FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 199–219
University of Białystok, 2005

The Fashoda Meet Theorem for Rectangles

Yatsuka Nakamura
Shinshu University
Nagano, Japan

Andrzej Trybulec
Institute of Computer Science
University of Białystok
Sosnowa 64, 15-887 Białystok
Poland

Summary. Here, the so called Fashoda Meet Theorem is proven in the case of rectangles. All cases of proper location of arcs are listed up, and it is shown that the theorem is valid in each case. Such a list of cases will be useful when one wants to apply the theorem.

MML identifier: `JGRAPH_7`, version: `7.5.01 4.39.921`

The articles [1], [6], [15], [17], [5], [2], [3], [16], [7], [14], [13], [10], [11], [8], [4], [9], and [12] provide the notation and terminology for this paper.

One can prove the following propositions:

(1) For all real numbers a, b, d and for every point p of $\mathcal{E}^2_{\rm T}$ such that $a < b$ and $p_{\mathbf{2}} = d$ and $a \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq b$ holds $p \in \mathcal{L}([a,d],[b,d])$.

(2) Let n be a natural number, P be a subset of $\mathcal{E}^n_{\rm T}$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. Suppose P is an arc from p_1 to p_2. Then there exists a map f from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$ such that f is continuous and one-to-one and $\operatorname{rng} f = P$ and $f(0) = p_1$ and $f(1) = p_2$.

(3) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and b, c, d be real numbers. If $(p_1)_{\mathbf{1}} < b$ and $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$, then $p_1 \leq_{{\rm Rectangle}((p_1)_{\mathbf{1}},b,c,d)} p_2$.

(4) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and b, c be real numbers. Suppose $(p_1)_{\mathbf{1}} < b$ and $c < (p_2)_{\mathbf{2}}$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq (p_2)_{\mathbf{2}}$ and $(p_1)_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$. Then $p_1 \leq_{{\rm Rectangle}((p_1)_{\mathbf{1}},b,c,(p_2)_{\mathbf{2}})} p_2$.

(5) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and c, d be real numbers. Suppose $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $c < d$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $c \leq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$. Then $p_1 \leq_{{\rm Rectangle}((p_1)_{\mathbf{1}},(p_2)_{\mathbf{1}},c,d)} p_2$.

ISSN 1426–2630

(6) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and b, d be real numbers. If $(p_2)_{\bf 2} < d$ and $(p_2)_{\bf 2} \leq (p_1)_{\bf 2}$ and $(p_1)_{\bf 2} \leq d$ and $(p_1)_{\bf 1} < (p_2)_{\bf 1}$ and $(p_2)_{\bf 1} \leq b$, then $p_1 \leq_{{\rm Rectangle}((p_1)_{\bf 1},b,(p_2)_{\bf 2},d)} p_2$.

(7) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\bf 2} = d$ and $(p_2)_{\bf 2} = d$ and $a \leq (p_1)_{\bf 1}$ and $(p_1)_{\bf 1} < (p_2)_{\bf 1}$ and $(p_2)_{\bf 1} \leq b$. Then $p_1 \leq_{{\rm Rectangle}(a,b,c,d)} p_2$.

(8) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\bf 2} = d$ and $(p_2)_{\bf 1} = b$ and $a \leq (p_1)_{\bf 1}$ and $(p_1)_{\bf 1} \leq b$ and $c \leq (p_2)_{\bf 2}$ and $(p_2)_{\bf 2} \leq d$. Then $p_1 \leq_{{\rm Rectangle}(a,b,c,d)} p_2$.

(9) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\bf 2} = d$ and $(p_2)_{\bf 2} = c$ and $a \leq (p_1)_{\bf 1}$ and $(p_1)_{\bf 1} \leq b$ and $a < (p_2)_{\bf 1}$ and $(p_2)_{\bf 1} \leq b$. Then $p_1 \leq_{{\rm Rectangle}(a,b,c,d)} p_2$.

(10) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\bf 1} = b$ and $(p_2)_{\bf 1} = b$ and $c \leq (p_2)_{\bf 2}$ and $(p_2)_{\bf 2} < (p_1)_{\bf 2}$ and $(p_1)_{\bf 2} \leq d$. Then $p_1 \leq_{{\rm Rectangle}(a,b,c,d)} p_2$.

(11) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\bf 1} = b$ and $(p_2)_{\bf 2} = c$ and $c \leq (p_1)_{\bf 2}$ and $(p_1)_{\bf 2} \leq d$ and $a < (p_2)_{\bf 1}$ and $(p_2)_{\bf 1} \leq b$. Then $p_1 \leq_{{\rm Rectangle}(a,b,c,d)} p_2$.

(12) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\bf 2} = c$ and $(p_2)_{\bf 2} = c$ and $a < (p_2)_{\bf 1}$ and $(p_2)_{\bf 1} < (p_1)_{\bf 1}$ and $(p_1)_{\bf 1} \leq b$. Then $p_1 \leq_{{\rm Rectangle}(a,b,c,d)} p_2$.

(13) Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\bf 2} = d$ and $(p_2)_{\bf 1} = b$ and $a \leq (p_1)_{\bf 1}$ and $(p_1)_{\bf 1} \leq b$ and $c \leq (p_2)_{\bf 2}$ and $(p_2)_{\bf 2} \leq d$. Then $p_1 \leq_{{\rm Rectangle}(a,b,c,d)} p_2$.

(14) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\bf 1} = a$ and $(p_2)_{\bf 1} = a$ and $(p_3)_{\bf 1} = a$ and $(p_4)_{\bf 1} = a$ and $c \leq (p_1)_{\bf 2}$ and $(p_1)_{\bf 2} < (p_2)_{\bf 2}$ and $(p_2)_{\bf 2} < (p_3)_{\bf 2}$ and $(p_3)_{\bf 2} < (p_4)_{\bf 2}$ and $(p_4)_{\bf 2} \leq d$. Then p_1, p_2, p_3, p_4 are in this order on ${\rm Rectangle}(a,b,c,d)$.

(15) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\bf 1} = a$ and $(p_2)_{\bf 1} = a$ and $(p_3)_{\bf 1} = a$ and $(p_4)_{\bf 2} = d$ and $c \leq (p_1)_{\bf 2}$ and $(p_1)_{\bf 2} < (p_2)_{\bf 2}$ and $(p_2)_{\bf 2} < (p_3)_{\bf 2}$ and $(p_3)_{\bf 2} \leq d$ and $a \leq (p_4)_{\bf 1}$ and $(p_4)_{\bf 1} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on ${\rm Rectangle}(a,b,c,d)$.

(16) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\bf 1} = a$ and $(p_2)_{\bf 1} = a$ and $(p_3)_{\bf 1} = a$ and $(p_4)_{\bf 1} = b$ and $c \leq (p_1)_{\bf 2}$ and $(p_1)_{\bf 2} < (p_2)_{\bf 2}$ and $(p_2)_{\bf 2} < (p_3)_{\bf 2}$ and $(p_3)_{\bf 2} \leq d$ and $c \leq (p_4)_{\bf 2}$ and $(p_4)_{\bf 2} \leq d$. Then p_1, p_2, p_3, p_4 are in this order on ${\rm Rectangle}(a,b,c,d)$.

(17) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\bf 1} = a$ and $(p_2)_{\bf 1} = a$ and $(p_3)_{\bf 1} = a$ and

$(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(18) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = d$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a \leq (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(19) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a \leq (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(20) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a \leq (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(21) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(22) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $c \leq (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(23) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(24) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = d$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(25) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and

$(p_3)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(26) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(27) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(28) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $(p_1)_{\mathbf{1}} \neq (p_3)_{\mathbf{1}}$ and $(p_4)_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq (p_2)_{\mathbf{2}}$ and $(p_1)_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq (p_3)_{\mathbf{1}}$ and $(p_4)_{\mathbf{2}} \leq (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq (p_2)_{\mathbf{2}}$ and $(p_1)_{\mathbf{1}} < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq (p_3)_{\mathbf{1}}$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle$((p_1)_{\mathbf{1}}, (p_3)_{\mathbf{1}}, (p_4)_{\mathbf{2}}, (p_2)_{\mathbf{2}})$.

(29) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(30) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $d \geq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} > (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \geq c$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(31) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $d \geq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \geq c$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(32) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $c \leq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(33) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = c$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on

Rectangle(a, b, c, d).

(34) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = d$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(35) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{1}} = b$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(36) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(37) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(38) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $c \leq (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(39) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(40) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leq b$ and $d \geq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} > (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \geq c$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(41) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leq b$ and $d \geq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \geq c$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(42) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leq b$ and $c \leq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(43) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = c$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(44) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = b$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $d \geq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} > (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} > (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \geq c$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(45) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = b$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $d \geq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} > (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \geq c$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(46) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = b$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $d \geq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} > (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \geq c$ and $b \geq (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} > (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} > a$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(47) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = b$ and $(p_2)_{\mathbf{2}} = c$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $b \geq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} > (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} > (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} > a$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(48) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$ and a, b, c, d be real numbers. Suppose $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = c$ and $(p_2)_{\mathbf{2}} = c$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $b \geq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} > (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} > (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} > (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} > a$. Then p_1, p_2, p_3, p_4 are in this order on Rectangle(a, b, c, d).

(49) Let A, B, C, D be real numbers and h, g be maps from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. Suppose $A > 0$ and $C > 0$ and $h = \mathrm{AffineMap}(A, B, C, D)$ and $g = \mathrm{AffineMap}(\frac{1}{A}, -\frac{B}{A}, \frac{1}{C}, -\frac{D}{C})$. Then $g = h^{-1}$ and $h = g^{-1}$.

(50) Let A, B, C, D be real numbers and h be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. Suppose $A > 0$ and $C > 0$ and $h = \mathrm{AffineMap}(A, B, C, D)$. Then h is a homeomorphism and for all points p_1, p_2 of $\mathcal{E}^2_{\rm T}$ such that $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ holds $h(p_1)_{\mathbf{1}} < h(p_2)_{\mathbf{1}}$.

(51) Let A, B, C, D be real numbers and h be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $A > 0$ and $C > 0$ and $h = \mathrm{AffineMap}(A, B, C, D)$. Then h is a homeomorphism and for all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ holds $h(p_1)_{\mathbf{2}} < h(p_2)_{\mathbf{2}}$.

(52) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $a < b$ and $c < d$ and $h = \mathrm{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}(h \cdot f) \subseteq \mathrm{ClosedInsideOfRectangle}(-1, 1, -1, 1)$.

(53) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose $a < b$ and $c < d$ and $h = \mathrm{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and f is continuous and one-to-one. Then $h \cdot f$ is continuous and one-to-one.

(54) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, and O be a point of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \mathrm{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $f(O)_{\mathbf{1}} = a$. Then $(h \cdot f)(O)_{\mathbf{1}} = -1$.

(55) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, and I be a point of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \mathrm{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $f(I)_{\mathbf{2}} = d$. Then $(h \cdot f)(I)_{\mathbf{2}} = 1$.

(56) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, and I be a point of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \mathrm{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $f(I)_{\mathbf{1}} = b$. Then $(h \cdot f)(I)_{\mathbf{1}} = 1$.

(57) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, and I be a point of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \mathrm{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $f(I)_{\mathbf{2}} = c$. Then $(h \cdot f)(I)_{\mathbf{2}} = -1$.

(58) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \mathrm{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $c \leq f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} < f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \leq d$. Then $-1 \leq (h \cdot f)(O)_{\mathbf{2}}$ and $(h \cdot f)(O)_{\mathbf{2}} < (h \cdot f)(I)_{\mathbf{2}}$ and $(h \cdot f)(I)_{\mathbf{2}} \leq 1$.

(59) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \mathrm{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $c \leq f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leq d$ and $a \leq f(I)_{\mathbf{1}}$ and $f(I)_{\mathbf{1}} \leq b$. Then $-1 \leq (h \cdot f)(O)_{\mathbf{2}}$ and $(h \cdot f)(O)_{\mathbf{2}} \leq 1$ and $-1 \leq (h \cdot f)(I)_{\mathbf{1}}$ and $(h \cdot f)(I)_{\mathbf{1}} \leq 1$.

(60) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\mathrm{T}}$ into $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$, and O, I be points of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \mathrm{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $c \leq f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leq d$ and $c \leq f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \leq d$. Then $-1 \leq (h \cdot f)(O)_{\mathbf{2}}$ and $(h \cdot f)(O)_{\mathbf{2}} \leq 1$ and $-1 \leq (h \cdot f)(I)_{\mathbf{2}}$ and $(h \cdot f)(I)_{\mathbf{2}} \leq 1$.

(61) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \text{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $c \leq f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leq d$ and $a < f(I)_{\mathbf{1}}$ and $f(I)_{\mathbf{1}} \leq b$. Then $-1 \leq (h \cdot f)(O)_{\mathbf{2}}$ and $(h \cdot f)(O)_{\mathbf{2}} \leq 1$ and $-1 < (h \cdot f)(I)_{\mathbf{1}}$ and $(h \cdot f)(I)_{\mathbf{1}} \leq 1$.

(62) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \text{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $a \leq f(O)_{\mathbf{1}}$ and $f(O)_{\mathbf{1}} < f(I)_{\mathbf{1}}$ and $f(I)_{\mathbf{1}} \leq b$. Then $-1 \leq (h \cdot f)(O)_{\mathbf{1}}$ and $(h \cdot f)(O)_{\mathbf{1}} < (h \cdot f)(I)_{\mathbf{1}}$ and $(h \cdot f)(I)_{\mathbf{1}} \leq 1$.

(63) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \text{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $a \leq f(O)_{\mathbf{1}}$ and $f(O)_{\mathbf{1}} \leq b$ and $c \leq f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \leq d$. Then $-1 \leq (h \cdot f)(O)_{\mathbf{1}}$ and $(h \cdot f)(O)_{\mathbf{1}} \leq 1$ and $-1 \leq (h \cdot f)(I)_{\mathbf{2}}$ and $(h \cdot f)(I)_{\mathbf{2}} \leq 1$.

(64) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \text{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $a \leq f(O)_{\mathbf{1}}$ and $f(O)_{\mathbf{1}} \leq b$ and $a < f(I)_{\mathbf{1}}$ and $f(I)_{\mathbf{1}} \leq b$. Then $-1 \leq (h \cdot f)(O)_{\mathbf{1}}$ and $(h \cdot f)(O)_{\mathbf{1}} \leq 1$ and $-1 < (h \cdot f)(I)_{\mathbf{1}}$ and $(h \cdot f)(I)_{\mathbf{1}} \leq 1$.

(65) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \text{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $d \geq f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} > f(I)_{\mathbf{2}}$ and $f(I)_{\mathbf{2}} \geq c$. Then $1 \geq (h \cdot f)(O)_{\mathbf{2}}$ and $(h \cdot f)(O)_{\mathbf{2}} > (h \cdot f)(I)_{\mathbf{2}}$ and $(h \cdot f)(I)_{\mathbf{2}} \geq -1$.

(66) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \text{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $c \leq f(O)_{\mathbf{2}}$ and $f(O)_{\mathbf{2}} \leq d$ and $a < f(I)_{\mathbf{1}}$ and $f(I)_{\mathbf{1}} \leq b$. Then $-1 \leq (h \cdot f)(O)_{\mathbf{2}}$ and $(h \cdot f)(O)_{\mathbf{2}} \leq 1$ and $-1 < (h \cdot f)(I)_{\mathbf{1}}$ and $(h \cdot f)(I)_{\mathbf{1}} \leq 1$.

(67) Let a, b, c, d be real numbers, h be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$, f be a map from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, and O, I be points of $\mathbb{I}$. Suppose $a < b$ and $c < d$ and $h = \text{AffineMap}(\frac{2}{b-a}, -\frac{b+a}{b-a}, \frac{2}{d-c}, -\frac{d+c}{d-c})$ and $a < f(I)_{\mathbf{1}}$ and $f(I)_{\mathbf{1}} < f(O)_{\mathbf{1}}$ and $f(O)_{\mathbf{1}} \leq b$. Then $-1 < (h \cdot f)(I)_{\mathbf{1}}$ and $(h \cdot f)(I)_{\mathbf{1}} < (h \cdot f)(O)_{\mathbf{1}}$ and $(h \cdot f)(O)_{\mathbf{1}} \leq 1$.

One can prove the following propositions:

(68) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = a$ and $(p_4)_{\mathbf{1}} = a$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} < (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is

continuous and one-to-one and $\operatorname{rng} f \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(69) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = a$ and $(p_4)_{\mathbf{1}} = a$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} < (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(70) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = a$ and $(p_4)_{\mathbf{2}} = d$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $a \leq (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(71) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = a$ and $(p_4)_{\mathbf{2}} = d$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $a \leq (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(72) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = a$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(73) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = a$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(74) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = a$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$

and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(75) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = a$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(76) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = d$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a \leq (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(77) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = d$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a \leq (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(78) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a \leq (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(79) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a \leq (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(80) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a \leq (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \text{ClosedInsideOfRectangle}(a,b,c,d)$ and $\operatorname{rng} g \subseteq \text{ClosedInsideOfRectangle}(a,b,c,d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(81) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a \leq (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \text{ClosedInsideOfRectangle}(a,b,c,d)$ and $Q \subseteq \text{ClosedInsideOfRectangle}(a,b,c,d)$. Then P meets Q.

(82) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \text{ClosedInsideOfRectangle}(a,b,c,d)$ and $\operatorname{rng} g \subseteq \text{ClosedInsideOfRectangle}(a,b,c,d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(83) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \text{ClosedInsideOfRectangle}(a,b,c,d)$ and $Q \subseteq \text{ClosedInsideOfRectangle}(a,b,c,d)$. Then P meets Q.

