleft}p{\upharpoonleft}(x{\upharpoonleft}x{\upharpoonleft}(x{\upharpoonleft}x){\upharpoonleft}p)$.

(86) For all x, p holds $(p{\upharpoonleft}x){\upharpoonleft}(p{\upharpoonleft}x) = x{\upharpoonleft}p{\upharpoonleft}(x{\upharpoonleft}x{\upharpoonleft}(x{\upharpoonleft}x){\upharpoonleft}p)$.

(87) For all x, p holds $(p{\upharpoonleft}x){\upharpoonleft}(p{\upharpoonleft}x) = x{\upharpoonleft}p{\upharpoonleft}(x{\upharpoonleft}p)$.

(88) For all y, q, w holds $(w{\upharpoonleft}q{\upharpoonleft}(y{\upharpoonleft}y{\upharpoonleft}y)){\upharpoonleft}(w{\upharpoonleft}q{\upharpoonleft}(w{\upharpoonleft}q)) = w{\upharpoonleft}w{\upharpoonleft}(w{\upharpoonleft}q){\upharpoonleft}(q{\upharpoonleft}q{\upharpoonleft}(w{\upharpoonleft}q))$.

(89) For all q, w holds $w{\upharpoonleft}q = w{\upharpoonleft}w{\upharpoonleft}(w{\upharpoonleft}q){\upharpoonleft}(q{\upharpoonleft}q{\upharpoonleft}(w{\upharpoonleft}q))$.

(90) For all q, p holds $(p{\upharpoonleft}p){\upharpoonleft}(p{\upharpoonleft}(q{\upharpoonleft}q{\upharpoonleft}q)) = q{\upharpoonleft}q{\upharpoonleft}(q{\upharpoonleft}q){\upharpoonleft}p{\upharpoonleft}(q{\upharpoonleft}q{\upharpoonleft}p)$.

(91) For all p, q holds $p = q{\upharpoonleft}q{\upharpoonleft}(q{\upharpoonleft}q){\upharpoonleft}p{\upharpoonleft}(q{\upharpoonleft}q{\upharpoonleft}p)$.

(92) For all p, q holds $p = q{\upharpoonleft}p{\upharpoonleft}(q{\upharpoonleft}q{\upharpoonleft}p)$.

(93) For all z, w, x holds $(x{\upharpoonleft}x{\upharpoonleft}w{\upharpoonleft}(z{\upharpoonleft}z{\upharpoonleft}w)){\upharpoonleft}(w{\upharpoonleft}(x{\upharpoonleft}z){\upharpoonleft}(w{\upharpoonleft}(x{\upharpoonleft}z))) = w{\upharpoonleft}w{\upharpoonleft}(w{\upharpoonleft}(x{\upharpoonleft}z)){\upharpoonleft}(x{\upharpoonleft}z{\upharpoonleft}(x{\upharpoonleft}z){\upharpoonleft}(w{\upharpoonleft}(x{\upharpoonleft}z)))$.

(94) For all z, w, x holds $(x{\upharpoonleft}x{\upharpoonleft}w{\upharpoonleft}(z{\upharpoonleft}z{\upharpoonleft}w)){\upharpoonleft}(w{\upharpoonleft}(x{\upharpoonleft}z){\upharpoonleft}(w{\upharpoonleft}(x{\upharpoonleft}z))) = w{\upharpoonleft}(x{\upharpoonleft}z)$.

(95) For all w, p holds $(p{\upharpoonleft}p){\upharpoonleft}(p{\upharpoonleft}(w{\upharpoonleft}(w{\upharpoonleft}w))) = w{\upharpoonleft}w{\upharpoonleft}p{\upharpoonleft}(w{\upharpoonleft}w{\upharpoonleft}(w{\upharpoonleft}w){\upharpoonleft}p)$.

(96) For all p, w holds $p = w{\upharpoonleft}w{\upharpoonleft}p{\upharpoonleft}(w{\upharpoonleft}w{\upharpoonleft}(w{\upharpoonleft}w){\upharpoonleft}p)$.

(97) For all p, w holds $p = w{\upharpoonleft}w{\upharpoonleft}p{\upharpoonleft}(w{\upharpoonleft}p)$.

