FORMALIZED MATHEMATICS
Volume 12, Number 3, 2004
University of Białystok

The Fundamental Group of Convex Subspaces of $\mathcal{E}^n_{\mathrm{T}}$

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Summary. The triviality of the fundamental group of subspaces of $\mathcal{E}^n_{\mathrm{T}}$ and $\mathbb{R}^{\mathbf{1}}$ have been shown.

MML Identifier: TOPALG_2.

The notation and terminology used in this paper have been introduced in the following articles: [20], [6], [23], [1], [17], [24], [4], [5], [3], [2], [19], [11], [16], [22], [21], [18], [14], [8], [7], [15], [13], [9], [10], and [12].

1. Convex subspaces of $\mathcal{E}^n_{\mathrm{T}}$

In this paper n denotes a natural number and a, b denote real numbers.

Let us consider n. One can verify that there exists a subset of $\mathcal{E}^n_{\mathrm{T}}$ which is non empty and convex.

Let n be a natural number and let T be a subspace of $\mathcal{E}^n_{\mathrm{T}}$. We say that T is convex if and only if:

(Def. 1) Ω_T is a convex subset of $\mathcal{E}^n_{\mathrm{T}}$.

Let n be a natural number. Note that every non empty subspace of $\mathcal{E}^n_{\mathrm{T}}$ which is convex is also arcwise connected.

Let n be a natural number. One can verify that there exists a subspace of $\mathcal{E}^n_{\mathrm{T}}$ which is strict, non empty, and convex.

The following proposition is true

[1]The paper was written during author's post-doctoral fellowship granted by Shinshu University, Japan. This work has been partially supported by KBN grant 4 T11C 039 24.

ISSN 1426–2630

(1) Let X be a non empty topological space, Y be a non empty subspace of X, x_1, x_2 be points of X, y_1, y_2 be points of Y, and f be a path from y_1 to y_2. Suppose $x_1 = y_1$ and $x_2 = y_2$ and y_1, y_2 are connected. Then f is a path from x_1 to x_2.

Let n be a natural number, let T be a non empty convex subspace of $\mathcal{E}^n_{\mathrm{T}}$, let a, b be points of T, and let P, Q be paths from a to b. The functor ConvexHomotopy(P, Q) yielding a map from $[\!:\mathbb{I}, \mathbb{I}:\!]$ into T is defined as follows:

(Def. 2) For all elements s, t of $\mathbb{I}$ and for all points a_1, b_1 of $\mathcal{E}^n_{\mathrm{T}}$ such that $a_1 = P(s)$ and $b_1 = Q(s)$ holds $(\mathrm{ConvexHomotopy}(P,Q))(s, t) = (1 - t) \cdot a_1 + t \cdot b_1$.

Next we state the proposition

(2) Let T be a non empty convex subspace of $\mathcal{E}^n_{\mathrm{T}}$, a, b be points of T, and P, Q be paths from a to b. Then P, Q are homotopic.

Let n be a natural number, let T be a non empty convex subspace of $\mathcal{E}^n_{\mathrm{T}}$, let a, b be points of T, and let P, Q be paths from a to b. Then ConvexHomotopy(P, Q) is a homotopy between P and Q.

Let n be a natural number, let T be a non empty convex subspace of $\mathcal{E}^n_{\mathrm{T}}$, let a, b be points of T, and let P, Q be paths from a to b. Note that every homotopy between P and Q is continuous.

We now state the proposition

(3) Let T be a non empty convex subspace of $\mathcal{E}^n_{\mathrm{T}}$, a be a point of T, and C be a loop of a. Then the carrier of $\pi_1(T, a) = \{[C]_{\mathrm{EqRel}(T,a)}\}$.

Let n be a natural number, let T be a non empty convex subspace of $\mathcal{E}^n_{\mathrm{T}}$, and let a be a point of T. Observe that $\pi_1(T, a)$ is trivial.

2. Convex subspaces of $\mathbb{R}^{\mathbf{1}}$

We now state the proposition

(4) $\mathrm{Proj}(|[a]|, 1) = a$.

One can verify that every subspace of $\mathbb{R}^{\mathbf{1}}$ is real-membered.

Next we state three propositions:

(5) If $a \leqslant b$, then $[a, b] = \{(1 - l) \cdot a + l \cdot b; l$ ranges over real numbers: $0 \leqslant l \ \wedge \ l \leqslant 1\}$.

(6) Let F be a map from $[\!:\mathbb{R}^{\mathbf{1}}, \mathbb{I}:\!]$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that for every point x of $\mathbb{R}^{\mathbf{1}}$ and for every point i of $\mathbb{I}$ holds $F(x, i) = (1 - i) \cdot x$. Then F is continuous.