(84) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $c \leq (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \text{ClosedInsideOfRectangle}(a,b,c,d)$ and $\operatorname{rng} g \subseteq \text{ClosedInsideOfRectangle}(a,b,c,d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(85) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $c \leq (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and

$a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(86) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(87) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = a$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(88) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = d$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(89) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = d$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(90) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(91) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P,

Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(92) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(93) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(94) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(95) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(96) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $c \leq (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and

$g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(97) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $c \leq (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(98) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(99) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a \leq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(100) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(101) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(102) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and

$(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $c \leq (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(103) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $c \leq (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} < (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(104) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $c \leq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(105) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $c \leq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(106) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = c$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(107) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = a$ and $(p_2)_{\mathbf{2}} = c$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$

and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(108) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = d$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(109) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = d$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(110) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{1}} = b$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(111) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{1}} = b$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \leq d$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(112) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(113) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = d$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and

$(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$. Then P meets Q.

(114) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(115) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $c \leq (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} < (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$. Then P meets Q.

(116) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $c \leq (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(117) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $c \leq (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$. Then P meets Q.

(118) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a,b,c,d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(119) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = d$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(120) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leq b$ and $d \geq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} > (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \geq c$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(121) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leq b$ and $d \geq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} > (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \geq c$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(122) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leq b$ and $d \geq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \geq c$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $\mathrm{rng}\, g \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\mathrm{rng}\, f$ meets $\mathrm{rng}\, g$.

(123) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leq b$ and $d \geq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \geq c$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \mathrm{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(124) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leq b$ and $c \leq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continu-

ous and one-to-one and $\operatorname{rng} f \subseteq \operatorname{ClosedInsideOfRectangle}(a,b,c,d)$ and $\operatorname{rng} g \subseteq \operatorname{ClosedInsideOfRectangle}(a,b,c,d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(125) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leq b$ and $c \leq (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \leq d$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \operatorname{ClosedInsideOfRectangle}(a,b,c,d)$ and $Q \subseteq \operatorname{ClosedInsideOfRectangle}(a,b,c,d)$. Then P meets Q.

(126) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = c$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \operatorname{ClosedInsideOfRectangle}(a,b,c,d)$ and $\operatorname{rng} g \subseteq \operatorname{ClosedInsideOfRectangle}(a,b,c,d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(127) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = d$ and $(p_2)_{\mathbf{2}} = c$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $a \leq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} \leq b$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} < (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} < (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \operatorname{ClosedInsideOfRectangle}(a,b,c,d)$ and $Q \subseteq \operatorname{ClosedInsideOfRectangle}(a,b,c,d)$. Then P meets Q.

(128) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = b$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $d \geq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} > (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} > (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \geq c$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \operatorname{ClosedInsideOfRectangle}(a,b,c,d)$ and $\operatorname{rng} g \subseteq \operatorname{ClosedInsideOfRectangle}(a,b,c,d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(129) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = b$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{1}} = b$ and $d \geq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} > (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} > (p_4)_{\mathbf{2}}$ and $(p_4)_{\mathbf{2}} \geq c$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \operatorname{ClosedInsideOfRectangle}(a,b,c,d)$ and $Q \subseteq \operatorname{ClosedInsideOfRectangle}(a,b,c,d)$. Then P meets Q.

(130) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = b$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $d \geq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} > (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \geq c$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and

$g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(131) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = b$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{1}} = b$ and $(p_4)_{\mathbf{2}} = c$ and $d \geq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} > (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} > (p_3)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} \geq c$ and $a < (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} \leq b$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(132) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = b$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $d \geq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} > (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \geq c$ and $b \geq (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} > (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} > a$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(133) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = b$ and $(p_2)_{\mathbf{1}} = b$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $d \geq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} > (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} \geq c$ and $b \geq (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} > (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} > a$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(134) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = b$ and $(p_2)_{\mathbf{2}} = c$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $b \geq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} > (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} > (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} > a$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(135) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{1}} = b$ and $(p_2)_{\mathbf{2}} = c$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $c \leq (p_1)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} \leq d$ and $b \geq (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} > (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} > (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} > a$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \operatorname{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

(136) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = c$ and $(p_2)_{\mathbf{2}} = c$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $b \geq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} > (p_2)_{\mathbf{1}}$ and

$(p_2)_{\mathbf{1}} > (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} > (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} > a$ and $f(0) = p_1$ and $f(1) = p_3$ and $g(0) = p_2$ and $g(1) = p_4$ and f is continuous and one-to-one and g is continuous and one-to-one and $\operatorname{rng} f \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $\operatorname{rng} g \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

(137) Let p_1, p_2, p_3, p_4 be points of $\mathcal{E}^2_{\mathrm{T}}$, a, b, c, d be real numbers, and P, Q be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that $a < b$ and $c < d$ and $(p_1)_{\mathbf{2}} = c$ and $(p_2)_{\mathbf{2}} = c$ and $(p_3)_{\mathbf{2}} = c$ and $(p_4)_{\mathbf{2}} = c$ and $b \geq (p_1)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} > (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} > (p_3)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} > (p_4)_{\mathbf{1}}$ and $(p_4)_{\mathbf{1}} > a$ and P is an arc from p_1 to p_3 and Q is an arc from p_2 to p_4 and $P \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$ and $Q \subseteq \text{ClosedInsideOfRectangle}(a, b, c, d)$. Then P meets Q.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.
[5] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[7] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[8] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.
[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[10] Artur Korniłowicz. The ordering of points on a curve. Part III. *Formalized Mathematics*, 10(**3**):169–171, 2002.
[11] Yatsuka Nakamura. On Outside Fashoda Meet Theorem. *Formalized Mathematics*, 9(**4**):697–704, 2001.
[12] Yatsuka Nakamura. General Fashoda meet theorem for unit circle and square. *Formalized Mathematics*, 11(**3**):213–224, 2003.
[13] Yatsuka Nakamura and Andrzej Trybulec. A decomposition of a simple closed curves and the order of their points. *Formalized Mathematics*, 6(**4**):563–572, 1997.
[14] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[17] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received January 3, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 221–225
University of Białystok, 2005

Preliminaries to Mathematical Morphology and Its Properties

Yuzhong Ding
QingDao Science and Technology University
China

Xiquan Liang
QingDao Science and Technology University
China

Summary. The article is a translation of chapter 2 of the book *Mathematical Morphological Method and Application* by Changqing Tang, Hongbo Lu, Zheng Huang, Fang Zhang, Science Press, China, 1990. In this article, the basic mathematical morphological operators such as Erosion, Dilation, Adjunction Opening, Adjunction Closing and their properties are given. And these operators are usually used in processing and analysing the images.

MML identifier: `MATHMORP`, version: `7.5.01 4.39.921`

The terminology and notation used here are introduced in the following articles: [5], [1], [2], [6], [4], and [3].

1. The Definition of Erosion and Dilation and Their Algebraic Properties

In this paper n denotes a natural number and q, y, b denote points of $\mathcal{E}^n_{\rm T}$.

Let us consider n, let p be a point of $\mathcal{E}^n_{\rm T}$, and let X be a subset of $\mathcal{E}^n_{\rm T}$. The functor $X + p$ yielding a subset of $\mathcal{E}^n_{\rm T}$ is defined by:

(Def. 1) $X + p = \{q + p : q \in X\}$.

Let us consider n and let X be a subset of $\mathcal{E}^n_{\rm T}$. The functor $X!$ yielding a subset of $\mathcal{E}^n_{\rm T}$ is defined as follows:

(Def. 2) $X! = \{-q : q \in X\}$.

Let us consider n and let X, B be subsets of $\mathcal{E}^n_{\rm T}$. The functor $X \ominus B$ yields a subset of $\mathcal{E}^n_{\rm T}$ and is defined as follows:

ISSN 1426–2630

(Def. 3) $X \ominus B = \{y : B + y \subseteq X\}$.

Let us consider n and let X, B be subsets of $\mathcal{E}^n_{\rm T}$. The functor $X \oplus B$ yields a subset of $\mathcal{E}^n_{\rm T}$ and is defined as follows:

(Def. 4) $X \oplus B = \{y + b : y \in X \ \wedge \ b \in B\}$.

We follow the rules: n is a natural number, X, Y, Z, B, C, B_1, B_2 are subsets of $\mathcal{E}^n_{\rm T}$, and x, y, p are points of $\mathcal{E}^n_{\rm T}$.

One can prove the following propositions:

(1) $B!! = B$.

(2) $\{0_{\mathcal{E}^n_{\rm T}}\} + x = \{x\}$.

(3) If $B_1 \subseteq B_2$, then $B_1 + p \subseteq B_2 + p$.

(4) For every X such that $X = \emptyset$ holds $X + x = \emptyset$.

(5) $X \ominus \{0_{\mathcal{E}^n_{\rm T}}\} = X$.

(6) $X \oplus \{0_{\mathcal{E}^n_{\rm T}}\} = X$.

(7) $X \oplus \{x\} = X + x$.

(8) For all X, Y such that $Y = \emptyset$ holds $X \ominus Y = \mathcal{R}^n$.

(9) If $X \subseteq Y$, then $X \ominus B \subseteq Y \ominus B$ and $X \oplus B \subseteq Y \oplus B$.

(10) If $B_1 \subseteq B_2$, then $X \ominus B_2 \subseteq X \ominus B_1$ and $X \oplus B_1 \subseteq X \oplus B_2$.

(11) If $0_{\mathcal{E}^n_{\rm T}} \in B$, then $X \ominus B \subseteq X$ and $X \subseteq X \oplus B$.

(12) $X \oplus Y = Y \oplus X$.

(13) $Y + y \subseteq X + x$ iff $Y + (y - x) \subseteq X$.

(14) $(X + p) \ominus Y = X \ominus Y + p$.

(15) $(X + p) \oplus Y = X \oplus Y + p$.

(16) $(X + x) + y = X + (x + y)$.

(17) $X \ominus (Y + p) = X \ominus Y + -p$.

(18) $X \oplus (Y + p) = X \oplus Y + p$.

(19) If $x \in X$, then $B + x \subseteq B \oplus X$.

(20) $X \subseteq (X \oplus B) \ominus B$.

(21) $X + 0_{\mathcal{E}^n_{\rm T}} = X$.

(22) $X \ominus \{x\} = X + -x$.

(23) $X \ominus (Y \oplus Z) = X \ominus Y \ominus Z$.

(24) $X \ominus (Y \oplus Z) = X \ominus Z \ominus Y$.

(25) $X \oplus (Y \ominus Z) \subseteq (X \oplus Y) \ominus Z$.

(26) $X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z$.

(27) $(B \cup C) + y = (B + y) \cup (C + y)$.

(28) $B \cap C + y = (B + y) \cap (C + y)$.

(29) $X \ominus (B \cup C) = (X \ominus B) \cap (X \ominus C)$.

(30) $X \oplus (B \cup C) = X \oplus B \cup X \oplus C$.

(31) $X \ominus B \cup Y \ominus B \subseteq (X \cup Y) \ominus B$.

(32) $(X \cup Y) \oplus B = X \oplus B \cup Y \oplus B$.

(33) $X \cap Y \ominus B = (X \ominus B) \cap (Y \ominus B)$.

(34) $X \cap Y \oplus B \subseteq (X \oplus B) \cap (Y \oplus B)$.

(35) $B \oplus X \cap Y \subseteq (B \oplus X) \cap (B \oplus Y)$.

(36) $B \ominus X \cup B \ominus Y \subseteq B \ominus X \cap Y$.

(37) $(X^{\mathrm{c}} \ominus B)^{\mathrm{c}} = X \oplus B!$.

(38) $(X \ominus B)^{\mathrm{c}} = X^{\mathrm{c}} \oplus B!$.

2. The Definition of Adjunction Opening and Closing and Their Algebraic Properties

Let n be a natural number and let X, B be subsets of $\mathcal{E}^n_{\mathrm{T}}$. The functor $X \bigcirc B$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 5) $X \bigcirc B = (X \ominus B) \oplus B$.

Let n be a natural number and let X, B be subsets of $\mathcal{E}^n_{\mathrm{T}}$. The functor $X \circledcirc B$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 6) $X \circledcirc B = (X \oplus B) \ominus B$.

We now state a number of propositions:

(39) $(X^{\mathrm{c}} \bigcirc B!)^{\mathrm{c}} = X \circledcirc B$.

(40) $(X^{\mathrm{c}} \circledcirc B!)^{\mathrm{c}} = X \bigcirc B$.

(41) $X \bigcirc B \subseteq X$ and $X \subseteq X \circledcirc B$.

(42) $X \bigcirc X = X$.

(43) $X \bigcirc B \ominus B \subseteq X \ominus B$ and $X \bigcirc B \oplus B \subseteq X \oplus B$.

(44) $X \ominus B \subseteq X \circledcirc B \ominus B$ and $X \oplus B \subseteq X \circledcirc B \oplus B$.

(45) If $X \subseteq Y$, then $X \bigcirc B \subseteq Y \bigcirc B$ and $X \circledcirc B \subseteq Y \circledcirc B$.

(46) $(X + p) \bigcirc Y = X \bigcirc Y + p$.

(47) $(X + p) \circledcirc Y = X \circledcirc Y + p$.

(48) If $C \subseteq B$, then $X \bigcirc B \subseteq (X \ominus C) \oplus B$.

(49) If $B \subseteq C$, then $X \circledcirc B \subseteq (X \oplus C) \ominus B$.

(50) $X \oplus Y = X \circledcirc Y \oplus Y$ and $X \ominus Y = X \bigcirc Y \ominus Y$.

(51) $X \oplus Y = (X \oplus Y) \bigcirc Y$ and $X \ominus Y = (X \ominus Y) \circledcirc Y$.

(52) $X \bigcirc B \bigcirc B = X \bigcirc B$.

(53) $X \circledcirc B \circledcirc B = X \circledcirc B$.

(54) $X \bigcirc B \subseteq (X \cup Y) \bigcirc B$.

(55) If $B = B \bigcirc B_1$, then $X \bigcirc B \subseteq X \bigcirc B_1$.

3. The Definition of Scaling Transformation and Its Algebraic Properties

In the sequel a is a point of $\mathcal{E}^n_{\mathrm{T}}$.

Let t be a real number, let us consider n, and let A be a subset of $\mathcal{E}^n_{\mathrm{T}}$. The functor $t \odot A$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 7) $t \odot A = \{t \cdot a : a \in A\}$.

In the sequel t, s denote real numbers.

One can prove the following propositions:

(56) For every subset X of $\mathcal{E}^n_{\mathrm{T}}$ such that $X = \emptyset$ holds $0 \odot X = \emptyset$.

(57) For every non empty subset X of $\mathcal{E}^n_{\mathrm{T}}$ holds $0 \odot X = \{0_{\mathcal{E}^n_{\mathrm{T}}}\}$.

(58) $1 \odot X = X$.

(59) $2 \odot X \subseteq X \oplus X$.

(60) $(t \cdot s) \odot X = t \odot (s \odot X)$.

(61) If $X \subseteq Y$, then $t \odot X \subseteq t \odot Y$.

(62) $t \odot (X + x) = t \odot X + t \cdot x$.

(63) $t \odot (X \oplus Y) = t \odot X \oplus t \odot Y$.

(64) If $t \neq 0$, then $t \odot (X \ominus Y) = t \odot X \ominus t \odot Y$.

(65) If $t \neq 0$, then $t \odot (X \bigcirc Y) = (t \odot X) \bigcirc (t \odot Y)$.

(66) If $t \neq 0$, then $t \odot (X \circledcirc Y) = (t \odot X) \circledcirc (t \odot Y)$.

4. The Definition of Thinning and Thickening and Their Algebraic Properties

Let n be a natural number and let X, B_1, B_2 be subsets of $\mathcal{E}^n_{\mathrm{T}}$. The functor $X \circledast (B_1, B_2)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 8) $X \circledast (B_1, B_2) = (X \ominus B_1) \cap (X^{\mathrm{c}} \ominus B_2)$.

Let n be a natural number and let X, B_1, B_2 be subsets of $\mathcal{E}^n_{\mathrm{T}}$. The functor $X \otimes (B_1, B_2)$ yields a subset of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 9) $X \otimes (B_1, B_2) = X \cup (X \circledast (B_1, B_2))$.

Let n be a natural number and let X, B_1, B_2 be subsets of $\mathcal{E}^n_{\mathrm{T}}$. The functor $X \circledast (B_1, B_2)$ yielding a subset of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def. 10) $X \circledast (B_1, B_2) = X \setminus (X \circledast (B_1, B_2))$.

The following propositions are true:

(67) If $B_1 = \emptyset$, then $X \circledast (B_1, B_2) = X^{\mathrm{c}} \ominus B_2$.

(68) If $B_2 = \emptyset$, then $X \circledast (B_1, B_2) = X \ominus B_1$.

(69) If $0_{\mathcal{E}^n_{\mathrm{T}}} \in B_1$, then $X \circledast (B_1, B_2) \subseteq X$.

(70) If $0_{\mathcal{E}^n_{\mathrm{T}}} \in B_2$, then $(X \circledast (B_1, B_2)) \cap X = \emptyset$.

(71) If $0_{\mathcal{E}^n_{\mathrm{T}}} \in B_1$, then $X \otimes (B_1, B_2) = X$.

(72) If $0_{\mathcal{E}^n_{\mathrm{T}}} \in B_2$, then $X \circledast (B_1, B_2) = X$.

(73) $X \otimes (B_2, B_1) = (X^{\mathrm{c}} \circledast (B_1, B_2))^{\mathrm{c}}$.

(74) $X \circledast (B_2, B_1) = (X^{\mathrm{c}} \otimes (B_1, B_2))^{\mathrm{c}}$.

5. Properties of Erosion, Dilation, Adjunction Opening, Adjunction Closing on Convex Sets

One can prove the following proposition

(75) Let n be a natural number and B be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Then B is convex if and only if for all points x, y of $\mathcal{E}^n_{\mathrm{T}}$ and for every real number r such that $0 \leq r$ and $r \leq 1$ and $x \in B$ and $y \in B$ holds $r \cdot x + (1 - r) \cdot y \in B$.

Let n be a natural number and let B be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Let us observe that B is convex if and only if:

(Def. 11) For all points x, y of $\mathcal{E}^n_{\mathrm{T}}$ and for every real number r such that $0 \leq r$ and $r \leq 1$ and $x \in B$ and $y \in B$ holds $r \cdot x + (1 - r) \cdot y \in B$.

One can prove the following propositions:

(76) If X is convex, then $X!$ is convex.