(98) For all z, q, x holds $(x{\upharpoonleft}x{\upharpoonleft}q{\upharpoonleft}(z{\upharpoonleft}z{\upharpoonleft}q)){\upharpoonleft}(q{\upharpoonleft}(x{\upharpoonleft}z){\upharpoonleft}(q{\upharpoonleft}(x{\upharpoonleft}z))) = z{\upharpoonleft}z{\upharpoonleft}(z{\upharpoonleft}z){\upharpoonleft}(x{\upharpoonleft}x{\upharpoonleft}q){\upharpoonleft}(q{\upharpoonleft}q{\upharpoonleft}(x{\upharpoonleft}x{\upharpoonleft}q))$.

(99) For all q, z, x holds $q{\upharpoonleft}(x{\upharpoonleft}z) = (z{\upharpoonleft}z{\upharpoonleft}(z{\upharpoonleft}z){\upharpoonleft}(x{\upharpoonleft}x{\upharpoonleft}q)){\upharpoonleft}(q{\upharpoonleft}q{\upharpoonleft}(x{\upharpoonleft}x{\upharpoonleft}q))$.

(100) For all q, z, x holds $q{\upharpoonleft}(x{\upharpoonleft}z) = (z{\upharpoonleft}(x{\upharpoonleft}x{\upharpoonleft}q)){\upharpoonleft}(q{\upharpoonleft}q{\upharpoonleft}(x{\upharpoonleft}x{\upharpoonleft}q))$.

(101) For all w, y holds $w{\upharpoonleft}w = y{\upharpoonleft}y{\upharpoonleft}y{\upharpoonleft}w$.

(102) For all w, p holds $p{\upharpoonleft}w{\upharpoonleft}(w{\upharpoonleft}w{\upharpoonleft}p) = p$.

(103) For all y, w holds $w{\upharpoonleft}w{\upharpoonleft}(w{\upharpoonleft}w{\upharpoonleft}(y{\upharpoonleft}y{\upharpoonleft}y)) = (y{\upharpoonleft}y){\upharpoonleft}y$.

(104) For all y, w holds $w{\upharpoonleft}w{\upharpoonleft}w = y{\upharpoonleft}y{\upharpoonleft}y$.

(105) For all p, w holds $w{\upharpoonleft}p{\upharpoonleft}(p{\upharpoonleft}(w{\upharpoonleft}w)) = p$.

(106) For all w, p holds $p{\upharpoonleft}(w{\upharpoonleft}w){\upharpoonleft}(w{\upharpoonleft}p) = p$.

(107) For all p, w holds $w{\upharpoonleft}p{\upharpoonleft}(w{\upharpoonleft}(p{\upharpoonleft}p)) = w$.

(108) For all x, y holds $y{\upharpoonleft}(y{\upharpoonleft}(x{\upharpoonleft}x){\upharpoonleft}(y{\upharpoonleft}(x{\upharpoonleft}x)){\upharpoonleft}(x{\upharpoonleft}y)) = x{\upharpoonleft}y$.

(109) For all p, w holds $w{\upharpoonleft}(p{\upharpoonleft}p){\upharpoonleft}(w{\upharpoonleft}p) = w$.

(110) For all p, w, q, y holds $(y{\upharpoonleft}y{\upharpoonleft}y{\upharpoonleft}q){\upharpoonleft}(w{\upharpoonleft}w{\upharpoonleft}q) = q{\upharpoonleft}(p{\upharpoonleft}(p{\upharpoonleft}p){\upharpoonleft}(p{\upharpoonleft}(p{\upharpoonleft}p)){\upharpoonleft}w){\upharpoonleft}(q{\upharpoonleft}(p{\upharpoonleft}(p{\upharpoonleft}p){\upharpoonleft}(p{\upharpoonleft}(p{\upharpoonleft}p)){\upharpoonleft}w))$.

(111) For all q, w, p holds $(q{\upharpoonleft}q){\upharpoonleft}(w{\upharpoonleft}w{\upharpoonleft}q) = q{\upharpoonleft}(p{\upharpoonleft}(p{\upharpoonleft}p){\upharpoonleft}(p{\upharpoonleft}(p{\upharpoonleft}p)){\upharpoonleft}w){\upharpoonleft}(q{\upharpoonleft}(p{\upharpoonleft}(p{\upharpoonleft}p){\upharpoonleft}(p{\upharpoonleft}(p{\upharpoonleft}p)){\upharpoonleft}w))$.

(112) For all w, y, p holds $w{\upharpoonleft}p{\upharpoonleft}(w{\upharpoonleft}(p{\upharpoonleft}(y{\upharpoonleft}(y{\upharpoonleft}y)))) = w$.