(7) Let F be a map from $[\!:\mathbb{R}^{\mathbf{1}}, \mathbb{I}:\!]$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that for every point x of $\mathbb{R}^{\mathbf{1}}$ and for every point i of $\mathbb{I}$ holds $F(x, i) = i \cdot x$. Then F is continuous.

Let P be a subset of $\mathbb{R}^{\mathbf{1}}$. We say that P is convex if and only if:

(Def. 3) For all points a, b of $\mathbb{R}^{\mathbf{1}}$ such that $a \in P$ and $b \in P$ holds $[a, b] \subseteq P$.

One can check that there exists a subset of $\mathbb{R}^{\mathbf{1}}$ which is non empty and convex and every subset of $\mathbb{R}^{\mathbf{1}}$ which is empty is also convex.

We now state four propositions:

(8) $[a, b]$ is a convex subset of $\mathbb{R}^{\mathbf{1}}$.

(9) $]a, b[$ is a convex subset of $\mathbb{R}^{\mathbf{1}}$.

(10) $[a, b[$ is a convex subset of $\mathbb{R}^{\mathbf{1}}$.

(11) $]a, b]$ is a convex subset of $\mathbb{R}^{\mathbf{1}}$.

Let T be a subspace of $\mathbb{R}^{\mathbf{1}}$. We say that T is convex if and only if:

(Def. 4) Ω_T is a convex subset of $\mathbb{R}^{\mathbf{1}}$.

Let us note that there exists a subspace of $\mathbb{R}^{\mathbf{1}}$ which is strict, non empty, and convex.

$\mathbb{R}^{\mathbf{1}}$ is a strict convex subspace of $\mathbb{R}^{\mathbf{1}}$.

The following proposition is true

(12) For every non empty convex subspace T of $\mathbb{R}^{\mathbf{1}}$ and for all points a, b of T holds $[a, b] \subseteq$ the carrier of T.

Let us note that every non empty subspace of $\mathbb{R}^{\mathbf{1}}$ which is convex is also arcwise connected.

One can prove the following propositions:

(13) If $a \leqslant b$, then $[a, b]_{\mathrm{T}}$ is convex.

(14) $\mathbb{I}$ is convex.

(15) If $a \leqslant b$, then $[a, b]_{\mathrm{T}}$ is arcwise connected.

Let T be a non empty convex subspace of $\mathbb{R}^{\mathbf{1}}$, let a, b be points of T, and let P, Q be paths from a to b. The functor R1Homotopy(P, Q) yields a map from $[\!:\mathbb{I}, \mathbb{I}:\!]$ into T and is defined by:

(Def. 5) For all elements s, t of $\mathbb{I}$ holds $(\text{R1Homotopy}(P, Q))(s, t) = (1 - t) \cdot P(s) + t \cdot Q(s)$.

Next we state the proposition

(16) Let T be a non empty convex subspace of $\mathbb{R}^{\mathbf{1}}$, a, b be points of T, and P, Q be paths from a to b. Then P, Q are homotopic.

Let T be a non empty convex subspace of $\mathbb{R}^{\mathbf{1}}$, let a, b be points of T, and let P, Q be paths from a to b. Then R1Homotopy(P, Q) is a homotopy between P and Q.

Let T be a non empty convex subspace of $\mathbb{R}^{\mathbf{1}}$, let a, b be points of T, and let P, Q be paths from a to b. Note that every homotopy between P and Q is continuous.

The following proposition is true

(17) Let T be a non empty convex subspace of $\mathbb{R}^{\mathbf{1}}$, a be a point of T, and C be a loop of a. Then the carrier of $\pi_1(T, a) = \{[C]_{\mathrm{EqRel}(T, a)}\}$.

Let T be a non empty convex subspace of $\mathbb{R}^{\mathbf{1}}$ and let a be a point of T. Observe that $\pi_1(T, a)$ is trivial.

One can prove the following four propositions:

(18) If $a \leqslant b$, then for all points x, y of $[a, b]_{\rm T}$ and for all paths P, Q from x to y holds P, Q are homotopic.

(19) If $a \leqslant b$, then for every point x of $[a, b]_{\rm T}$ and for every loop C of x holds the carrier of $\pi_1([a, b]_{\rm T}, x) = \{[C]_{{\rm EqRel}([a,b]_{\rm T}, x)}\}$.

(20) For all points x, y of $\mathbb{I}$ and for all paths P, Q from x to y holds P, Q are homotopic.

(21) For every point x of $\mathbb{I}$ and for every loop C of x holds the carrier of $\pi_1(\mathbb{I}, x) = \{[C]_{{\rm EqRel}(\mathbb{I}, x)}\}$.

Let x be a point of $\mathbb{I}$. Observe that $\pi_1(\mathbb{I}, x)$ is trivial.

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Received April 20, 2004