(77) If X is convex and B is convex, then $X \oplus B$ is convex and $X \ominus B$ is convex.

(78) If X is convex and B is convex, then $X \bigcirc B$ is convex and $X \circledcirc B$ is convex.

(79) If B is convex and $0 < t$ and $0 < s$, then $(s + t) \odot B = s \odot B \oplus t \odot B$.

Acknowledgments

The authors would like to acknowledge Prof. Andrzej Trybulec and Prof. Yatsuka Nakamura for their help.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[3] Yatsuka Nakamura and Jarosław Kotowicz. The Jordan's property for certain subsets of the plane. *Formalized Mathematics*, 3(**2**):137–142, 1992.
[4] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[6] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received January 7, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 227–233
University of Białystok, 2005

Subsequences of Almost, Weakly and Poorly One-to-one Finite Sequences[1]

Robert Milewski
Institute of Computer Science
University of Białystok
Akademicka 2, 15-267 Białystok, Poland

MML identifier: JORDAN23, version: 7.5.01 4.39.921

The articles [21], [24], [1], [3], [2], [23], [4], [11], [9], [22], [16], [20], [19], [6], [7], [12], [8], [13], [17], [14], [15], [5], [18], and [10] provide the terminology and notation for this paper.

In this paper n is a natural number.

The following three propositions are true:

(1) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $\mathrm{len} \downharpoonright p, f \geq 1$.

(2) For every non empty finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for every point p of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{len} \downharpoonleft f, p \geq 1$.

(3) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for all points p, q of $\mathcal{E}^2_{\rm T}$ holds $\downharpoonright \downharpoonleft p, f, q \neq \emptyset$.

Let x be a set. One can check that $\langle x \rangle$ is one-to-one.

Let f be a finite sequence. We say that f is almost one-to-one if and only if:

(Def. 1) For all natural numbers i, j such that $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $i \neq 1$ or $j \neq \mathrm{len}\, f$ and $i \neq \mathrm{len}\, f$ or $j \neq 1$ and $f(i) = f(j)$ holds $i = j$.

Let f be a finite sequence. We say that f is weakly one-to-one if and only if:

(Def. 2) For every natural number i such that $1 \leq i$ and $i < \mathrm{len}\, f$ holds $f(i) \neq f(i+1)$.

[1]This work has been partially supported by the KBN grant 4 T11C 039 24.

ISSN 1426–2630

Let f be a finite sequence. We say that f is poorly one-to-one if and only if:

(Def. 3)(i) For every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $f(i) \neq f(i+1)$ if $\operatorname{len} f \neq 2$,

(ii) TRUE, otherwise.

The following three propositions are true:

(4) Let D be a set and f be a finite sequence of elements of D. Then f is almost one-to-one if and only if for all natural numbers i, j such that $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $i \neq 1$ or $j \neq \operatorname{len} f$ and $i \neq \operatorname{len} f$ or $j \neq 1$ and $f_i = f_j$ holds $i = j$.

(5) Let D be a set and f be a finite sequence of elements of D. Then f is weakly one-to-one if and only if for every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $f_i \neq f_{i+1}$.

(6) Let D be a set and f be a finite sequence of elements of D. Then f is poorly one-to-one if and only if if $\operatorname{len} f \neq 2$, then for every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $f_i \neq f_{i+1}$.

Let us note that every finite sequence which is one-to-one is also almost one-to-one.

One can check that every finite sequence which is almost one-to-one is also poorly one-to-one.

The following proposition is true

(7) For every finite sequence f such that $\operatorname{len} f \neq 2$ holds f is weakly one-to-one iff f is poorly one-to-one.

Let us note that $\emptyset$ is weakly one-to-one.

Let x be a set. One can verify that $\langle x \rangle$ is weakly one-to-one.

Let x, y be sets. Observe that $\langle x, y \rangle$ is poorly one-to-one.

Let us mention that there exists a finite sequence which is weakly one-to-one and non empty.

Let D be a non empty set. Observe that there exists a finite sequence of elements of D which is weakly one-to-one, circular, and non empty.

We now state three propositions:

(8) For every finite sequence f such that f is almost one-to-one holds $\operatorname{Rev}(f)$ is almost one-to-one.

(9) For every finite sequence f such that f is weakly one-to-one holds $\operatorname{Rev}(f)$ is weakly one-to-one.

(10) For every finite sequence f such that f is poorly one-to-one holds $\operatorname{Rev}(f)$ is poorly one-to-one.

Let us observe that there exists a finite sequence which is one-to-one and non empty.

Let f be an almost one-to-one finite sequence. Observe that $\operatorname{Rev}(f)$ is almost one-to-one.

Let f be a weakly one-to-one finite sequence. Observe that $\mathrm{Rev}(f)$ is weakly one-to-one.

Let f be a poorly one-to-one finite sequence. Observe that $\mathrm{Rev}(f)$ is poorly one-to-one.

One can prove the following three propositions:

(11) Let D be a non empty set and f be a finite sequence of elements of D. Suppose f is almost one-to-one. Let p be an element of D. Then $f \circlearrowleft p$ is almost one-to-one.

(12) Let D be a non empty set and f be a finite sequence of elements of D. Suppose f is weakly one-to-one and circular. Let p be an element of D. Then $f \circlearrowleft p$ is weakly one-to-one.

(13) Let D be a non empty set and f be a finite sequence of elements of D. Suppose f is poorly one-to-one and circular. Let p be an element of D. Then $f \circlearrowleft p$ is poorly one-to-one.

Let D be a non empty set. One can check that there exists a finite sequence of elements of D which is one-to-one, circular, and non empty.

Let D be a non empty set, let f be an almost one-to-one finite sequence of elements of D, and let p be an element of D. Note that $f \circlearrowleft p$ is almost one-to-one.

Let D be a non empty set, let f be a circular weakly one-to-one finite sequence of elements of D, and let p be an element of D. Note that $f \circlearrowleft p$ is weakly one-to-one.

Let D be a non empty set, let f be a circular poorly one-to-one finite sequence of elements of D, and let p be an element of D. One can verify that $f \circlearrowleft p$ is poorly one-to-one.

The following proposition is true

(14) Let D be a non empty set and f be a finite sequence of elements of D. Then f is almost one-to-one if and only if $f_{\restriction 1}$ is one-to-one and $f \restriction (\mathrm{len}\, f -' 1)$ is one-to-one.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Observe that $\mathrm{Cage}(C, n)$ is almost one-to-one.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. One can check that $\mathrm{Cage}(C, n)$ is weakly one-to-one.

The following propositions are true:

(15) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is weakly one-to-one, then $\downharpoonleft \downharpoonleft p, f, p = \langle p \rangle$.

(16) For every finite sequence f such that f is one-to-one holds f is weakly one-to-one.

One can check that every finite sequence which is one-to-one is also weakly one-to-one.

The following propositions are true:

(17) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is weakly one-to-one. Let p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonleft\downharpoonright p, f, q = \mathrm{Rev}(\downharpoonleft\downharpoonright q, f, p)$.

(18) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, p be a point of $\mathcal{E}^2_{\rm T}$, and i_1 be a natural number. Suppose f is poorly one-to-one, unfolded, and s.n.c. and $1 < i_1$ and $i_1 \leq \mathrm{len}\, f$ and $p = f(i_1)$. Then $\mathrm{Index}(p, f) + 1 = i_1$.

(19) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is weakly one-to-one. Let p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$, then $(\downharpoonleft\downharpoonright p, f, q)_1 = p$.

(20) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is weakly one-to-one. Let p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$, then $(\downharpoonleft\downharpoonright p, f, q)_{\mathrm{len}\downharpoonleft\downharpoonright p,f,q} = q$.

(21) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $\widetilde{\mathcal{L}}(\downharpoonleft p, f) \subseteq \widetilde{\mathcal{L}}(f)$.

(22) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and f is weakly one-to-one, then $\widetilde{\mathcal{L}}(\downharpoonleft\downharpoonright p, f, q) \subseteq \widetilde{\mathcal{L}}(f)$.

(23) For all finite sequences f, g holds $\mathrm{dom}\, f \subseteq \mathrm{dom}(f \frown g)$.

(24) For every non empty finite sequence f and for every finite sequence g holds $\mathrm{dom}\, g \subseteq \mathrm{dom}(f \frown g)$.

(25) For all finite sequences f, g such that $f \frown g$ is constant holds f is constant.

(26) For all finite sequences f, g such that $f \frown g$ is constant and $f(\mathrm{len}\, f) = g(1)$ and $f \neq \emptyset$ holds g is constant.

(27) For every special finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for all natural numbers i, j holds $\mathrm{mid}(f, i, j)$ is special.

(28) For every unfolded finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for all natural numbers i, j holds $\mathrm{mid}(f, i, j)$ is unfolded.

(29) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is special. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonleft p, f$ is special.

(30) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is special. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonright f, p$ is special.

(31) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is special and weakly one-to-one. Let p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonleft\downharpoonright p, f, q$ is special.

(32) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is unfolded. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonleft p, f$ is unfolded.

(33) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is unfolded. Let

p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonleft f, p$ is unfolded.

(34) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is unfolded and weakly one-to-one. Let p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonright\downharpoonleft p, f, q$ is unfolded.

(35) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is almost one-to-one, special, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$ and $g = (\mathrm{mid}(f, 1, \mathrm{Index}(p, f))) \frown \langle p \rangle$. Then g is a special sequence joining f_1, p.

(36) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is poorly one-to-one, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p = f(\mathrm{Index}(p, f)+1)$ and $p \neq f(\mathrm{len}\, f)$. Then $\mathrm{Index}(p, \mathrm{Rev}(f)) + \mathrm{Index}(p, f) + 1 = \mathrm{len}\, f$.

(37) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If f is weakly one-to-one and $\mathrm{len}\, f \geq 2$, then $\downharpoonright f_1, f = f$.

(38) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is poorly one-to-one, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$. Then $\downharpoonright p, \mathrm{Rev}(f) = \mathrm{Rev}(\downharpoonleft f, p)$.

(39) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is almost one-to-one, special, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $\downharpoonleft f, p$ is a special sequence joining f_1, p.

(40) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is almost one-to-one, special, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$ and $p \neq f(1)$. Then $\downharpoonright p, f$ is a special sequence joining p, $f_{\mathrm{len}\, f}$.

(41) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is almost one-to-one, special, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $\downharpoonleft f, p$ is a special sequence.

(42) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is almost one-to-one, special, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$ and $p \neq f(1)$. Then $\downharpoonright p, f$ is a special sequence.

(43) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. Suppose that f is almost one-to-one, special, unfolded, and s.n.c. and $\mathrm{len}\, f \neq 2$ and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$ and $p \neq f(1)$ and $q \neq f(1)$. Then $\downharpoonright\downharpoonleft p, f, q$ is a special sequence joining p, q.

(44) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. Suppose that f is almost one-to-one, special, unfolded, and s.n.c. and $\mathrm{len}\, f \neq 2$ and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$ and $p \neq f(1)$ and $q \neq f(1)$. Then $\downharpoonright\downharpoonleft p, f, q$ is a special sequence.

(45) Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. Suppose $p \in \mathrm{BDD}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$. Then there exists a

S-sequence B in $\mathbb{R}^2$ such that

(i) $B = \downharpoonleft\downharpoonright$ South-Bound$(p, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$, $(\mathrm{Cage}(C,n) \circlearrowleft (\mathrm{Cage}(C,n))_{\mathrm{Index}(\text{South-Bound}(p,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))),\mathrm{Cage}(C,n))})\upharpoonright(\mathrm{len}\,(\mathrm{Cage}(C,n) \circlearrowleft (\mathrm{Cage}(C,n))_{\mathrm{Index}(\text{South-Bound}(p,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))),\mathrm{Cage}(C,n))}) -' 1)$, North-Bound$(p, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$, and

(ii) there exists a S-sequence P in $\mathbb{R}^2$ such that P is a sequence which elements belong to the Go-board of $B \frown \langle$North-Bound$(p, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$, South-Bound$(p, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))\rangle$ and $\widetilde{\mathcal{L}}(\langle$North-Bound$(p, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$, South-Bound$(p, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))\rangle) = \widetilde{\mathcal{L}}(P)$ and $P_1 =$ North-Bound$(p, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$ and $P_{\mathrm{len}\,P} =$ South-Bound$(p, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$ and $\mathrm{len}\,P \geq 2$ and there exists a S-sequence B_1 in $\mathbb{R}^2$ such that B_1 is a sequence which elements belong to the Go-board of $B \frown \langle$North-Bound$(p, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$, South-Bound$(p, \widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))\rangle$ and $\widetilde{\mathcal{L}}(B) = \widetilde{\mathcal{L}}(B_1)$ and $B_1 = (B_1)_1$ and $B_{\mathrm{len}\,B} = (B_1)_{\mathrm{len}\,B_1}$ and $\mathrm{len}\,B \leq \mathrm{len}\,B_1$ and there exists a non constant standard special circular sequence g such that $g = B_1 \frown P$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Some properties of restrictions of finite sequences. *Formalized Mathematics*, 5(**2**):241–245, 1996.

[5] Czesław Byliński and Mariusz Żynel. Cages – the external approximation of Jordan's curve. *Formalized Mathematics*, 9(**1**):19–24, 2001.

[6] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[7] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[8] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[9] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.

[10] Artur Korniłowicz. The ordering of points on a curve. Part IV. *Formalized Mathematics*, 10(**3**):173–177, 2002.

[11] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Formalized Mathematics*, 3(**2**):275–278, 1992.

[12] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board – part I. *Formalized Mathematics*, 3(**1**):107–115, 1992.

[13] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board – part II. *Formalized Mathematics*, 3(**1**):117–121, 1992.

[14] Yatsuka Nakamura and Czesław Byliński. Extremal properties of vertices on special polygons. Part I. *Formalized Mathematics*, 5(**1**):97–102, 1996.

[15] Yatsuka Nakamura and Roman Matuszewski. Reconstructions of special sequences. *Formalized Mathematics*, 6(**2**):255–263, 1997.

[16] Yatsuka Nakamura and Piotr Rudnicki. Vertex sequences induced by chains. *Formalized Mathematics*, 5(**3**):297–304, 1996.

[17] Yatsuka Nakamura and Andrzej Trybulec. Decomposing a Go-board into cells. *Formalized Mathematics*, 5(**3**):323–328, 1996.

[18] Yatsuka Nakamura, Andrzej Trybulec, and Czesław Byliński. Bounded domains and unbounded domains. *Formalized Mathematics*, 8(**1**):1–13, 1999.
[19] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.
[20] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[22] Andrzej Trybulec. On the decomposition of finite sequences. *Formalized Mathematics*, 5(**3**):317–322, 1996.
[23] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[24] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received February 1, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 235–252
University of Białystok, 2005

Alternative Graph Structures[1]

Gilbert Lee[2]
University of Victoria
Victoria, Canada

Piotr Rudnicki
University of Alberta
Edmonton, Canada

Summary. We define the notion of a graph anew without using the available Mizar structures. In our approach, we model graph structure as a finite function whose domain is a subset of natural numbers. The elements of the domain of the function play the role of selectors for accessing the components of the structure. As these selectors are first class objects, many future extensions of the new graph structure turned out to be easier to formalize in Mizar than with the traditional Mizar structures.

After introducing graph structure, we define its selectors and then conditions that the structure needs to satisfy to form a directed graph (in the spirit of [13]). For these graphs we define a collection of basic graph notions; the presentation of these notions is continued in articles [16, 15, 17].

We have tried to follow a number of graph theory books in choosing graph terminology but since the terminology is not commonly agreed upon, we had to make a number of compromises, see [14].

MML identifier: `GLIB_000`, version: `7.5.01 4.39.921`

The papers [20], [19], [22], [21], [24], [2], [1], [25], [7], [5], [12], [3], [8], [6], [23], [9], [4], [10], [11], and [18] provide the terminology and notation for this paper.

1. Definitions

A finite function is called a graph structure if:

(Def. 1) $\operatorname{dom} \mathrm{it} \subseteq \mathbb{N}$.

The natural number VertexSelector is defined as follows:

(Def. 2) VertexSelector = 1.

[1]This work has been partially supported by NSERC, Alberta Ingenuity Fund and iCORE.
[2]Part of author's MSc work.

ISSN 1426–2630

The natural number EdgeSelector is defined as follows:

(Def. 3) EdgeSelector = 2.

The natural number SourceSelector is defined by:

(Def. 4) SourceSelector = 3.

The natural number TargetSelector is defined by:

(Def. 5) TargetSelector = 4.

The non empty subset the graph selectors of $\mathbb{N}$ is defined by:

(Def. 6) The graph selectors = {VertexSelector, EdgeSelector, SourceSelector, TargetSelector}.

Let G be a graph structure. The vertices of G is defined by:

(Def. 7) The vertices of $G = G$(VertexSelector).

The edges of G is defined by:

(Def. 8) The edges of $G = G$(EdgeSelector).

The source of G is defined by:

(Def. 9) The source of $G = G$(SourceSelector).

The target of G is defined by:

(Def. 10) The target of $G = G$(TargetSelector).

Let G be a graph structure. We say that G is graph-like if and only if the conditions (Def. 11) are satisfied.

(Def. 11) VertexSelector $\in \mathrm{dom}\, G$ and EdgeSelector $\in \mathrm{dom}\, G$ and SourceSelector $\in \mathrm{dom}\, G$ and TargetSelector $\in \mathrm{dom}\, G$ and the vertices of G is a non empty set and the source of G is a function from the edges of G into the vertices of G and the target of G is a function from the edges of G into the vertices of G.

Let us note that there exists a graph structure which is graph-like.

A graph is a graph-like graph structure.

Let G be a graph. Observe that the vertices of G is non empty.

Let G be a graph. Then the source of G is a function from the edges of G into the vertices of G. Then the target of G is a function from the edges of G into the vertices of G.

Let V be a non empty set, let E be a set, and let S, T be functions from E into V. The functor createGraph(V, E, S, T) yielding a graph is defined by:

(Def. 12) createGraph$(V, E, S, T) = \langle V, E, S, T \rangle$.

Let x, y be sets. One can verify that $x \mapsto y$ is finite.

