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On the Isomorphism of Fundamental Groups[1]

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The terminology and notation used here have been introduced in the following articles: [24], [7], [27], [28], [22], [4], [29], [5], [2], [18], [23], [3], [6], [21], [19], [26], [25], [9], [8], [20], [16], [11], [10], [1], [13], [14], [12], [15], and [17].

1. Preliminaries

One can prove the following propositions:

(1) Let A, B, a, b be sets and f be a function from A into B. If $a \in A$ and $b \in B$, then $f +\cdot (a \dot{\longmapsto} b)$ is a function from A into B.

(2) For every function f and for all sets X, x such that $f \restriction X$ is one-to-one and $x \in \mathrm{rng}(f \restriction X)$ holds $(f \cdot (f \restriction X)^{-1})(x) = x$.

(3) Let x, y, X, Y, Z be sets, f be a function from $[\!: X, Y :\!]$ into Z, and g be a function. If $Z \neq \emptyset$ and $x \in X$ and $y \in Y$, then $(g \cdot f)(x, y) = g(f(x, y))$.

(4) For all sets X, a, b and for every function f from X into $\{a, b\}$ holds $X = f^{-1}(\{a\}) \cup f^{-1}(\{b\})$.

(5) For all non empty 1-sorted structures S, T and for every point s of S and for every point t of T holds $(S \longmapsto t)(s) = t$.

(6) Let T be a non empty topological structure, t be a point of T, and A be a subset of T. If $A = \{t\}$, then $\mathrm{Sspace}(t) = T \restriction A$.

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(7) Let T be a topological space, A, B be subsets of T, and C, D be subsets of the topological structure of T. Suppose $A = C$ and $B = D$. Then A and B are separated if and only if C and D are separated.

(8) For every non empty topological space T holds T is connected iff there exists no map from T into $\{\{0,1\}\}_{\rm top}$ which is continuous and onto.

One can verify that every topological structure which is empty is also connected.

We now state the proposition

(9) For every topological space T such that the topological structure of T is connected holds T is connected.

Let T be a connected topological space. One can check that the topological structure of T is connected.

One can prove the following proposition

(10) Let S, T be non empty topological spaces. Suppose S and T are homeomorphic and S is arcwise connected. Then T is arcwise connected.

One can verify that every non empty topological space which is trivial is also arcwise connected.

One can prove the following propositions:

(11) For every subspace T of $\mathcal{E}^2_{\rm T}$ such that the carrier of T is a simple closed curve holds T is arcwise connected.

(12) Let T be a topological space. Then there exists a family F of subsets of T such that $F = \{\text{the carrier of } T\}$ and F is a cover of T and open.

Let T be a topological space. Note that there exists a family of subsets of T which is non empty, mutually-disjoint, open, and closed.

The following proposition is true

(13) Let T be a topological space, D be a mutually-disjoint open family of subsets of T, A be a subset of T, and X be a set. If A is connected and $X \in D$ and X meets A and D is a cover of A, then $A \subseteq X$.

2. On the Product of Topologies

One can prove the following three propositions:

(14) Let S, T be topological spaces. Then the topological structure of $[\!:S, T:\!] = [\!:\text{the topological structure of } S, \text{ the topological structure of } T:\!]$.

(15) For all topological spaces S, T and for every subset A of S and for every subset B of T holds $\overline{[\!:A, B:\!]} = [\!:\overline{A}, \overline{B}:\!]$.

(16) Let S, T be topological spaces, A be a closed subset of S, and B be a closed subset of T. Then $[\!:A, B:\!]$ is closed.

Let A, B be connected topological spaces. One can check that $[\!: A, B :\!]$ is connected.

One can prove the following propositions:

(17) Let S, T be topological spaces, A be a subset of S, and B be a subset of T. If A is connected and B is connected, then $[\!: A, B :\!]$ is connected.

(18) Let S, T be topological spaces, Y be a non empty topological space, A be a subset of S, f be a map from $[\!: S, T :\!]$ into Y, and g be a map from $[\!: S \restriction A, T :\!]$ into Y. If $g = f \restriction [\!: A$, the carrier of $T :\!]$ and f is continuous, then g is continuous.

(19) Let S, T be topological spaces, Y be a non empty topological space, A be a subset of T, f be a map from $[\!: S, T :\!]$ into Y, and g be a map from $[\!: S, T \restriction A :\!]$ into Y. If $g = f \restriction [\!:$ the carrier of S, $A :\!]$ and f is continuous, then g is continuous.

(20) Let S, T, T_1, T_2, Y be non empty topological spaces, f be a map from $[\!: Y, T_1 :\!]$ into S, g be a map from $[\!: Y, T_2 :\!]$ into S, and F_1, F_2 be closed subsets of T. Suppose that T_1 is a subspace of T and T_2 is a subspace of T and $F_1 = \Omega_{(T_1)}$ and $F_2 = \Omega_{(T_2)}$ and $\Omega_{(T_1)} \cup \Omega_{(T_2)} = \Omega_T$ and f is continuous and g is continuous and for every set p such that $p \in \Omega_{[\!: Y, T_1 :\!]} \cap \Omega_{[\!: Y, T_2 :\!]}$ holds $f(p) = g(p)$. Then there exists a map h from $[\!: Y, T :\!]$ into S such that $h = f {+\cdot} g$ and h is continuous.