(113) For all w, y, p holds $w{\upharpoonleft}(p{\upharpoonleft}(y{\upharpoonleft}(y{\upharpoonleft}y))){\upharpoonleft}(w{\upharpoonleft}p) = w$.

(114) For all q, p, y holds $(y{\upharpoonleft}y{\upharpoonleft}y{\upharpoonleft}p){\upharpoonleft}(q{\upharpoonleft}q{\upharpoonleft}p) = p{\upharpoonleft}(p{\upharpoonleft}p{\upharpoonleft}q){\upharpoonleft}(p{\upharpoonleft}(p{\upharpoonleft}p{\upharpoonleft}q))$.

(115) For all q, z, x holds $(q\upharpoonleft(x\upharpoonleft x\upharpoonleft z)\upharpoonleft(q\upharpoonleft(x\upharpoonleft x\upharpoonleft z)))\upharpoonleft(x\upharpoonleft q\upharpoonleft(z\upharpoonleft z\upharpoonleft q)) = z\upharpoonleft z\upharpoonleft(z\upharpoonleft z)\upharpoonleft(x\upharpoonleft q)\upharpoonleft(q\upharpoonleft q\upharpoonleft(x\upharpoonleft q))$.

(116) For all q, z, x holds $(q\upharpoonleft(x\upharpoonleft x\upharpoonleft z)\upharpoonleft(q\upharpoonleft(x\upharpoonleft x\upharpoonleft z)))\upharpoonleft(x\upharpoonleft q\upharpoonleft(z\upharpoonleft z\upharpoonleft q)) = z\upharpoonleft(x\upharpoonleft q)\upharpoonleft(q\upharpoonleft q\upharpoonleft(x\upharpoonleft q))$.

(117) For all w, q, z holds $(w\upharpoonleft w\upharpoonleft(z\upharpoonleft z\upharpoonleft q))\upharpoonleft(q\upharpoonleft(q\upharpoonleft q\upharpoonleft z)\upharpoonleft(q\upharpoonleft(q\upharpoonleft q\upharpoonleft z))) = z\upharpoonleft z\upharpoonleft q\upharpoonleft(w\upharpoonleft q)\upharpoonleft(z\upharpoonleft z\upharpoonleft q\upharpoonleft(w\upharpoonleft q))$.

(118) For all q, p, x holds $p\upharpoonleft(x\upharpoonleft p)\upharpoonleft(p\upharpoonleft(x\upharpoonleft p))\upharpoonleft(q\upharpoonleft(q\upharpoonleft q)) = (x\upharpoonleft x)\upharpoonleft p$.

(119) For all p, x holds $p\upharpoonleft(x\upharpoonleft p) = (x\upharpoonleft x)\upharpoonleft p$.

(120) For all p, y holds $(y\upharpoonleft p)\upharpoonleft(y\upharpoonleft y\upharpoonleft p) = p\upharpoonleft p\upharpoonleft(y\upharpoonleft p)$.

(121) For all x, y holds $x = x\upharpoonleft x\upharpoonleft(y\upharpoonleft x)$.

(122) For all x, y holds $(y\upharpoonleft x)\upharpoonleft x = x\upharpoonleft(y\upharpoonleft y)\upharpoonleft(x\upharpoonleft(y\upharpoonleft y))\upharpoonleft(y\upharpoonleft x)$.

(123) For all x, z, y holds $x\upharpoonleft(y\upharpoonleft y\upharpoonleft z)\upharpoonleft(x\upharpoonleft(y\upharpoonleft y\upharpoonleft z))\upharpoonleft(y\upharpoonleft x\upharpoonleft(z\upharpoonleft z\upharpoonleft x)) = (z\upharpoonleft(y\upharpoonleft x))\upharpoonleft x$.

(124) For all x, y, z holds $x\upharpoonleft(z\upharpoonleft(z\upharpoonleft z)\upharpoonleft(z\upharpoonleft(z\upharpoonleft z))\upharpoonleft y)\upharpoonleft(x\upharpoonleft(z\upharpoonleft(z\upharpoonleft z)\upharpoonleft(z\upharpoonleft(z\upharpoonleft z))\upharpoonleft y)) = x$.

(125) For all x, z, y holds $(x\upharpoonleft(y\upharpoonleft y\upharpoonleft z))\upharpoonleft z = z\upharpoonleft(y\upharpoonleft x)$.

(126) For all x, y holds $x\upharpoonleft(y\upharpoonleft x\upharpoonleft x) = y\upharpoonleft x$.