Let G be a graph structure, let n be a natural number, and let x be a set. The functor G.set(n, x) yielding a graph structure is defined as follows:

(Def. 13) G.set$(n, x) = G {+\cdot} (n \mapsto x)$.

Let G be a graph structure and let X be a set. The functor G.strict(X) yielding a graph structure is defined by:

(Def. 14) G.strict$(X) = G \restriction X$.

Let G be a graph. Observe that G.strict(the graph selectors) is graph-like.

Let G be a graph and let x, y, e be sets. We say that e joins x and y in G if and only if the conditions (Def. 15) are satisfied.

(Def. 15)(i) $e \in$ the edges of G, and

(ii) (the source of $G)(e) = x$ and (the target of $G)(e) = y$ or (the source of $G)(e) = y$ and (the target of $G)(e) = x$.

Let G be a graph and let x, y, e be sets. We say that e joins x to y in G if and only if:

(Def. 16) $e \in$ the edges of G and (the source of $G)(e) = x$ and (the target of $G)(e) = y$.

Let G be a graph and let X, Y, e be sets. We say that e joins a vertex from X and a vertex from Y in G if and only if the conditions (Def. 17) are satisfied.

(Def. 17)(i) $e \in$ the edges of G, and

(ii) (the source of $G)(e) \in X$ and (the target of $G)(e) \in Y$ or (the source of $G)(e) \in Y$ and (the target of $G)(e) \in X$.

We say that e joins a vertex from X to a vertex from Y in G if and only if:

(Def. 18) $e \in$ the edges of G and (the source of $G)(e) \in X$ and (the target of $G)(e) \in Y$.

Let G be a graph. We say that G is finite if and only if:

(Def. 19) The vertices of G is finite and the edges of G is finite.

We say that G is loopless if and only if:

(Def. 20) It is not true that there exists a set e such that $e \in$ the edges of G and (the source of $G)(e) =$ (the target of $G)(e)$.

We say that G is trivial if and only if:

(Def. 21) $\overline{\overline{\text{the vertices of } G}} = \mathbf{1}$.

We say that G is non-multi if and only if:

(Def. 22) For all sets e_1, e_2, v_1, v_2 such that e_1 joins v_1 and v_2 in G and e_2 joins v_1 and v_2 in G holds $e_1 = e_2$.

We say that G is non-directed-multi if and only if:

(Def. 23) For all sets e_1, e_2, v_1, v_2 such that e_1 joins v_1 to v_2 in G and e_2 joins v_1 to v_2 in G holds $e_1 = e_2$.

Let G be a graph. We say that G is simple if and only if:

(Def. 24) G is loopless and non-multi.

We say that G is directed-simple if and only if:

(Def. 25) G is loopless and non-directed-multi.

One can verify the following observations:

* every graph which is non-multi is also non-directed-multi,

* every graph which is simple is also loopless and non-multi,
* every graph which is loopless and non-multi is also simple,
* every graph which is loopless and non-directed-multi is also directed-simple,
* every graph which is directed-simple is also loopless and non-directed-multi,
* every graph which is trivial and loopless is also finite, and
* every graph which is trivial and non-directed-multi is also finite.

Let us note that there exists a graph which is trivial and simple and there exists a graph which is finite, non trivial, and simple.

Let G be a finite graph. Observe that the vertices of G is finite and the edges of G is finite.

Let G be a trivial graph. One can verify that the vertices of G is finite.

Let V be a non empty finite set, let E be a finite set, and let S, T be functions from E into V. One can check that createGraph(V, E, S, T) is finite.

Let V be a non empty set, let E be an empty set, and let S, T be functions from E into V. One can check that createGraph(V, E, S, T) is simple.

Let v be a set, let E be a set, and let S, T be functions from E into $\{v\}$. Observe that createGraph$(\{v\}, E, S, T)$ is trivial.

Let G be a graph. The functor G.order() yielding a cardinal number is defined as follows:

(Def. 26) G.order() $= \overline{\overline{\text{the vertices of } G}}$.

Let G be a finite graph. Then G.order() is a non empty natural number.

Let G be a graph. The functor G.size() yields a cardinal number and is defined by:

(Def. 27) G.size() $= \overline{\overline{\text{the edges of } G}}$.

Let G be a finite graph. Then G.size() is a natural number.

Let G be a graph and let X be a set. The functor G.edgesInto(X) yields a subset of the edges of G and is defined as follows:

(Def. 28) For every set e holds $e \in G$.edgesInto(X) iff $e \in$ the edges of G and (the target of $G)(e) \in X$.

The functor G.edgesOutOf(X) yields a subset of the edges of G and is defined by:

(Def. 29) For every set e holds $e \in G$.edgesOutOf(X) iff $e \in$ the edges of G and (the source of $G)(e) \in X$.

Let G be a graph and let X be a set. The functor G.edgesInOut(X) yields a subset of the edges of G and is defined by:

(Def. 30) G.edgesInOut$(X) = G$.edgesInto$(X) \cup G$.edgesOutOf(X).

The functor G.edgesBetween(X) yielding a subset of the edges of G is defined as follows:

(Def. 31) G.edgesBetween(X) = G.edgesInto(X) $\cap$ G.edgesOutOf(X).

Let G be a graph and let X, Y be sets. The functor G.edgesBetween(X, Y) yielding a subset of the edges of G is defined by:

(Def. 32) For every set e holds $e \in G$.edgesBetween(X, Y) iff e joins a vertex from X and a vertex from Y in G.

The functor G.edgesDBetween(X, Y) yields a subset of the edges of G and is defined as follows:

(Def. 33) For every set e holds $e \in G$.edgesDBetween(X, Y) iff e joins a vertex from X to a vertex from Y in G.

In this article we present several logical schemes. The scheme *FinGraphOrderInd* concerns a unary predicate $\mathcal{P}$, and states that:

For every finite graph G holds $\mathcal{P}[G]$

provided the following conditions are met:

- For every finite graph G such that G.order() = 1 holds $\mathcal{P}[G]$, and
- Let k be a non empty natural number. Suppose that for every finite graph G_1 such that G_1.order() = k holds $\mathcal{P}[G_1]$. Let G_2 be a finite graph. If G_2.order() = $k + 1$, then $\mathcal{P}[G_2]$.

The scheme *FinGraphSizeInd* concerns a unary predicate $\mathcal{P}$, and states that:

For every finite graph G holds $\mathcal{P}[G]$

provided the following requirements are met:

- For every finite graph G such that G.size() = 0 holds $\mathcal{P}[G]$, and
- Let k be a natural number. Suppose that for every finite graph G_1 such that G_1.size() = k holds $\mathcal{P}[G_1]$. Let G_2 be a finite graph. If G_2.size() = $k + 1$, then $\mathcal{P}[G_2]$.

Let G be a graph. A graph is called a subgraph of G if it satisfies the conditions (Def. 34).

(Def. 34)(i) The vertices of it $\subseteq$ the vertices of G,

(ii) the edges of it $\subseteq$ the edges of G, and

(iii) for every set e such that $e \in$ the edges of it holds (the source of it)(e) = (the source of G)(e) and (the target of it)(e) = (the target of G)(e).

Let G_3 be a graph and let G_4 be a subgraph of G_3. Then the vertices of G_4 is a non empty subset of the vertices of G_3. Then the edges of G_4 is a subset of the edges of G_3.

Let G be a graph. Note that there exists a subgraph of G which is trivial and simple.

Let G be a finite graph. Note that every subgraph of G is finite.

Let G be a loopless graph. Observe that every subgraph of G is loopless.

Let G be a trivial graph. One can check that every subgraph of G is trivial.

Let G be a non-multi graph. Observe that every subgraph of G is non-multi.

Let G_3 be a graph and let G_4 be a subgraph of G_3. We say that G_4 is spanning if and only if:

(Def. 35) The vertices of G_4 = the vertices of G_3.

Let G be a graph. One can verify that there exists a subgraph of G which is spanning.

Let G_3, G_4 be graphs. The predicate $G_3 =_G G_4$ is defined by the conditions (Def. 36).

(Def. 36)(i) The vertices of G_3 = the vertices of G_4,

(ii) the edges of G_3 = the edges of G_4,

(iii) the source of G_3 = the source of G_4, and

(iv) the target of G_3 = the target of G_4.

Let us notice that the predicate $G_3 =_G G_4$ is reflexive and symmetric.

Let G_3, G_4 be graphs. We introduce $G_3 \neq_G G_4$ as an antonym of $G_3 =_G G_4$.

Let G_3, G_4 be graphs. The predicate $G_3 \subseteq G_4$ is defined as follows:

(Def. 37) G_3 is a subgraph of G_4.

Let us note that the predicate $G_3 \subseteq G_4$ is reflexive.

Let G_3, G_4 be graphs. The predicate $G_3 \subset G_4$ is defined as follows:

(Def. 38) $G_3 \subseteq G_4$ and $G_3 \neq_G G_4$.

Let us note that the predicate $G_3 \subset G_4$ is irreflexive.

Let G be a graph and let V, E be sets. A subgraph of G is called a subgraph of G induced by V and E if:

(Def. 39)(i) The vertices of it = V and the edges of it = E if V is a non empty subset of the vertices of G and $E \subseteq G$.edgesBetween(V),

(ii) it $=_G G$, otherwise.

Let G be a graph and let V be a set. A subgraph of G induced by V is a subgraph of G induced by V and G.edgesBetween(V).

Let G be a graph, let V be a finite non empty subset of the vertices of G, and let E be a finite subset of G.edgesBetween(V). Observe that every subgraph of G induced by V and E is finite.

Let G be a graph, let v be an element of the vertices of G, and let E be a subset of G.edgesBetween($\{v\}$). Note that every subgraph of G induced by $\{v\}$ and E is trivial.

Let G be a graph and let v be an element of the vertices of G. Note that every subgraph of G induced by $\{v\}$ and $\emptyset$ is finite and trivial.

Let G be a graph and let V be a non empty subset of the vertices of G. Note that every subgraph of G induced by V and $\emptyset$ is simple.

Let G be a graph and let E be a subset of the edges of G. Observe that every subgraph of G induced by the vertices of G and E is spanning.

Let G be a graph. One can check that every subgraph of G induced by the vertices of G and $\emptyset$ is spanning.

Let G be a graph and let v be a set. A subgraph of G with vertex v removed is a subgraph of G induced by (the vertices of G) $\setminus \{v\}$.

Let G be a graph and let V be a set. A subgraph of G with vertices V removed is a subgraph of G induced by (the vertices of G) $\setminus V$.

Let G be a graph and let e be a set. A subgraph of G with edge e removed is a subgraph of G induced by the vertices of G and (the edges of G) $\setminus \{e\}$.

Let G be a graph and let E be a set. A subgraph of G with edges E removed is a subgraph of G induced by the vertices of G and (the edges of G) $\setminus E$.

Let G be a graph and let e be a set. Observe that every subgraph of G with edge e removed is spanning.

Let G be a graph and let E be a set. Observe that every subgraph of G with edges E removed is spanning.

Let G be a graph. A vertex of G is an element of the vertices of G.

Let G be a graph and let v be a vertex of G. The functor v.edgesIn() yielding a subset of the edges of G is defined as follows:

(Def. 40) $v.\mathrm{edgesIn}() = G.\mathrm{edgesInto}(\{v\})$.

The functor v.edgesOut() yields a subset of the edges of G and is defined as follows:

(Def. 41) $v.\mathrm{edgesOut}() = G.\mathrm{edgesOutOf}(\{v\})$.

The functor v.edgesInOut() yields a subset of the edges of G and is defined by:

(Def. 42) $v.\mathrm{edgesInOut}() = G.\mathrm{edgesInOut}(\{v\})$.

Let G be a graph, let v be a vertex of G, and let e be a set. The functor $v.\mathrm{adj}(e)$ yields a vertex of G and is defined by:

(Def. 43) $$v.\mathrm{adj}(e) = \begin{cases} (\text{the source of } G)(e), & \text{if } e \in \text{the edges of } G \text{ and} \\ \quad (\text{the target of } G)(e) = v, & \\ (\text{the target of } G)(e), & \text{if } e \in \text{the edges of } G \text{ and} \\ \quad (\text{the source of } G)(e) = v \text{ and } (\text{the target of } G)(e) \neq v, & \\ v, & \text{otherwise.} \end{cases}$$

Let G be a graph and let v be a vertex of G. The functor v.inDegree() yields a cardinal number and is defined as follows:

(Def. 44) $v.\mathrm{inDegree}() = \overline{\overline{v.\mathrm{edgesIn}()}}$.

The functor v.outDegree() yielding a cardinal number is defined as follows:

(Def. 45) $v.\mathrm{outDegree}() = \overline{\overline{v.\mathrm{edgesOut}()}}$.

Let G be a finite graph and let v be a vertex of G. Then v.inDegree() is a natural number. Then v.outDegree() is a natural number.

Let G be a graph and let v be a vertex of G. The functor v.degree() yielding a cardinal number is defined as follows:

(Def. 46) $v.\mathrm{degree}() = v.\mathrm{inDegree}() + v.\mathrm{outDegree}()$.

Let G be a finite graph and let v be a vertex of G. Then v.degree() is a natural number and it can be characterized by the condition:

(Def. 47) v.degree() = v.inDegree() + v.outDegree().

Let G be a graph and let v be a vertex of G. The functor v.inNeighbors() yields a subset of the vertices of G and is defined as follows:

(Def. 48) v.inNeighbors() = (the source of G)$^\circ v$.edgesIn().

The functor v.outNeighbors() yielding a subset of the vertices of G is defined by:

(Def. 49) v.outNeighbors() = (the target of G)$^\circ v$.edgesOut().

Let G be a graph and let v be a vertex of G. The functor v.allNeighbors() yields a subset of the vertices of G and is defined by:

(Def. 50) v.allNeighbors() = v.inNeighbors() $\cup$ v.outNeighbors().

Let G be a graph and let v be a vertex of G. We say that v is isolated if and only if:

(Def. 51) v.edgesInOut() = $\emptyset$.

Let G be a finite graph and let v be a vertex of G. Let us observe that v is isolated if and only if:

(Def. 52) v.degree() = 0.

Let G be a graph and let v be a vertex of G. We say that v is endvertex if and only if:

(Def. 53) There exists a set e such that v.edgesInOut() = $\{e\}$ and e does not join v and v in G.

Let G be a finite graph and let v be a vertex of G. Let us observe that v is endvertex if and only if:

(Def. 54) v.degree() = 1.

Let F be a many sorted set indexed by $\mathbb{N}$. We say that F is graph-yielding if and only if:

(Def. 55) For every natural number n holds $F(n)$ is a graph.

We say that F is halting if and only if:

(Def. 56) There exists a natural number n such that $F(n) = F(n+1)$.

Let F be a many sorted set indexed by $\mathbb{N}$. The functor F.Lifespan() yielding a natural number is defined by:

(Def. 57)(i) $F(F$.Lifespan()$) = F(F$.Lifespan()$+1)$ and for every natural number n such that $F(n) = F(n+1)$ holds F.Lifespan() $\leq n$ if F is halting,

(ii) F.Lifespan() = 0, otherwise.

Let F be a many sorted set indexed by $\mathbb{N}$. The functor F.Result() yielding a set is defined by:

(Def. 58) F.Result() = $F(F$.Lifespan()$)$.

Let us mention that there exists a many sorted set indexed by $\mathbb{N}$ which is graph-yielding.

A graph sequence is a graph-yielding many sorted set indexed by $\mathbb{N}$.

Let G_5 be a graph sequence and let x be a natural number. The functor $G_5 {\rightarrow} x$ yields a graph and is defined by:

(Def. 59) $G_5 {\rightarrow} x = G_5(x)$.

Let G_5 be a graph sequence. We say that G_5 is finite if and only if:

(Def. 60) For every natural number x holds $G_5 {\rightarrow} x$ is finite.

We say that G_5 is loopless if and only if:

(Def. 61) For every natural number x holds $G_5 {\rightarrow} x$ is loopless.

We say that G_5 is trivial if and only if:

(Def. 62) For every natural number x holds $G_5 {\rightarrow} x$ is trivial.

We say that G_5 is non-trivial if and only if:

(Def. 63) For every natural number x holds $G_5 {\rightarrow} x$ is non trivial.

We say that G_5 is non-multi if and only if:

(Def. 64) For every natural number x holds $G_5 {\rightarrow} x$ is non-multi.

We say that G_5 is non-directed-multi if and only if:

(Def. 65) For every natural number x holds $G_5 {\rightarrow} x$ is non-directed-multi.

We say that G_5 is simple if and only if:

(Def. 66) For every natural number x holds $G_5 {\rightarrow} x$ is simple.

We say that G_5 is directed-simple if and only if:

(Def. 67) For every natural number x holds $G_5 {\rightarrow} x$ is directed-simple.

Let G_5 be a graph sequence. Let us observe that G_5 is halting if and only if:

(Def. 68) There exists a natural number n such that $G_5 {\rightarrow} n = G_5 {\rightarrow} (n+1)$.

One can verify that there exists a graph sequence which is halting, finite, loopless, trivial, non-multi, non-directed-multi, simple, and directed-simple and there exists a graph sequence which is halting, finite, loopless, non-trivial, non-multi, non-directed-multi, simple, and directed-simple.

Let G_5 be a finite graph sequence and let x be a natural number. One can check that $G_5 {\rightarrow} x$ is finite.

Let G_5 be a loopless graph sequence and let x be a natural number. Note that $G_5 {\rightarrow} x$ is loopless.

Let G_5 be a trivial graph sequence and let x be a natural number. Observe that $G_5 {\rightarrow} x$ is trivial.

Let G_5 be a non-trivial graph sequence and let x be a natural number. Observe that $G_5 {\rightarrow} x$ is non trivial.

Let G_5 be a non-multi graph sequence and let x be a natural number. Note that $G_5 {\rightarrow} x$ is non-multi.

Let G_5 be a non-directed-multi graph sequence and let x be a natural number. Observe that $G_5 {\rightarrow} x$ is non-directed-multi.

Let G_5 be a simple graph sequence and let x be a natural number. Note that $G_5 \rightarrow x$ is simple.