(21) Let S, T, T_1, T_2, Y be non empty topological spaces, f be a map from $[\!: T_1, Y :\!]$ into S, g be a map from $[\!: T_2, Y :\!]$ into S, and F_1, F_2 be closed subsets of T. Suppose that T_1 is a subspace of T and T_2 is a subspace of T and $F_1 = \Omega_{(T_1)}$ and $F_2 = \Omega_{(T_2)}$ and $\Omega_{(T_1)} \cup \Omega_{(T_2)} = \Omega_T$ and f is continuous and g is continuous and for every set p such that $p \in \Omega_{[\!: T_1, Y :\!]} \cap \Omega_{[\!: T_2, Y :\!]}$ holds $f(p) = g(p)$. Then there exists a map h from $[\!: T, Y :\!]$ into S such that $h = f {+\cdot} g$ and h is continuous.

3. On the Fundamental Groups

Let T be a non empty topological space and let t be a point of T. Observe that every loop of t is continuous.

We now state a number of propositions:

(22) Let T be a non empty topological space, t be a point of T, x be a point of $\mathbb{I}$, and P be a constant loop of t. Then $P(x) = t$.

(23) For every non empty topological space T and for every point t of T and for every loop P of t holds $P(0) = t$ and $P(1) = t$.

(24) Let S, T be non empty topological spaces, f be a continuous map from S into T, and a, b be points of S. If a, b are connected, then $f(a)$, $f(b)$ are connected.

(25) Let S, T be non empty topological spaces, f be a continuous map from S into T, a, b be points of S, and P be a path from a to b. If a, b are connected, then $f \cdot P$ is a path from $f(a)$ to $f(b)$.

(26) Let S be a non empty arcwise connected topological space, T be a non empty topological space, f be a continuous map from S into T, a, b be points of S, and P be a path from a to b. Then $f \cdot P$ is a path from $f(a)$ to $f(b)$.

(27) Let S, T be non empty topological spaces, f be a continuous map from S into T, a be a point of S, and P be a loop of a. Then $f \cdot P$ is a loop of $f(a)$.

(28) Let S, T be non empty topological spaces, f be a continuous map from S into T, a, b be points of S, P, Q be paths from a to b, and P_1, Q_1 be paths from $f(a)$ to $f(b)$. Suppose P, Q are homotopic and $P_1 = f \cdot P$ and $Q_1 = f \cdot Q$. Then P_1, Q_1 are homotopic.

(29) Let S, T be non empty topological spaces, f be a continuous map from S into T, a, b be points of S, P, Q be paths from a to b, P_1, Q_1 be paths from $f(a)$ to $f(b)$, and F be a homotopy between P and Q. Suppose P, Q are homotopic and $P_1 = f \cdot P$ and $Q_1 = f \cdot Q$. Then $f \cdot F$ is a homotopy between P_1 and Q_1.

(30) Let S, T be non empty topological spaces, f be a continuous map from S into T, a, b, c be points of S, P be a path from a to b, Q be a path from b to c, P_1 be a path from $f(a)$ to $f(b)$, and Q_1 be a path from $f(b)$ to $f(c)$. Suppose a, b are connected and b, c are connected and $P_1 = f \cdot P$ and $Q_1 = f \cdot Q$. Then $P_1 + Q_1 = f \cdot (P + Q)$.

(31) Let S be a non empty topological space, s be a point of S, x, y be elements of $\pi_1(S, s)$, and P, Q be loops of s. If $x = [P]_{\mathrm{EqRel}(S,s)}$ and $y = [Q]_{\mathrm{EqRel}(S,s)}$, then $x \cdot y = [P + Q]_{\mathrm{EqRel}(S,s)}$.

Let S, T be non empty topological spaces, let s be a point of S, and let f be a map from S into T. Let us assume that f is continuous. The functor $\mathrm{FundGrIso}(f, s)$ yielding a map from $\pi_1(S, s)$ into $\pi_1(T, f(s))$ is defined by the condition (Def. 1).

(Def. 1) Let x be an element of $\pi_1(S, s)$. Then there exists a loop l_1 of s and there exists a loop l_2 of $f(s)$ such that $x = [l_1]_{\mathrm{EqRel}(S,s)}$ and $l_2 = f \cdot l_1$ and $(\mathrm{FundGrIso}(f, s))(x) = [l_2]_{\mathrm{EqRel}(T,f(s))}$.

The following proposition is true

(32) Let S, T be non empty topological spaces, s be a point of S, f be a continuous map from S into T, x be an element of $\pi_1(S, s)$, l_1 be a loop of s, and l_2 be a loop of $f(s)$. If $x = [l_1]_{\mathrm{EqRel}(S,s)}$ and $l_2 = f \cdot l_1$, then $(\mathrm{FundGrIso}(f, s))(x) = [l_2]_{\mathrm{EqRel}(T,f(s))}$.

Let S, T be non empty topological spaces, let s be a point of S, and let f

be a continuous map from S into T. Then FundGrIso(f, s) is a homomorphism from $\pi_1(S, s)$ to $\pi_1(T, f(s))$.

We now state three propositions:

(33) Let S, T be non empty topological spaces, s be a point of S, and f be a continuous map from S into T. If f is a homeomorphism, then FundGrIso(f, s) is an isomorphism.

(34) Let S, T be non empty topological spaces, s be a point of S, t be a point of T, f be a continuous map from S into T, P be a path from t to $f(s)$, and h be a homomorphism from $\pi_1(S, s)$ to $\pi_1(T, t)$. Suppose f is a homeomorphism and $f(s)$, t are connected and $h = \pi_1\text{-iso}(P) \cdot$ FundGrIso(f, s). Then h is an isomorphism.

(35) Let S be a non empty topological space, T be a non empty arcwise connected topological space, s be a point of S, and t be a point of T. If S and T are homeomorphic, then $\pi_1(S, s)$ and $\pi_1(T, t)$ are isomorphic.

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