(127) For all z, y, x holds $y = x\upharpoonleft x\upharpoonleft x\upharpoonleft y\upharpoonleft(z\upharpoonleft z\upharpoonleft y)$.

(128) For all z, y holds $y\upharpoonleft(y\upharpoonleft y\upharpoonleft z)\upharpoonleft(y\upharpoonleft(y\upharpoonleft y\upharpoonleft z)) = y$.

(129) For all x, z, y holds $y\upharpoonleft y\upharpoonleft z\upharpoonleft(x\upharpoonleft z)\upharpoonleft(y\upharpoonleft y\upharpoonleft z\upharpoonleft(x\upharpoonleft z)) = (x\upharpoonleft x\upharpoonleft(y\upharpoonleft y\upharpoonleft z))\upharpoonleft z$.

(130) For all x, z, y holds $(y\upharpoonleft y\upharpoonleft z\upharpoonleft(x\upharpoonleft z))\upharpoonleft(y\upharpoonleft y\upharpoonleft z\upharpoonleft(x\upharpoonleft z)) = z\upharpoonleft(y\upharpoonleft(x\upharpoonleft x))$.

(131) For all y, x holds $x\upharpoonleft y\upharpoonleft(x\upharpoonleft y)\upharpoonleft x = x\upharpoonleft y$.

(132) For all p, w holds $w\upharpoonleft w\upharpoonleft(w\upharpoonleft p) = w$.

(133) For all w, p holds $p\upharpoonleft w\upharpoonleft(w\upharpoonleft w) = w$.

(134) For all p, y, w holds $w\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft(w\upharpoonleft p) = w$.

(135) For all p, w holds $w\upharpoonleft p\upharpoonleft(w\upharpoonleft w) = w$.

(136) For all y, p, w holds $w\upharpoonleft p\upharpoonleft(w\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))) = w$.

(137) For all p, q, w, y, x holds $(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft w\upharpoonleft(q\upharpoonleft q\upharpoonleft w))\upharpoonleft(w\upharpoonleft(x\upharpoonleft q)\upharpoonleft(w\upharpoonleft(x\upharpoonleft q))) = w\upharpoonleft(p\upharpoonleft(p\upharpoonleft p))\upharpoonleft(w\upharpoonleft(x\upharpoonleft q))\upharpoonleft(x\upharpoonleft q\upharpoonleft(x\upharpoonleft q)\upharpoonleft(w\upharpoonleft(x\upharpoonleft q)))$.

(138) For all q, w, y, x holds $(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft w\upharpoonleft(q\upharpoonleft q\upharpoonleft w))\upharpoonleft(w\upharpoonleft(x\upharpoonleft q)\upharpoonleft(w\upharpoonleft(x\upharpoonleft q))) = w\upharpoonleft(x\upharpoonleft q\upharpoonleft(x\upharpoonleft q)\upharpoonleft(w\upharpoonleft(x\upharpoonleft q)))$.

(139) For all q, w, y, x holds $(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft w\upharpoonleft(q\upharpoonleft q\upharpoonleft w))\upharpoonleft(w\upharpoonleft(x\upharpoonleft q)\upharpoonleft(w\upharpoonleft(x\upharpoonleft q))) = w\upharpoonleft(x\upharpoonleft q)$.

(140) For all z, p, q, y, x holds $(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q\upharpoonleft(z\upharpoonleft z\upharpoonleft q))\upharpoonleft(q\upharpoonleft(x\upharpoonleft z)\upharpoonleft(q\upharpoonleft(x\upharpoonleft z))) = z\upharpoonleft z\upharpoonleft(p\upharpoonleft(p\upharpoonleft p))\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q)\upharpoonleft(q\upharpoonleft q\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q))$.

(141) For all z, p, q, y, x holds $q\upharpoonleft(x\upharpoonleft z) = (z\upharpoonleft z\upharpoonleft(p\upharpoonleft(p\upharpoonleft p))\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q))\upharpoonleft(q\upharpoonleft q\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q))$.

(142) For all z, q, y, x holds $q\upharpoonleft(x\upharpoonleft z) = (z\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q))\upharpoonleft(q\upharpoonleft q\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft q))$.

(143) For all v, p, y, x holds $p\upharpoonleft(x\upharpoonleft v) = (v\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft p))\upharpoonleft p$.

(144) For all y, w, z, v, x holds $(w\upharpoonleft(z\upharpoonleft(x\upharpoonleft v)))\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft z\upharpoonleft(v\upharpoonleft v\upharpoonleft z)) = z\upharpoonleft(x\upharpoonleft v)$.