Let G_5 be a directed-simple graph sequence and let x be a natural number. Note that $G_5 \rightarrow x$ is directed-simple.

One can check that every graph sequence which is non-multi is also non-directed-multi.

Let us observe that every graph sequence which is simple is also loopless and non-multi.

One can verify that every graph sequence which is loopless and non-multi is also simple.

Let us note that every graph sequence which is loopless and non-directed-multi is also directed-simple.

One can verify that every graph sequence which is directed-simple is also loopless and non-directed-multi.

Let us note that every graph sequence which is trivial and loopless is also finite.

Let us observe that every graph sequence which is trivial and non-directed-multi is also finite.

2. Theorems

For simplicity, we adopt the following convention: G_6 denotes a graph structure, G, G_3, G_4, G_7 denote graphs, e, x, x_1, x_2, y, y_1, y_2, E, V, X, Y denote sets, n, n_1, n_2 denote natural numbers, and v, v_1, v_2 denote vertices of G.

We now state a number of propositions:

(1) VertexSelector $= 1$ and EdgeSelector $= 2$ and SourceSelector $= 3$ and TargetSelector $= 4$.

(2) $x \in$ the graph selectors iff $x =$ VertexSelector or $x =$ EdgeSelector or $x =$ SourceSelector or $x =$ TargetSelector.

(3) The graph selectors $\subseteq \operatorname{dom} G$.

(4) The vertices of $G_6 = G_6$(VertexSelector) and the edges of $G_6 = G_6$(EdgeSelector) and the source of $G_6 = G_6$(SourceSelector) and the target of $G_6 = G_6$(TargetSelector).

(5)(i) dom (the source of G) = the edges of G,

(ii) dom (the target of G) = the edges of G,

(iii) rng (the source of G) $\subseteq$ the vertices of G, and

(iv) rng (the target of G) $\subseteq$ the vertices of G.

(7)[3] G_6 is graph-like if and only if the following conditions are satisfied:

(i) the graph selectors $\subseteq \operatorname{dom} G_6$,

[3]The proposition (6) has been removed.

(ii) the vertices of G_6 is non empty,
(iii) the source of G_6 is a function from the edges of G_6 into the vertices of G_6, and
(iv) the target of G_6 is a function from the edges of G_6 into the vertices of G_6.

(8) Let V be a non empty set, E be a set, and S, T be functions from E into V. Then
(i) the vertices of createGraph$(V, E, S, T) = V$,
(ii) the edges of createGraph$(V, E, S, T) = E$,
(iii) the source of createGraph$(V, E, S, T) = S$, and
(iv) the target of createGraph$(V, E, S, T) = T$.

(9) $\mathrm{dom}(G_6.\mathrm{set}(n, x)) = \mathrm{dom}\, G_6 \cup \{n\}$.

(10) $\mathrm{dom}\, G_6 \subseteq \mathrm{dom}(G_6.\mathrm{set}(n, x))$.

(11) $(G_6.\mathrm{set}(n, x))(n) = x$.

(12) If $n_1 \neq n_2$, then $G_6(n_2) = (G_6.\mathrm{set}(n_1, x))(n_2)$.

(13) Suppose $n \notin$ the graph selectors. Then
(i) the vertices of $G =$ the vertices of $G.\mathrm{set}(n, x)$,
(ii) the edges of $G =$ the edges of $G.\mathrm{set}(n, x)$,
(iii) the source of $G =$ the source of $G.\mathrm{set}(n, x)$,
(iv) the target of $G =$ the target of $G.\mathrm{set}(n, x)$, and
(v) $G.\mathrm{set}(n, x)$ is a graph.

(14) The vertices of $G_6.\mathrm{set}(\mathrm{VertexSelector}, x) = x$ and the edges of $G_6.\mathrm{set}(\mathrm{EdgeSelector}, x) = x$ and the source of $G_6.\mathrm{set}(\mathrm{SourceSelector}, x) = x$ and the target of $G_6.\mathrm{set}(\mathrm{TargetSelector}, x) = x$.

(15) If $n_1 \neq n_2$, then $n_1 \in \mathrm{dom}(G_6.\mathrm{set}(n_1, x).\mathrm{set}(n_2, y))$ and $n_2 \in \mathrm{dom}(G_6.\mathrm{set}(n_1, x).\mathrm{set}(n_2, y))$ and $(G_6.\mathrm{set}(n_1, x).\mathrm{set}(n_2, y))(n_1) = x$ and $(G_6.\mathrm{set}(n_1, x).\mathrm{set}(n_2, y))(n_2) = y$.

(16) If e joins x and y in G, then $x \in$ the vertices of G and $y \in$ the vertices of G.

(17) If e joins x and y in G, then e joins y and x in G.

(18) If e joins x_1 and y_1 in G and e joins x_2 and y_2 in G, then $x_1 = x_2$ and $y_1 = y_2$ or $x_1 = y_2$ and $y_1 = x_2$.

(19) e joins x and y in G iff e joins x to y in G or e joins y to x in G.

(20) Suppose e joins x and y in G but $x \in X$ and $y \in Y$ or $x \in Y$ and $y \in X$. Then e joins a vertex from X and a vertex from Y in G.

(21) G is loopless iff for every set v it is not true that there exists a set e such that e joins v and v in G.

(22) For every finite loopless graph G and for every vertex v of G holds $v.\mathrm{degree}() = \mathrm{card}(v.\mathrm{edgesInOut}())$.

(23) For every non trivial graph G and for every vertex v of G holds (the vertices of $G) \setminus \{v\}$ is non empty.

(24) For every non trivial graph G there exist vertices v_1, v_2 of G such that $v_1 \neq v_2$.

(25) For every trivial graph G there exists a vertex v of G such that the vertices of $G = \{v\}$.

(26) For every trivial loopless graph G holds the edges of $G = \emptyset$.

(27) If the edges of $G = \emptyset$, then G is simple.

(28) For every finite graph G holds G.order() ≥ 1.

(29) For every finite graph G holds G.order() $= 1$ iff G is trivial.

(30) For every finite graph G holds G.order() $= 1$ iff there exists a vertex v of G such that the vertices of $G = \{v\}$.

(31) $e \in$ the edges of G but (the source of $G)(e) \in X$ or (the target of $G)(e) \in X$ iff $e \in G$.edgesInOut(X).

(32) G.edgesInto$(X) \subseteq G$.edgesInOut(X) and G.edgesOutOf$(X) \subseteq G$.edgesInOut(X).

(33) The edges of $G = G$.edgesInOut(the vertices of G).

(34) $e \in$ the edges of G and (the source of $G)(e) \in X$ and (the target of $G)(e) \in X$ iff $e \in G$.edgesBetween(X).

(35) If $x \in X$ and $y \in X$ and e joins x and y in G, then $e \in G$.edgesBetween(X).

(36) G.edgesBetween$(X) \subseteq G$.edgesInOut(X).

(37) The edges of $G = G$.edgesBetween(the vertices of G).

(38) (The edges of $G) \setminus G$.edgesInOut$(X) = G$.edgesBetween((the vertices of $G) \setminus X)$.

(39) If $X \subseteq Y$, then G.edgesBetween$(X) \subseteq G$.edgesBetween(Y).

(40) For every graph G and for all sets X_1, X_2, Y_1, Y_2 such that $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ holds G.edgesBetween$(X_1, Y_1) \subseteq G$.edgesBetween(X_2, Y_2).

(41) For every graph G and for all sets X_1, X_2, Y_1, Y_2 such that $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ holds G.edgesDBetween$(X_1, Y_1) \subseteq G$.edgesDBetween(X_2, Y_2).

(42) For every graph G and for every vertex v of G holds v.edgesIn() $= G$.edgesDBetween(the vertices of G, $\{v\})$ and v.edgesOut() $= G$.edgesDBetween$(\{v\},$ the vertices of $G)$.

(43) G is a subgraph of G.

(44) G_3 is a subgraph of G_4 and G_4 is a subgraph of G_3 if and only if the following conditions are satisfied:
  (i) the vertices of $G_3 =$ the vertices of G_4,
  (ii) the edges of $G_3 =$ the edges of G_4,
  (iii) the source of $G_3 =$ the source of G_4, and

(iv) the target of G_3 = the target of G_4.

(45) Let G_3 be a graph, G_4 be a subgraph of G_3, and x be a set. Then

(i) if $x \in$ the vertices of G_4, then $x \in$ the vertices of G_3, and

(ii) if $x \in$ the edges of G_4, then $x \in$ the edges of G_3.

(46) For every graph G_3 and for every subgraph G_4 of G_3 holds every subgraph of G_4 is a subgraph of G_3.

(47) Let G be a graph and G_3, G_4 be subgraphs of G. Suppose the vertices of $G_3 \subseteq$ the vertices of G_4 and the edges of $G_3 \subseteq$ the edges of G_4. Then G_3 is a subgraph of G_4.

(48) Let G_3 be a graph and G_4 be a subgraph of G_3. Then

(i) the source of G_4 = (the source of G_3)$\restriction$(the edges of G_4), and

(ii) the target of G_4 = (the target of G_3)$\restriction$(the edges of G_4).

(49) Let G be a graph, V_1, V_2, E_1, E_2 be sets, G_3 be a subgraph of G induced by V_1 and E_1, and G_4 be a subgraph of G induced by V_2 and E_2. Suppose $V_2 \subseteq V_1$ and $E_2 \subseteq E_1$ and V_2 is a non empty subset of the vertices of G and $E_2 \subseteq G$.edgesBetween(V_2). Then G_4 is a subgraph of G_3.

(50) Let G_3 be a non trivial graph, v be a vertex of G_3, and G_4 be a subgraph of G_3 with vertex v removed. Then the vertices of G_4 = (the vertices of G_3) $\setminus \{v\}$ and the edges of G_4 = G_3.edgesBetween((the vertices of G_3) $\setminus \{v\}$).

(51) Let G_3 be a finite non trivial graph, v be a vertex of G_3, and G_4 be a subgraph of G_3 with vertex v removed. Then G_4.order() + 1 = G_3.order() and G_4.size() + card(v.edgesInOut()) = G_3.size().

(52) Let G_3 be a graph, V be a set, and G_4 be a subgraph of G_3 with vertices V removed. Suppose $V \subset$ the vertices of G_3. Then the vertices of G_4 = (the vertices of G_3) $\setminus V$ and the edges of G_4 = G_3.edgesBetween((the vertices of G_3) $\setminus V$).

(53) Let G_3 be a finite graph, V be a subset of the vertices of G_3, and G_4 be a subgraph of G_3 with vertices V removed. If $V \neq$ the vertices of G_3, then G_4.order() + card V = G_3.order() and G_4.size() + card(G_3.edgesInOut(V)) = G_3.size().

(54) Let G_3 be a graph, e be a set, and G_4 be a subgraph of G_3 with edge e removed. Then the vertices of G_4 = the vertices of G_3 and the edges of G_4 = (the edges of G_3) $\setminus \{e\}$.

(55) Let G_3 be a finite graph, e be a set, and G_4 be a subgraph of G_3 with edge e removed. Then G_3.order() = G_4.order() and if $e \in$ the edges of G_3, then G_4.size() + 1 = G_3.size().

(56) Let G_3 be a graph, E be a set, and G_4 be a subgraph of G_3 with edges E removed. Then the vertices of G_4 = the vertices of G_3 and the edges of G_4 = (the edges of G_3) $\setminus E$.

(57) For every finite graph G_3 and for every set E and for every subgraph G_4 of G_3 with edges E removed holds G_3.order() = G_4.order().

(58) Let G_3 be a finite graph, E be a subset of the edges of G_3, and G_4 be a subgraph of G_3 with edges E removed. Then G_4.size() + card E = G_3.size().

(59) $e \in v$.edgesIn() iff $e \in$ the edges of G and (the target of G)(e) = v.

(60) $e \in v$.edgesIn() iff there exists a set x such that e joins x to v in G.

(61) $e \in v$.edgesOut() iff $e \in$ the edges of G and (the source of G)(e) = v.

(62) $e \in v$.edgesOut() iff there exists a set x such that e joins v to x in G.

(63) v.edgesInOut() = v.edgesIn() $\cup$ v.edgesOut().

(64) $e \in v$.edgesInOut() iff $e \in$ the edges of G but (the source of G)(e) = v or (the target of G)(e) = v.

(65) If e joins v_1 and x in G, then $e \in v_1$.edgesInOut().

(66) If e joins v_1 and v_2 in G, then $e \in v_1$.edgesIn() and $e \in v_2$.edgesOut() or $e \in v_2$.edgesIn() and $e \in v_1$.edgesOut().

(67) $e \in v_1$.edgesInOut() iff there exists a vertex v_2 of G such that e joins v_1 and v_2 in G.

(68) If $e \in v$.edgesInOut() and e joins x and y in G, then $v = x$ or $v = y$.

(69) If e joins v_1 and v_2 in G, then v_1.adj(e) = v_2 and v_2.adj(e) = v_1.

(70) $e \in v$.edgesInOut() iff e joins v and v.adj(e) in G.

(71) Let G be a finite graph, e be a set, and v_1, v_2 be vertices of G. If e joins v_1 and v_2 in G, then $1 \leq v_1$.degree() and $1 \leq v_2$.degree().

(72) $x \in v$.inNeighbors() iff there exists a set e such that e joins x to v in G.

(73) $x \in v$.outNeighbors() iff there exists a set e such that e joins v to x in G.

(74) $x \in v$.allNeighbors() iff there exists a set e such that e joins v and x in G.

(75) Let G_3 be a graph, G_4 be a subgraph of G_3, and x, y, e be sets. Then
- (i) if e joins x and y in G_4, then e joins x and y in G_3,
- (ii) if e joins x to y in G_4, then e joins x to y in G_3,
- (iii) if e joins a vertex from x and a vertex from y in G_4, then e joins a vertex from x and a vertex from y in G_3, and
- (iv) if e joins a vertex from x to a vertex from y in G_4, then e joins a vertex from x to a vertex from y in G_3.

(76) Let G_3 be a graph, G_4 be a subgraph of G_3, and x, y, e be sets such that $e \in$ the edges of G_4. Then
- (i) if e joins x and y in G_3, then e joins x and y in G_4,
- (ii) if e joins x to y in G_3, then e joins x to y in G_4,

(iii) if e joins a vertex from x and a vertex from y in G_3, then e joins a vertex from x and a vertex from y in G_4, and

(iv) if e joins a vertex from x to a vertex from y in G_3, then e joins a vertex from x to a vertex from y in G_4.

(77) For every graph G_3 and for every spanning subgraph G_4 of G_3 holds every spanning subgraph of G_4 is a spanning subgraph of G_3.

(78) For every finite graph G_3 and for every subgraph G_4 of G_3 holds G_4.order() $\leq G_3$.order() and G_4.size() $\leq G_3$.size().

(79) Let G_3 be a graph, G_4 be a subgraph of G_3, and X be a set. Then G_4.edgesInto(X) $\subseteq G_3$.edgesInto(X) and G_4.edgesOutOf(X) $\subseteq G_3$.edgesOutOf(X) and G_4.edgesInOut(X) $\subseteq G_3$.edgesInOut(X) and G_4.edgesBetween(X) $\subseteq G_3$.edgesBetween(X).

(80) For every graph G_3 and for every subgraph G_4 of G_3 and for all sets X, Y holds G_4.edgesBetween(X, Y) $\subseteq G_3$.edgesBetween(X, Y) and G_4.edgesDBetween(X, Y) $\subseteq G_3$.edgesDBetween(X, Y).

(81) Let G_3 be a graph, G_4 be a subgraph of G_3, v_1 be a vertex of G_3, and v_2 be a vertex of G_4. If $v_1 = v_2$, then v_2.edgesIn() $\subseteq v_1$.edgesIn() and v_2.edgesOut() $\subseteq v_1$.edgesOut() and v_2.edgesInOut() $\subseteq v_1$.edgesInOut().

(82) Let G_3 be a graph, G_4 be a subgraph of G_3, v_1 be a vertex of G_3, and v_2 be a vertex of G_4. Suppose $v_1 = v_2$. Then v_2.edgesIn() $= v_1$.edgesIn() $\cap$ the edges of G_4 and v_2.edgesOut() $= v_1$.edgesOut() $\cap$ the edges of G_4 and v_2.edgesInOut() $= v_1$.edgesInOut() $\cap$ the edges of G_4.

(83) Let G_3 be a graph, G_4 be a subgraph of G_3, v_1 be a vertex of G_3, v_2 be a vertex of G_4, and e be a set. If $v_1 = v_2$ and $e \in$ the edges of G_4, then v_1.adj(e) $= v_2$.adj(e).

(84) Let G_3 be a finite graph, G_4 be a subgraph of G_3, v_1 be a vertex of G_3, and v_2 be a vertex of G_4. If $v_1 = v_2$, then v_2.inDegree() $\leq v_1$.inDegree() and v_2.outDegree() $\leq v_1$.outDegree() and v_2.degree() $\leq v_1$.degree().

(85) Let G_3 be a graph, G_4 be a subgraph of G_3, v_1 be a vertex of G_3, and v_2 be a vertex of G_4. If $v_1 = v_2$, then v_2.inNeighbors() $\subseteq v_1$.inNeighbors() and v_2.outNeighbors() $\subseteq v_1$.outNeighbors() and v_2.allNeighbors() $\subseteq v_1$.allNeighbors().

(86) Let G_3 be a graph, G_4 be a subgraph of G_3, v_1 be a vertex of G_3, and v_2 be a vertex of G_4. If $v_1 = v_2$ and v_1 is isolated, then v_2 is isolated.

(87) Let G_3 be a graph, G_4 be a subgraph of G_3, v_1 be a vertex of G_3, and v_2 be a vertex of G_4. If $v_1 = v_2$ and v_1 is endvertex, then v_2 is endvertex or isolated.

(88) If $G_3 =_G G_4$ and $G_4 =_G G_7$, then $G_3 =_G G_7$.

(89) Let G be a graph and G_3, G_4 be subgraphs of G. Suppose the vertices of $G_3 =$ the vertices of G_4 and the edges of $G_3 =$ the edges of G_4. Then

$G_3 =_G G_4$.