(145) For all y, z, x holds $(y\upharpoonleft(x\upharpoonleft x\upharpoonleft z)\upharpoonleft(y\upharpoonleft(x\upharpoonleft x\upharpoonleft z)))\upharpoonleft(x\upharpoonleft y\upharpoonleft(z\upharpoonleft z\upharpoonleft y)) = y\upharpoonleft(x\upharpoonleft x\upharpoonleft z)$.

(146) For all z, y, x holds $(z\upharpoonleft(x\upharpoonleft y))\upharpoonleft y = y\upharpoonleft(x\upharpoonleft x\upharpoonleft z)$.

(147) For all x, w, y, z holds $(x\upharpoonleft x\upharpoonleft w\upharpoonleft(z\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft w))\upharpoonleft w = w\upharpoonleft(x\upharpoonleft z)$.

(148) For all z, w, x holds $w\upharpoonleft(z\upharpoonleft(x\upharpoonleft x\upharpoonleft w)) = w\upharpoonleft(x\upharpoonleft z)$.

(149) For all p, z, y, x holds $(z\upharpoonleft(x\upharpoonleft p)\upharpoonleft(z\upharpoonleft(x\upharpoonleft p)))\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft z\upharpoonleft(p\upharpoonleft p\upharpoonleft z)) = p\upharpoonleft p\upharpoonleft z\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft z)\upharpoonleft(p\upharpoonleft p\upharpoonleft z\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft z))$.

(150) For all p, z, y, x holds $z\upharpoonleft(x\upharpoonleft p) = (p\upharpoonleft p\upharpoonleft z\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft z))\upharpoonleft(p\upharpoonleft p\upharpoonleft z\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft z))$.

(151) For all z, p, y, x holds $z\upharpoonleft(x\upharpoonleft p) = z\upharpoonleft(p\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y)))))$.

(152) For all z, p, x holds $z\upharpoonleft(x\upharpoonleft p) = z\upharpoonleft(p\upharpoonleft x)$.

(153) For all w, q, p holds $(p\upharpoonleft q)\upharpoonleft w = w\upharpoonleft(q\upharpoonleft p)$.

(154) For all w, p, q holds $(q\upharpoonleft p\upharpoonleft w)\upharpoonleft q = q\upharpoonleft(p\upharpoonleft p\upharpoonleft w)$.

(155) For all z, w, y, x holds $w\upharpoonleft x = w\upharpoonleft(x\upharpoonleft z\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y))\upharpoonleft(x\upharpoonleft(y\upharpoonleft(y\upharpoonleft y)))\upharpoonleft w))$.

(156) For all w, z, x holds $w\upharpoonleft x = w\upharpoonleft(x\upharpoonleft z\upharpoonleft(x\upharpoonleft w))$.

(157) For all q, x, z, y holds $(x\upharpoonleft y)\upharpoonleft(x\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft z)))\upharpoonleft q\upharpoonleft x) = x\upharpoonleft y\upharpoonleft(x\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft z))))$.

(158) For all x, q, z, y holds $(x\upharpoonleft y)\upharpoonleft(x\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft z))\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft z)))\upharpoonleft q)) = x\upharpoonleft y\upharpoonleft(x\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft z))))$.

(159) For all z, x, q, y holds $(x\upharpoonleft y)\upharpoonleft(x\upharpoonleft(y\upharpoonleft q)) = x\upharpoonleft y\upharpoonleft(x\upharpoonleft(y\upharpoonleft(z\upharpoonleft(z\upharpoonleft z))))$.

(160) For all x, q, y holds $x\upharpoonleft y\upharpoonleft(x\upharpoonleft(y\upharpoonleft q)) = x$.

(161) L satisfies (Sh_1).

Let us mention that every non empty Sheffer structure which satisfies ($\mathrm{Sheffer}_1$), ($\mathrm{Sheffer}_2$), and ($\mathrm{Sheffer}_3$) satisfies also (Sh_1) and every non empty Sheffer structure which satisfies (Sh_1) satisfies also ($\mathrm{Sheffer}_1$), ($\mathrm{Sheffer}_2$), and ($\mathrm{Sheffer}_3$).

Let us observe that every non empty Sheffer ortholattice structure which is properly defined satisfies (Sh_1) is also Boolean and lattice-like and every non empty Sheffer ortholattice structure which is Boolean, lattice-like, well-complemented, and properly defined satisfies also (Sh_1).

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