(90) $G_3 =_G G_4$ iff G_3 is a subgraph of G_4 and G_4 is a subgraph of G_3.

(91) Suppose $G_3 =_G G_4$. Then

(i) if e joins x and y in G_3, then e joins x and y in G_4,

(ii) if e joins x to y in G_3, then e joins x to y in G_4,

(iii) if e joins a vertex from X and a vertex from Y in G_3, then e joins a vertex from X and a vertex from Y in G_4, and

(iv) if e joins a vertex from X to a vertex from Y in G_3, then e joins a vertex from X to a vertex from Y in G_4.

(92) Suppose $G_3 =_G G_4$. Then

(i) if G_3 is finite, then G_4 is finite,

(ii) if G_3 is loopless, then G_4 is loopless,

(iii) if G_3 is trivial, then G_4 is trivial,

(iv) if G_3 is non-multi, then G_4 is non-multi,

(v) if G_3 is non-directed-multi, then G_4 is non-directed-multi,

(vi) if G_3 is simple, then G_4 is simple, and

(vii) if G_3 is directed-simple, then G_4 is directed-simple.

(93) If $G_3 =_G G_4$, then G_3.order() = G_4.order() and G_3.size() = G_4.size() and G_3.edgesInto(X) = G_4.edgesInto(X) and G_3.edgesOutOf(X) = G_4.edgesOutOf(X) and G_3.edgesInOut(X) = G_4.edgesInOut(X) and G_3.edgesBetween(X) = G_4.edgesBetween(X) and G_3.edgesDBetween(X, Y) = G_4.edgesDBetween(X, Y).

(94) If $G_3 =_G G_4$ and G_7 is a subgraph of G_3, then G_7 is a subgraph of G_4.

(95) If $G_3 =_G G_4$ and G_3 is a subgraph of G_7, then G_4 is a subgraph of G_7.

(96) For all subgraphs G_3, G_4 of G induced by V and E holds $G_3 =_G G_4$.

(97) For every graph G_3 and for every subgraph G_4 of G_3 induced by the vertices of G_3 holds $G_3 =_G G_4$.

(98) Let G_3, G_4 be graphs, V, E be sets, and G_7 be a subgraph of G_3 induced by V and E. If $G_3 =_G G_4$, then G_7 is a subgraph of G_4 induced by V and E.

(99) Let v_1 be a vertex of G_3 and v_2 be a vertex of G_4. Suppose $v_1 = v_2$ and $G_3 =_G G_4$. Then v_1.edgesIn() = v_2.edgesIn() and v_1.edgesOut() = v_2.edgesOut() and v_1.edgesInOut() = v_2.edgesInOut() and v_1.adj(e) = v_2.adj(e) and v_1.inDegree() = v_2.inDegree() and v_1.outDegree() = v_2.outDegree() and v_1.degree() = v_2.degree() and v_1.inNeighbors() = v_2.inNeighbors() and v_1.outNeighbors() = v_2.outNeighbors() and v_1.allNeighbors() = v_2.allNeighbors().

(100) Let v_1 be a vertex of G_3 and v_2 be a vertex of G_4 such that $v_1 = v_2$ and $G_3 =_G G_4$. Then

(i) if v_1 is isolated, then v_2 is isolated, and

(ii) if v_1 is endvertex, then v_2 is endvertex.

(101) Let G be a graph and G_3, G_4 be subgraphs of G. Suppose $G_3 \subset G_4$. Then the vertices of $G_3 \subset$ the vertices of G_4 or the edges of $G_3 \subset$ the edges of G_4.

(102) Let G be a graph and G_3, G_4 be subgraphs of G. Suppose $G_3 \subset G_4$. Then

(i) there exists a set v such that $v \in$ the vertices of G_4 and $v \notin$ the vertices of G_3, or

(ii) there exists a set e such that $e \in$ the edges of G_4 and $e \notin$ the edges of G_3.

References

[1] Grzegorz Bancerek. Cardinal arithmetics. *Formalized Mathematics*, 1(**3**):543–547, 1990.

[2] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[4] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.

[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[6] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.

[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[9] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[10] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.

[11] Jing-Chao Chen. A small computer model with push-down stack. *Formalized Mathematics*, 8(**1**):175–182, 1999.

[12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[13] Krzysztof Hryniewiecki. Graphs. *Formalized Mathematics*, 2(**3**):365–370, 1991.

[14] Gilbert Lee. *Verification of graph algorithms in Mizar*. Dept. of Comp. Sci., University of Alberta, Edmonton, Canada, 2004. M Sc thesis, `http://www.cs.ualberta.ca/~piotr/Mizar/Doc/GL-thesis.ps`.

[15] Gilbert Lee. Trees and Graph Components. *Formalized Mathematics*, 13(**2**):271–277, 2005.

[16] Gilbert Lee. Walks in Graphs. *Formalized Mathematics*, 13(**2**):253–269, 2005.

[17] Gilbert Lee. Weighted and Labeled Graphs. *Formalized Mathematics*, 13(**2**):279–293, 2005.

[18] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*

[19] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[21] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.

[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[23] Josef Urban. Basic facts about inaccessible and measurable cardinals. *Formalized Mathematics*, 9(**2**):323–329, 2001.

[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[25] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 22, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 253–269
University of Białystok, 2005

Walks in Graphs[1]

Gilbert Lee[2]
University of Victoria, Victoria, Canada

Summary. We define walks for graphs introduced in [9], introduce walk attributes and functors for walk creation and modification of walks. Subwalks of a walk are also defined. In our rendition, walks are alternating finite sequences of vertices and edges.

MML identifier: `GLIB_001`, version: `7.5.01 4.39.921`

The notation and terminology used here are introduced in the following papers: [14], [12], [16], [13], [18], [6], [4], [5], [1], [10], [17], [7], [3], [19], [15], [8], [2], [9], and [11].

1. Preliminaries

The following propositions are true:

(1) For all odd natural numbers x, y holds $x < y$ iff $x + 2 \leq y$.

(2) Let X be a set and k be a natural number. Suppose $X \subseteq \mathrm{Seg}\, k$. Let m, n be natural numbers. If $m \in \mathrm{dom}\,\mathrm{Sgm}\, X$ and $n = (\mathrm{Sgm}\, X)(m)$, then $m \leq n$.

(3) For every set X and for every finite sequence f_2 of elements of X and for every FinSubsequence f_1 of f_2 holds $\mathrm{len}\,\mathrm{Seq}\, f_1 \leq \mathrm{len}\, f_2$.

(4) Let X be a set, f_2 be a finite sequence of elements of X, f_1 be a FinSubsequence of f_2, and m be a natural number. Suppose $m \in \mathrm{dom}\,\mathrm{Seq}\, f_1$. Then there exists a natural number n such that $n \in \mathrm{dom}\, f_2$ and $m \leq n$ and $(\mathrm{Seq}\, f_1)(m) = f_2(n)$.

[1]This work has been partially supported by NSERC, Alberta Ingenuity Fund and iCORE.
[2]Part of author's MSc work.

ISSN 1426–2630

(5) For every set X and for every finite sequence f_2 of elements of X and for every FinSubsequence f_1 of f_2 holds $\operatorname{len}\operatorname{Seq} f_1 = \operatorname{card} f_1$.

(6) Let X be a set, f_2 be a finite sequence of elements of X, and f_1 be a FinSubsequence of f_2. Then $\operatorname{dom}\operatorname{Seq} f_1 = \operatorname{dom}\operatorname{Sgm}\operatorname{dom} f_1$.

2. Walk Definitions

Let G be a graph. A finite sequence of elements of the vertices of G is said to be a vertex sequence of G if:

(Def. 1) For every natural number n such that $1 \leq n$ and $n < \operatorname{len}\mathrm{it}$ there exists a set e such that e joins $\mathrm{it}(n)$ and $\mathrm{it}(n+1)$ in G.

Let G be a graph. A finite sequence of elements of the edges of G is said to be a edge sequence of G if it satisfies the condition (Def. 2).

(Def. 2) There exists a finite sequence v_1 of elements of the vertices of G such that $\operatorname{len} v_1 = \operatorname{len}\mathrm{it} + 1$ and for every natural number n such that $1 \leq n$ and $n \leq \operatorname{len}\mathrm{it}$ holds $\mathrm{it}(n)$ joins $v_1(n)$ and $v_1(n+1)$ in G.

Let G be a graph. A finite sequence of elements of (the vertices of G) $\cup$ (the edges of G) is said to be a walk of G if it satisfies the conditions (Def. 3).

(Def. 3)(i) $\operatorname{len}\mathrm{it}$ is odd,

(ii) $\mathrm{it}(1) \in$ the vertices of G, and

(iii) for every odd natural number n such that $n < \operatorname{len}\mathrm{it}$ holds $\mathrm{it}(n+1)$ joins $\mathrm{it}(n)$ and $\mathrm{it}(n+2)$ in G.

Let G be a graph and let W be a walk of G. One can verify that $\operatorname{len} W$ is odd and non empty.

Let G be a graph and let v be a vertex of G. The functor $G.\mathrm{walkOf}(v)$ yielding a walk of G is defined as follows:

(Def. 4) $G.\mathrm{walkOf}(v) = \langle v \rangle$.

Let G be a graph and let x, y, e be sets. The functor $G.\mathrm{walkOf}(x, e, y)$ yielding a walk of G is defined as follows:

(Def. 5) $G.\mathrm{walkOf}(x, e, y) = \begin{cases} \langle x, e, y \rangle, & \text{if } e \text{ joins } x \text{ and } y \text{ in } G, \\ G.\mathrm{walkOf}(\mathrm{choose}(\text{the vertices of } G)), & \text{otherwise.} \end{cases}$

Let G be a graph and let W be a walk of G. The functor $W.\mathrm{first}()$ yields a vertex of G and is defined as follows:

(Def. 6) $W.\mathrm{first}() = W(1)$.

The functor $W.\mathrm{last}()$ yields a vertex of G and is defined by:

(Def. 7) $W.\mathrm{last}() = W(\operatorname{len} W)$.

Let G be a graph, let W be a walk of G, and let n be a natural number. The functor $W.\mathrm{vertexAt}(n)$ yielding a vertex of G is defined as follows:

(Def. 8) $W.\text{vertexAt}(n) = \begin{cases} W(n), & \text{if } n \text{ is odd and } n \leq \operatorname{len} W, \\ W.\text{first}(), & \text{otherwise.} \end{cases}$

Let G be a graph and let W be a walk of G. The functor $W.\text{reverse}()$ yielding a walk of G is defined as follows:

(Def. 9) $W.\text{reverse}() = \text{Rev}(W)$.

Let G be a graph and let W_1, W_2 be walks of G. The functor $W_1.\text{append}(W_2)$ yields a walk of G and is defined by:

(Def. 10) $W_1.\text{append}(W_2) = \begin{cases} W_1 \frown W_2, & \text{if } W_1.\text{last}() = W_2.\text{first}(), \\ W_1, & \text{otherwise.} \end{cases}$

Let G be a graph, let W be a walk of G, and let m, n be natural numbers. The functor $W.\text{cut}(m, n)$ yields a walk of G and is defined by:

(Def. 11) $W.\text{cut}(m, n) = \begin{cases} \langle W(m), \ldots, W(n) \rangle, & \text{if } m \text{ is odd and } n \text{ is odd and} \\ \quad m \leq n \text{ and } n \leq \operatorname{len} W, \\ W, & \text{otherwise.} \end{cases}$

Let G be a graph, let W be a walk of G, and let m, n be natural numbers. The functor $W.\text{remove}(m, n)$ yielding a walk of G is defined by:

(Def. 12) $W.\text{remove}(m, n) = \begin{cases} (W.\text{cut}(1, m)).\text{append}((W.\text{cut}(n, \operatorname{len} W))), \\ \quad \text{if } m \text{ is odd and } n \text{ is odd and } m \leq n \text{ and} \\ \quad n \leq \operatorname{len} W \text{ and } W(m) = W(n), \\ W, \text{ otherwise.} \end{cases}$

Let G be a graph, let W be a walk of G, and let e be a set. The functor $W.\text{addEdge}(e)$ yields a walk of G and is defined as follows:

(Def. 13) $W.\text{addEdge}(e) = W.\text{append}((G.\text{walkOf}(W.\text{last}(), e, W.\text{last}().\text{adj}(e))))$.

Let G be a graph and let W be a walk of G. The functor $W.\text{vertexSeq}()$ yielding a vertex sequence of G is defined by:

(Def. 14) $\operatorname{len} W + 1 = 2 \cdot \operatorname{len}(W.\text{vertexSeq}())$ and for every natural number n such that $1 \leq n$ and $n \leq \operatorname{len}(W.\text{vertexSeq}())$ holds $W.\text{vertexSeq}()(n) = W(2 \cdot n - 1)$.

Let G be a graph and let W be a walk of G. The functor $W.\text{edgeSeq}()$ yields a edge sequence of G and is defined by:

(Def. 15) $\operatorname{len} W = 2 \cdot \operatorname{len}(W.\text{edgeSeq}()) + 1$ and for every natural number n such that $1 \leq n$ and $n \leq \operatorname{len}(W.\text{edgeSeq}())$ holds $W.\text{edgeSeq}()(n) = W(2 \cdot n)$.

Let G be a graph and let W be a walk of G. The functor $W.\text{vertices}()$ yields a finite subset of the vertices of G and is defined as follows:

(Def. 16) $W.\text{vertices}() = \text{rng}(W.\text{vertexSeq}())$.

Let G be a graph and let W be a walk of G. The functor $W.\text{edges}()$ yields a finite subset of the edges of G and is defined by:

(Def. 17) $W.\text{edges}() = \text{rng}(W.\text{edgeSeq}())$.

Let G be a graph and let W be a walk of G. The functor $W.\text{length}()$ yielding a natural number is defined by:

(Def. 18) W.length() = len(W.edgeSeq()).

Let G be a graph, let W be a walk of G, and let v be a set. The functor W.find(v) yields an odd natural number and is defined by:

(Def. 19)(i) W.find(v) $\leq$ len W and W(W.find(v)) = v and for every odd natural number n such that $n \leq$ len W and $W(n) = v$ holds W.find(v) $\leq n$ if $v \in W$.vertices(),

(ii) W.find(v) = len W, otherwise.

Let G be a graph, let W be a walk of G, and let n be a natural number. The functor W.find(n) yielding an odd natural number is defined by:

(Def. 20)(i) W.find(n) $\leq$ len W and W(W.find(n)) = $W(n)$ and for every odd natural number k such that $k \leq$ len W and $W(k) = W(n)$ holds W.find(n) $\leq k$ if n is odd and $n \leq$ len W,

(ii) W.find(n) = len W, otherwise.

Let G be a graph, let W be a walk of G, and let v be a set. The functor W.rfind(v) yields an odd natural number and is defined as follows:

(Def. 21)(i) W.rfind(v) $\leq$ len W and W(W.rfind(v)) = v and for every odd natural number n such that $n \leq$ len W and $W(n) = v$ holds $n \leq W$.rfind(v) if $v \in W$.vertices(),

(ii) W.rfind(v) = len W, otherwise.

Let G be a graph, let W be a walk of G, and let n be a natural number. The functor W.rfind(n) yields an odd natural number and is defined by:

(Def. 22)(i) W.rfind(n) $\leq$ len W and W(W.rfind(n)) = $W(n)$ and for every odd natural number k such that $k \leq$ len W and $W(k) = W(n)$ holds $k \leq W$.rfind(n) if n is odd and $n \leq$ len W,

(ii) W.rfind(n) = len W, otherwise.

Let G be a graph, let u, v be sets, and let W be a walk of G. We say that W is walk from u to v if and only if:

(Def. 23) W.first() = u and W.last() = v.

Let G be a graph and let W be a walk of G. We say that W is closed if and only if:

(Def. 24) W.first() = W.last().

We say that W is directed if and only if:

(Def. 25) For every odd natural number n such that $n <$ len W holds (the source of G)($W(n+1)$) = $W(n)$.

We say that W is trivial if and only if:

(Def. 26) W.length() = 0.

We say that W is trail-like if and only if:

(Def. 27) W.edgeSeq() is one-to-one.

Let G be a graph and let W be a walk of G. We introduce W is open as an antonym of W is closed.

Let G be a graph and let W be a walk of G. We say that W is path-like if and only if the conditions (Def. 28) are satisfied.

(Def. 28)(i) W is trail-like, and

(ii) for all odd natural numbers m, n such that $m < n$ and $n \leq \operatorname{len} W$ holds if $W(m) = W(n)$, then $m = 1$ and $n = \operatorname{len} W$.

Let G be a graph and let W be a walk of G. We say that W is vertex-distinct if and only if:

(Def. 29) For all odd natural numbers m, n such that $m \leq \operatorname{len} W$ and $n \leq \operatorname{len} W$ and $W(m) = W(n)$ holds $m = n$.

Let G be a graph and let W be a walk of G. We say that W is circuit-like if and only if:

(Def. 30) W is closed, trail-like, and non trivial.

We say that W is cycle-like if and only if:

(Def. 31) W is closed, path-like, and non trivial.

Let G be a graph. One can verify the following observations:

* every walk of G which is path-like is also trail-like,
* every walk of G which is trivial is also path-like,
* every walk of G which is trivial is also vertex-distinct,
* every walk of G which is vertex-distinct is also path-like,
* every walk of G which is circuit-like is also closed, trail-like, and non trivial, and
* every walk of G which is cycle-like is also closed, path-like, and non trivial.

Let G be a graph. Observe that there exists a walk of G which is closed, directed, and trivial.

Let G be a graph. Observe that there exists a walk of G which is vertex-distinct.

Let G be a graph. A trail of G is a trail-like walk of G. A path of G is a path-like walk of G.

Let G be a graph. A dwalk of G is a directed walk of G. A dtrail of G is a directed trail of G. A dpath of G is a directed path of G.

Let G be a graph and let v be a vertex of G. Note that $G.\text{walkOf}(v)$ is closed, directed, and trivial.

Let G be a graph and let x, e, y be sets. One can check that $G.\text{walkOf}(x, e, y)$ is path-like.

Let G be a graph and let x, e be sets. Note that $G.\text{walkOf}(x, e, x)$ is closed.

Let G be a graph and let W be a closed walk of G. One can check that $W.\text{reverse}()$ is closed.

Let G be a graph and let W be a trivial walk of G. One can verify that W.reverse() is trivial.

Let G be a graph and let W be a trail of G. Note that W.reverse() is trail-like.

Let G be a graph and let W be a path of G. Observe that W.reverse() is path-like.

Let G be a graph and let W_1, W_2 be closed walks of G. Note that W_1.append(W_2) is closed.

Let G be a graph and let W_1, W_2 be dwalks of G. One can verify that W_1.append(W_2) is directed.

Let G be a graph and let W_1, W_2 be trivial walks of G. Observe that W_1.append(W_2) is trivial.

Let G be a graph, let W be a dwalk of G, and let m, n be natural numbers. Note that W.cut(m, n) is directed.

Let G be a graph, let W be a trivial walk of G, and let m, n be natural numbers. Observe that W.cut(m, n) is trivial.

Let G be a graph, let W be a trail of G, and let m, n be natural numbers. Note that W.cut(m, n) is trail-like.

Let G be a graph, let W be a path of G, and let m, n be natural numbers. Note that W.cut(m, n) is path-like.

Let G be a graph, let W be a vertex-distinct walk of G, and let m, n be natural numbers. One can verify that W.cut(m, n) is vertex-distinct.

Let G be a graph, let W be a closed walk of G, and let m, n be natural numbers. One can verify that W.remove(m, n) is closed.

Let G be a graph, let W be a dwalk of G, and let m, n be natural numbers. Note that W.remove(m, n) is directed.

Let G be a graph, let W be a trivial walk of G, and let m, n be natural numbers. One can check that W.remove(m, n) is trivial.

Let G be a graph, let W be a trail of G, and let m, n be natural numbers. Observe that W.remove(m, n) is trail-like.

Let G be a graph, let W be a path of G, and let m, n be natural numbers. Observe that W.remove(m, n) is path-like.

Let G be a graph and let W be a walk of G. A walk of G is called a subwalk of W if:

(Def. 32) It is walk from W.first() to W.last() and there exists a FinSubsequence e_1 of W.edgeSeq() such that it.edgeSeq() $=$ Seq e_1.

Let G be a graph, let W be a walk of G, and let m, n be natural numbers. Then W.remove(m, n) is a subwalk of W.

Let G be a graph and let W be a walk of G. Note that there exists a subwalk of W which is trail-like and path-like.

Let G be a graph and let W be a walk of G. A trail of W is a trail-like subwalk of W. A path of W is a path-like subwalk of W.

Let G be a graph and let W be a dwalk of G. One can verify that there exists a path of W which is directed.

Let G be a graph and let W be a dwalk of G. A dwalk of W is a directed subwalk of W. A dtrail of W is a directed trail of W. A dpath of W is a directed path of W.

Let G be a graph. The functor G.allWalks() yields a non empty subset of ((the vertices of G) $\cup$ (the edges of G))* and is defined by:

(Def. 33) G.allWalks() $= \{W : W$ ranges over walks of $G\}$.

Let G be a graph. The functor G.allTrails() yielding a non empty subset of G.allWalks() is defined by:

(Def. 34) G.allTrails() $= \{W : W$ ranges over trails of $G\}$.

Let G be a graph. The functor G.allPaths() yields a non empty subset of G.allTrails() and is defined as follows:

(Def. 35) G.allPaths() $= \{W : W$ ranges over paths of $G\}$.

Let G be a graph. The functor G.allDWalks() yields a non empty subset of G.allWalks() and is defined by:

(Def. 36) G.allDWalks() $= \{W : W$ ranges over dwalks of $G\}$.

Let G be a graph. The functor G.allDTrails() yields a non empty subset of G.allTrails() and is defined as follows:

(Def. 37) G.allDTrails() $= \{W : W$ ranges over dtrails of $G\}$.

Let G be a graph. The functor G.allDPaths() yields a non empty subset of G.allDTrails() and is defined by:

(Def. 38) G.allDPaths() $= \{W : W$ ranges over directed paths of $G\}$.

Let G be a finite graph. One can check that G.allTrails() is finite.

Let G be a graph and let X be a non empty subset of G.allWalks(). We see that the element of X is a walk of G.

Let G be a graph and let X be a non empty subset of G.allTrails(). We see that the element of X is a trail of G.

Let G be a graph and let X be a non empty subset of G.allPaths(). We see that the element of X is a path of G.

Let G be a graph and let X be a non empty subset of G.allDWalks(). We see that the element of X is a dwalk of G.

Let G be a graph and let X be a non empty subset of G.allDTrails(). We see that the element of X is a dtrail of G.

Let G be a graph and let X be a non empty subset of G.allDPaths(). We see that the element of X is a dpath of G.

3. Walk Theorems

For simplicity, we adopt the following rules: G, G_1, G_2 are graphs, W, W_1, W_2 are walks of G, e, x, y, z are sets, v is a vertex of G, and n, m are natural numbers.

We now state a number of propositions:

(8)[3] For every odd natural number n such that $n \leq \operatorname{len} W$ holds $W(n) \in$ the vertices of G.

(9) For every even natural number n such that $n \in \operatorname{dom} W$ holds $W(n) \in$ the edges of G.

(10) Let n be an even natural number. Suppose $n \in \operatorname{dom} W$. Then there exists an odd natural number n_1 such that $n_1 = n - 1$ and $n - 1 \in \operatorname{dom} W$ and $n + 1 \in \operatorname{dom} W$ and $W(n)$ joins $W(n_1)$ and $W(n+1)$ in G.

(11) For every odd natural number n such that $n < \operatorname{len} W$ holds $W(n+1) \in (W.\mathrm{vertexAt}(n)).\mathrm{edgesInOut}()$.

(12) For every odd natural number n such that $1 < n$ and $n \leq \operatorname{len} W$ holds $W(n-1) \in (W.\mathrm{vertexAt}(n)).\mathrm{edgesInOut}()$.

(13) For every odd natural number n such that $n < \operatorname{len} W$ holds $n \in \operatorname{dom} W$ and $n + 1 \in \operatorname{dom} W$ and $n + 2 \in \operatorname{dom} W$.

(14) $\operatorname{len}(G.\mathrm{walkOf}(v)) = 1$ and $(G.\mathrm{walkOf}(v))(1) = v$ and $(G.\mathrm{walkOf}(v)).\mathrm{first}() = v$ and $(G.\mathrm{walkOf}(v)).\mathrm{last}() = v$ and $G.\mathrm{walkOf}(v)$ is walk from v to v.

(15) If e joins x and y in G, then $\operatorname{len}(G.\mathrm{walkOf}(x, e, y)) = 3$.

(16) If e joins x and y in G, then $(G.\mathrm{walkOf}(x, e, y)).\mathrm{first}() = x$ and $(G.\mathrm{walkOf}(x, e, y)).\mathrm{last}() = y$ and $G.\mathrm{walkOf}(x, e, y)$ is walk from x to y.

(17) For every walk W_1 of G_1 and for every walk W_2 of G_2 such that $W_1 = W_2$ holds $W_1.\mathrm{first}() = W_2.\mathrm{first}()$ and $W_1.\mathrm{last}() = W_2.\mathrm{last}()$.

(18) W is walk from x to y iff $W(1) = x$ and $W(\operatorname{len} W) = y$.

(19) If W is walk from x to y, then x is a vertex of G and y is a vertex of G.

(20) Let W_1 be a walk of G_1 and W_2 be a walk of G_2. If $W_1 = W_2$, then W_1 is walk from x to y iff W_2 is walk from x to y.

(21) For every walk W_1 of G_1 and for every walk W_2 of G_2 such that $W_1 = W_2$ and for every natural number n holds $W_1.\mathrm{vertexAt}(n) = W_2.\mathrm{vertexAt}(n)$.

(22) $\operatorname{len} W = \operatorname{len}(W.\mathrm{reverse}())$ and $\operatorname{dom} W = \operatorname{dom}(W.\mathrm{reverse}())$ and $\operatorname{rng} W = \operatorname{rng}(W.\mathrm{reverse}())$.

(23) $W.\mathrm{first}() = W.\mathrm{reverse}().\mathrm{last}()$ and $W.\mathrm{last}() = W.\mathrm{reverse}().\mathrm{first}()$.

(24) W is walk from x to y iff $W.\mathrm{reverse}()$ is walk from y to x.

[3] The proposition (7) has been removed.

(25) If $n \in \operatorname{dom} W$, then $W(n) = W$.reverse()$((\operatorname{len} W - n) + 1)$ and $(\operatorname{len} W - n) + 1 \in \operatorname{dom}(W$.reverse()$)$.

(26) If $n \in \operatorname{dom}(W$.reverse()$)$, then W.reverse()$(n) = W((\operatorname{len} W - n) + 1)$ and $(\operatorname{len} W - n) + 1 \in \operatorname{dom} W$.

(27) W.reverse().reverse() $= W$.

(28) For every walk W_1 of G_1 and for every walk W_2 of G_2 such that $W_1 = W_2$ holds W_1.reverse() $= W_2$.reverse().

(29) If W_1.last() $= W_2$.first(), then $\operatorname{len}(W_1$.append$(W_2)) + 1 = \operatorname{len} W_1 + \operatorname{len} W_2$.

(30) If W_1.last() $= W_2$.first(), then $\operatorname{len} W_1 \leq \operatorname{len}(W_1$.append$(W_2))$ and $\operatorname{len} W_2 \leq \operatorname{len}(W_1$.append$(W_2))$.

(31) If W_1.last() $= W_2$.first(), then $(W_1$.append$(W_2))$.first() $= W_1$.first() and $(W_1$.append$(W_2))$.last() $= W_2$.last() and W_1.append(W_2) is walk from W_1.first() to W_2.last().

(32) If W_1 is walk from x to y and W_2 is walk from y to z, then W_1.append(W_2) is walk from x to z.

(33) If $n \in \operatorname{dom} W_1$, then $(W_1$.append$(W_2))(n) = W_1(n)$ and $n \in \operatorname{dom}(W_1$.append$(W_2))$.

(34) If W_1.last() $= W_2$.first(), then for every natural number n such that $n < \operatorname{len} W_2$ holds $(W_1$.append$(W_2))(\operatorname{len} W_1 + n) = W_2(n+1)$ and $\operatorname{len} W_1 + n \in \operatorname{dom}(W_1$.append$(W_2))$.

(35) If $n \in \operatorname{dom}(W_1$.append$(W_2))$, then $n \in \operatorname{dom} W_1$ or there exists a natural number k such that $k < \operatorname{len} W_2$ and $n = \operatorname{len} W_1 + k$.

(36) For all walks W_3, W_4 of G_1 and for all walks W_5, W_6 of G_2 such that $W_3 = W_5$ and $W_4 = W_6$ holds W_3.append$(W_4) = W_5$.append(W_6).

(37) Let m, n be odd natural numbers. Suppose $m \leq n$ and $n \leq \operatorname{len} W$. Then $\operatorname{len}(W.\mathrm{cut}(m,n)) + m = n + 1$ and for every natural number i such that $i < \operatorname{len}(W.\mathrm{cut}(m,n))$ holds $(W.\mathrm{cut}(m,n))(i+1) = W(m+i)$ and $m + i \in \operatorname{dom} W$.

(38) Let m, n be odd natural numbers. Suppose $m \leq n$ and $n \leq \operatorname{len} W$. Then $(W.\mathrm{cut}(m,n))$.first() $= W(m)$ and $(W.\mathrm{cut}(m,n))$.last() $= W(n)$ and $W.\mathrm{cut}(m,n)$ is walk from $W(m)$ to $W(n)$.

(39) For all odd natural numbers m, n, o such that $m \leq n$ and $n \leq o$ and $o \leq \operatorname{len} W$ holds $(W.\mathrm{cut}(m,n))$.append$((W.\mathrm{cut}(n,o))) = W.\mathrm{cut}(m,o)$.

(40) $W.\mathrm{cut}(1, \operatorname{len} W) = W$.

(41) For every odd natural number n such that $n < \operatorname{len} W$ holds G.walkOf$(W(n), W(n+1), W(n+2)) = W.\mathrm{cut}(n, n+2)$.

(42) For all odd natural numbers m, n such that $m \leq n$ and $n < \operatorname{len} W$ holds $(W.\mathrm{cut}(m,n))$.addEdge$(W(n+1)) = W.\mathrm{cut}(m, n+2)$.

(43) For every odd natural number n such that $n \leq \operatorname{len} W$ holds $W.\mathrm{cut}(n,n) = \langle W.\mathrm{vertexAt}(n)\rangle$.

(44) If m is odd and $m \leq n$, then $W.\mathrm{cut}(1,n).\mathrm{cut}(1,m) = W.\mathrm{cut}(1,m)$.

(45) For all odd natural numbers m, n such that $m \leq n$ and $n \leq \operatorname{len} W_1$ and $W_1.\mathrm{last}() = W_2.\mathrm{first}()$ holds $(W_1.\mathrm{append}(W_2)).\mathrm{cut}(m,n) = W_1.\mathrm{cut}(m,n)$.

(46) For every odd natural number m such that $m \leq \operatorname{len} W$ holds $\operatorname{len}(W.\mathrm{cut}(1,m)) = m$.

(47) For every odd natural number m and for every natural number x such that $x \in \operatorname{dom}(W.\mathrm{cut}(1,m))$ and $m \leq \operatorname{len} W$ holds $(W.\mathrm{cut}(1,m))(x) = W(x)$.

(48) Let m, n be odd natural numbers and i be a natural number. If $m \leq n$ and $n \leq \operatorname{len} W$ and $i \in \operatorname{dom}(W.\mathrm{cut}(m,n))$, then $(W.\mathrm{cut}(m,n))(i) = W((m+i)-1)$ and $(m+i)-1 \in \operatorname{dom} W$.

(49) For every walk W_1 of G_1 and for every walk W_2 of G_2 and for all natural numbers m, n such that $W_1 = W_2$ holds $W_1.\mathrm{cut}(m,n) = W_2.\mathrm{cut}(m,n)$.

(50) For all odd natural numbers m, n such that $m \leq n$ and $n \leq \operatorname{len} W$ and $W(m) = W(n)$ holds $\operatorname{len}(W.\mathrm{remove}(m,n)) + n = \operatorname{len} W + m$.

(51) If W is walk from x to y, then $W.\mathrm{remove}(m,n)$ is walk from x to y.

(52) $\operatorname{len}(W.\mathrm{remove}(m,n)) \leq \operatorname{len} W$.

(53) $W.\mathrm{remove}(m,m) = W$.

(54) For all odd natural numbers m, n such that $m \leq n$ and $n \leq \operatorname{len} W$ and $W(m) = W(n)$ holds $(W.\mathrm{cut}(1,m)).\mathrm{last}() = (W.\mathrm{cut}(n,\operatorname{len} W)).\mathrm{first}()$.

(55) Let m, n be odd natural numbers. Suppose $m \leq n$ and $n \leq \operatorname{len} W$ and $W(m) = W(n)$. Let x be a natural number. If $x \in \operatorname{Seg} m$, then $(W.\mathrm{remove}(m,n))(x) = W(x)$.

(56) Let m, n be odd natural numbers. Suppose $m \leq n$ and $n \leq \operatorname{len} W$ and $W(m) = W(n)$. Let x be a natural number. Suppose $m \leq x$ and $x \leq \operatorname{len}(W.\mathrm{remove}(m,n))$. Then $(W.\mathrm{remove}(m,n))(x) = W((x-m)+n)$ and $(x-m)+n$ is a natural number and $(x-m)+n \leq \operatorname{len} W$.

(57) For all odd natural numbers m, n such that $m \leq n$ and $n \leq \operatorname{len} W$ and $W(m) = W(n)$ holds $\operatorname{len}(W.\mathrm{remove}(m,n)) = (\operatorname{len} W + m) - n$.

(58) For every natural number m such that $W(m) = W.\mathrm{last}()$ holds $W.\mathrm{remove}(m,\operatorname{len} W) = W.\mathrm{cut}(1,m)$.

(59) For every natural number m such that $W.\mathrm{first}() = W(m)$ holds $W.\mathrm{remove}(1,m) = W.\mathrm{cut}(m,\operatorname{len} W)$.

(60) $(W.\mathrm{remove}(m,n)).\mathrm{first}() = W.\mathrm{first}()$ and $(W.\mathrm{remove}(m,n)).\mathrm{last}() = W.\mathrm{last}()$.

(61) Let m, n be odd natural numbers and x be a natural number. Suppose $m \leq n$ and $n \leq \operatorname{len} W$ and $W(m) = W(n)$ and $x \in \operatorname{dom}(W.\mathrm{remove}(m,n))$.

Then $x \in \operatorname{Seg} m$ or $m \leq x$ and $x \leq \operatorname{len}(W.\mathrm{remove}(m,n))$.

(62) For every walk W_1 of G_1 and for every walk W_2 of G_2 and for all natural numbers m, n such that $W_1 = W_2$ holds $W_1.\mathrm{remove}(m,n) = W_2.\mathrm{remove}(m,n)$.

(63) If e joins $W.\mathrm{last}()$ and x in G, then $W.\mathrm{addEdge}(e) = W \frown \langle e, x\rangle$.

(64) If e joins $W.\mathrm{last}()$ and x in G, then $(W.\mathrm{addEdge}(e)).\mathrm{first}() = W.\mathrm{first}()$ and $(W.\mathrm{addEdge}(e)).\mathrm{last}() = x$ and $W.\mathrm{addEdge}(e)$ is walk from $W.\mathrm{first}()$ to x.

(65) If e joins $W.\mathrm{last}()$ and x in G, then $\operatorname{len}(W.\mathrm{addEdge}(e)) = \operatorname{len} W + 2$.

(66) Suppose e joins $W.\mathrm{last}()$ and x in G. Then $(W.\mathrm{addEdge}(e))(\operatorname{len} W+1) = e$ and $(W.\mathrm{addEdge}(e))(\operatorname{len} W + 2) = x$ and for every natural number n such that $n \in \operatorname{dom} W$ holds $(W.\mathrm{addEdge}(e))(n) = W(n)$.

(67) If W is walk from x to y and e joins y and z in G, then $W.\mathrm{addEdge}(e)$ is walk from x to z.

(68) $1 \leq \operatorname{len}(W.\mathrm{vertexSeq}())$.

(69) For every odd natural number n such that $n \leq \operatorname{len} W$ holds $2 \cdot ((n+1) \div 2) - 1 = n$ and $1 \leq (n + 1) \div 2$ and $(n + 1) \div 2 \leq \operatorname{len}(W.\mathrm{vertexSeq}())$.

(70) $(G.\mathrm{walkOf}(v)).\mathrm{vertexSeq}() = \langle v\rangle$.

(71) If e joins x and y in G, then $(G.\mathrm{walkOf}(x, e, y)).\mathrm{vertexSeq}() = \langle x, y\rangle$.

(72) $W.\mathrm{first}() = W.\mathrm{vertexSeq}()(1)$ and $W.\mathrm{last}() = W.\mathrm{vertexSeq}()(\operatorname{len}(W.\mathrm{vertexSeq}()))$.

(73) For every odd natural number n such that $n \leq \operatorname{len} W$ holds $W.\mathrm{vertexAt}(n) = W.\mathrm{vertexSeq}()((n + 1) \div 2)$.

(74) $n \in \operatorname{dom}(W.\mathrm{vertexSeq}())$ iff $2 \cdot n - 1 \in \operatorname{dom} W$.

(75) $(W.\mathrm{cut}(1, n)).\mathrm{vertexSeq}() \subseteq W.\mathrm{vertexSeq}()$.

(76) If e joins $W.\mathrm{last}()$ and x in G, then $(W.\mathrm{addEdge}(e)).\mathrm{vertexSeq}() = W.\mathrm{vertexSeq}() \frown \langle x\rangle$.

(77) For every walk W_1 of G_1 and for every walk W_2 of G_2 such that $W_1 = W_2$ holds $W_1.\mathrm{vertexSeq}() = W_2.\mathrm{vertexSeq}()$.

(78) For every even natural number n such that $1 \leq n$ and $n \leq \operatorname{len} W$ holds $n \div 2 \in \operatorname{dom}(W.\mathrm{edgeSeq}())$ and $W(n) = W.\mathrm{edgeSeq}()(n \div 2)$.

(79) $n \in \operatorname{dom}(W.\mathrm{edgeSeq}())$ iff $2 \cdot n \in \operatorname{dom} W$.

(80) For every natural number n such that $n \in \operatorname{dom}(W.\mathrm{edgeSeq}())$ holds $W.\mathrm{edgeSeq}()(n) \in$ the edges of G.

(81) There exists an even natural number l_1 such that $l_1 = \operatorname{len} W - 1$ and $\operatorname{len}(W.\mathrm{edgeSeq}()) = l_1 \div 2$.

(82) $(W.\mathrm{cut}(1, n)).\mathrm{edgeSeq}() \subseteq W.\mathrm{edgeSeq}()$.

(83) If e joins $W.\mathrm{last}()$ and x in G, then $(W.\mathrm{addEdge}(e)).\mathrm{edgeSeq}() = W.\mathrm{edgeSeq}() \frown \langle e\rangle$.

(84) e joins x and y in G iff $(G.\mathrm{walkOf}(x, e, y)).\mathrm{edgeSeq}() = \langle e \rangle$.

(85) $W.\mathrm{reverse}().\mathrm{edgeSeq}() = \mathrm{Rev}(W.\mathrm{edgeSeq}())$.

(86) If $W_1.\mathrm{last}() = W_2.\mathrm{first}()$, then $(W_1.\mathrm{append}(W_2)).\mathrm{edgeSeq}() = W_1.\mathrm{edgeSeq}() \frown W_2.\mathrm{edgeSeq}()$.

(87) For every walk W_1 of G_1 and for every walk W_2 of G_2 such that $W_1 = W_2$ holds $W_1.\mathrm{edgeSeq}() = W_2.\mathrm{edgeSeq}()$.

(88) $x \in W.\mathrm{vertices}()$ iff there exists an odd natural number n such that $n \leq \mathrm{len}\, W$ and $W(n) = x$.

(89) $W.\mathrm{first}() \in W.\mathrm{vertices}()$ and $W.\mathrm{last}() \in W.\mathrm{vertices}()$.

(90) For every odd natural number n such that $n \leq \mathrm{len}\, W$ holds $W.\mathrm{vertexAt}(n) \in W.\mathrm{vertices}()$.

(91) $(G.\mathrm{walkOf}(v)).\mathrm{vertices}() = \{v\}$.

(92) If e joins x and y in G, then $(G.\mathrm{walkOf}(x, e, y)).\mathrm{vertices}() = \{x, y\}$.

(93) $W.\mathrm{vertices}() = W.\mathrm{reverse}().\mathrm{vertices}()$.

(94) If $W_1.\mathrm{last}() = W_2.\mathrm{first}()$, then $(W_1.\mathrm{append}(W_2)).\mathrm{vertices}() = W_1.\mathrm{vertices}() \cup W_2.\mathrm{vertices}()$.

(95) For all odd natural numbers m, n such that $m \leq n$ and $n \leq \mathrm{len}\, W$ holds $(W.\mathrm{cut}(m, n)).\mathrm{vertices}() \subseteq W.\mathrm{vertices}()$.

(96) If e joins $W.\mathrm{last}()$ and x in G, then $(W.\mathrm{addEdge}(e)).\mathrm{vertices}() = W.\mathrm{vertices}() \cup \{x\}$.

(97) Let G be a finite graph, W be a walk of G, and e, x be sets. If e joins $W.\mathrm{last}()$ and x in G and $x \notin W.\mathrm{vertices}()$, then $\mathrm{card}((W.\mathrm{addEdge}(e)).\mathrm{vertices}()) = \mathrm{card}(W.\mathrm{vertices}()) + 1$.

(98) If $x \in W.\mathrm{vertices}()$ and $y \in W.\mathrm{vertices}()$, then there exists a walk of G which is walk from x to y.

(99) For every walk W_1 of G_1 and for every walk W_2 of G_2 such that $W_1 = W_2$ holds $W_1.\mathrm{vertices}() = W_2.\mathrm{vertices}()$.

(100) $e \in W.\mathrm{edges}()$ iff there exists an even natural number n such that $1 \leq n$ and $n \leq \mathrm{len}\, W$ and $W(n) = e$.

(101) $e \in W.\mathrm{edges}()$ iff there exists an odd natural number n such that $n < \mathrm{len}\, W$ and $W(n + 1) = e$.

(102) $\mathrm{rng}\, W = W.\mathrm{vertices}() \cup W.\mathrm{edges}()$.

(103) If $W_1.\mathrm{last}() = W_2.\mathrm{first}()$, then $(W_1.\mathrm{append}(W_2)).\mathrm{edges}() = W_1.\mathrm{edges}() \cup W_2.\mathrm{edges}()$.

(104) Suppose $e \in W.\mathrm{edges}()$. Then there exist vertices v_2, v_3 of G and there exists an odd natural number n such that $n + 2 \leq \mathrm{len}\, W$ and $v_2 = W(n)$ and $e = W(n + 1)$ and $v_3 = W(n + 2)$ and e joins v_2 and v_3 in G.

(105) $e \in W.\mathrm{edges}()$ iff there exists a natural number n such that $n \in \mathrm{dom}(W.\mathrm{edgeSeq}())$ and $W.\mathrm{edgeSeq}()(n) = e$.

(106) If $e \in W$.edges() and e joins x and y in G, then $x \in W$.vertices() and $y \in W$.vertices().

(107) $(W$.cut$(m, n))$.edges() $\subseteq W$.edges().

(108) W.edges() $= W$.reverse().edges().

(109) e joins x and y in G iff $(G$.walkOf$(x, e, y))$.edges() $= \{e\}$.

(110) W.edges() $\subseteq G$.edgesBetween$(W$.vertices()).

(111) For every walk W_1 of G_1 and for every walk W_2 of G_2 such that $W_1 = W_2$ holds W_1.edges() $= W_2$.edges().

(112) If e joins W.last() and x in G, then $(W$.addEdge$(e))$.edges() $= W$.edges() $\cup \{e\}$.

(113) $\operatorname{len} W = 2 \cdot W$.length() $+ 1$.

(114) $\operatorname{len} W_1 = \operatorname{len} W_2$ iff W_1.length() $= W_2$.length().

(115) For every walk W_1 of G_1 and for every walk W_2 of G_2 such that $W_1 = W_2$ holds W_1.length() $= W_2$.length().

(116) For every odd natural number n such that $n \leq \operatorname{len} W$ holds W.find$(W(n)) \leq n$ and W.rfind$(W(n)) \geq n$.

(117) For every walk W_1 of G_1 and for every walk W_2 of G_2 and for every set v such that $W_1 = W_2$ holds W_1.find$(v) = W_2$.find(v) and W_1.rfind$(v) = W_2$.rfind(v).

(118) For every odd natural number n such that $n \leq \operatorname{len} W$ holds W.find$(n) \leq n$ and W.rfind$(n) \geq n$.

(119) W is closed iff $W(1) = W(\operatorname{len} W)$.

(120) W is closed iff there exists a set x such that W is walk from x to x.

(121) W is closed iff W.reverse() is closed.

(122) For every walk W_1 of G_1 and for every walk W_2 of G_2 such that $W_1 = W_2$ and W_1 is closed holds W_2 is closed.

(123) W is directed if and only if for every odd natural number n such that $n < \operatorname{len} W$ holds $W(n+1)$ joins $W(n)$ to $W(n+2)$ in G.

(124) Suppose W is directed and walk from x to y and e joins y to z in G. Then W.addEdge(e) is directed and W.addEdge(e) is walk from x to z.

(125) For every dwalk W of G and for all natural numbers m, n holds W.cut(m, n) is directed.

(126) W is non trivial iff $3 \leq \operatorname{len} W$.

(127) W is non trivial iff $\operatorname{len} W \neq 1$.

(128) If W.first() $\neq W$.last(), then W is non trivial.

(129) W is trivial iff there exists a vertex v of G such that $W = G$.walkOf(v).

(130) W is trivial iff W.reverse() is trivial.

(131) If W_2 is trivial, then W_1.append$(W_2) = W_1$.

(132) For all odd natural numbers m, n such that $m \leq n$ and $n \leq \operatorname{len} W$ holds W.cut(m, n) is trivial iff $m = n$.

(133) If e joins W.last() and x in G, then W.addEdge(e) is non trivial.

(134) If W is non trivial, then there exists an odd natural number l_2 such that $l_2 = \operatorname{len} W - 2$ and $(W.\text{cut}(1, l_2))$.addEdge$(W(l_2 + 1)) = W$.

(135) If W_2 is non trivial and W_2.edges() $\subseteq W_1$.edges(), then W_2.vertices() $\subseteq W_1$.vertices().

(136) If W is non trivial, then for every vertex v of G such that $v \in W$.vertices() holds v is not isolated.

(137) W is trivial iff W.edges() $= \emptyset$.

(138) For every walk W_1 of G_1 and for every walk W_2 of G_2 such that $W_1 = W_2$ and W_1 is trivial holds W_2 is trivial.

(139) W is trail-like iff for all even natural numbers m, n such that $1 \leq m$ and $m < n$ and $n \leq \operatorname{len} W$ holds $W(m) \neq W(n)$.

(140) If $\operatorname{len} W \leq 3$, then W is trail-like.

(141) W is trail-like iff W.reverse() is trail-like.

(142) For every trail W of G and for all natural numbers m, n holds W.cut(m, n) is trail-like.

(143) For every trail W of G and for every set e such that $e \in W$.last().edgesInOut() and $e \notin W$.edges() holds W.addEdge(e) is trail-like.

(144) For every trail W of G and for every vertex v of G such that $v \in W$.vertices() and v is endvertex holds $v = W$.first() or $v = W$.last().

(145) For every finite graph G and for every trail W of G holds len$(W$.edgeSeq()$) \leq G$.size().

(146) If $\operatorname{len} W \leq 3$, then W is path-like.

(147) If for all odd natural numbers m, n such that $m \leq \operatorname{len} W$ and $n \leq \operatorname{len} W$ and $W(m) = W(n)$ holds $m = n$, then W is path-like.

(148) Let W be a path of G. Suppose W is open. Let m, n be odd natural numbers. If $m < n$ and $n \leq \operatorname{len} W$, then $W(m) \neq W(n)$.

(149) W is path-like iff W.reverse() is path-like.

(150) For every path W of G and for all natural numbers m, n holds W.cut(m, n) is path-like.

(151) Let W be a path of G and e, v be sets. Suppose that
 (i) e joins W.last() and v in G,
 (ii) $e \notin W$.edges(),
 (iii) W is trivial or open, and
 (iv) for every odd natural number n such that $1 < n$ and $n \leq \operatorname{len} W$ holds $W(n) \neq v$.

Then W.addEdge(e) is path-like.

(152) Let W be a path of G and e, v be sets. Suppose e joins W.last() and v in G and $v \notin W$.vertices() and W is trivial or open. Then W.addEdge(e) is path-like.

(153) If for every odd natural number n such that $n \leq \operatorname{len} W$ holds W.find($W(n)$) $=$ W.rfind($W(n)$), then W is path-like.

(154) If for every odd natural number n such that $n \leq \operatorname{len} W$ holds W.rfind(n) $= n$, then W is path-like.

(155) For every finite graph G and for every path W of G holds len(W.vertexSeq()) $\leq G$.order() $+ 1$.

(156) Let G be a graph, W be a vertex-distinct walk of G, and e, v be sets. If e joins W.last() and v in G and $v \notin W$.vertices(), then W.addEdge(e) is vertex-distinct.

(157) If e joins x and x in G, then G.walkOf(x, e, x) is cycle-like.

(158) Suppose e joins x and y in G and $e \in W_1$.edges() and W_1 is cycle-like. Then there exists a walk W_2 of G such that W_2 is walk from x to y and $e \notin W_2$.edges().

(159) W is a subwalk of W.

(160) For every walk W_1 of G and for every subwalk W_2 of W_1 holds every subwalk of W_2 is a subwalk of W_1.

(161) If W_1 is a subwalk of W_2, then W_1 is walk from x to y iff W_2 is walk from x to y.

(162) If W_1 is a subwalk of W_2, then W_1.first() $=$ W_2.first() and W_1.last() $=$ W_2.last().

(163) If W_1 is a subwalk of W_2, then $\operatorname{len} W_1 \leq \operatorname{len} W_2$.

(164) If W_1 is a subwalk of W_2, then W_1.edges() $\subseteq$ W_2.edges() and W_1.vertices() $\subseteq$ W_2.vertices().

(165) Suppose W_1 is a subwalk of W_2. Let m be an odd natural number. Suppose $m \leq \operatorname{len} W_1$. Then there exists an odd natural number n such that $m \leq n$ and $n \leq \operatorname{len} W_2$ and $W_1(m) = W_2(n)$.

(166) Suppose W_1 is a subwalk of W_2. Let m be an even natural number. Suppose $1 \leq m$ and $m \leq \operatorname{len} W_1$. Then there exists an even natural number n such that $m \leq n$ and $n \leq \operatorname{len} W_2$ and $W_1(m) = W_2(n)$.

(167) For every trail W_1 of G such that W_1 is non trivial holds there exists a path of W_1 which is non trivial.

(168) For every graph G_1 and for every subgraph G_2 of G_1 holds every walk of G_2 is a walk of G_1.

(169) Let G_1 be a graph, G_2 be a subgraph of G_1, and W be a walk of G_1. If W is trivial and W.first() $\in$ the vertices of G_2, then W is a walk of G_2.

(170) Let G_1 be a graph, G_2 be a subgraph of G_1, and W be a walk of G_1. If W is non trivial and W.edges() $\subseteq$ the edges of G_2, then W is a walk of G_2.

(171) Let G_1 be a graph, G_2 be a subgraph of G_1, and W be a walk of G_1. Suppose W.vertices() $\subseteq$ the vertices of G_2 and W.edges() $\subseteq$ the edges of G_2. Then W is a walk of G_2.

(172) Let G_1 be a non trivial graph, W be a walk of G_1, v be a vertex of G_1, and G_2 be a subgraph of G_1 with vertex v removed. If $v \notin W$.vertices(), then W is a walk of G_2.

(173) Let G_1 be a graph, W be a walk of G_1, e be a set, and G_2 be a subgraph of G_1 with edge e removed. If $e \notin W$.edges(), then W is a walk of G_2.

(174) Let G_1 be a graph, G_2 be a subgraph of G_1, and x, y, e be sets. If e joins x and y in G_2, then G_1.walkOf$(x, e, y) = G_2$.walkOf(x, e, y).

(175) Let G_1 be a graph, G_2 be a subgraph of G_1, W_1 be a walk of G_1, W_2 be a walk of G_2, and e be a set. If $W_1 = W_2$ and $e \in W_2$.last().edgesInOut(), then W_1.addEdge$(e) = W_2$.addEdge(e).

(176) Let G_1 be a graph, G_2 be a subgraph of G_1, and W be a walk of G_2. Then

(i) if W is closed, then W is a closed walk of G_1,
(ii) if W is directed, then W is a directed walk of G_1,
(iii) if W is trivial, then W is a trivial walk of G_1,
(iv) if W is trail-like, then W is a trail-like walk of G_1,
(v) if W is path-like, then W is a path-like walk of G_1, and
(vi) if W is vertex-distinct, then W is a vertex-distinct walk of G_1.

(177) Let G_1 be a graph, G_2 be a subgraph of G_1, W_1 be a walk of G_1, and W_2 be a walk of G_2 such that $W_1 = W_2$. Then

(i) W_1 is closed iff W_2 is closed,
(ii) W_1 is directed iff W_2 is directed,
(iii) W_1 is trivial iff W_2 is trivial,
(iv) W_1 is trail-like iff W_2 is trail-like,
(v) W_1 is path-like iff W_2 is path-like, and
(vi) W_1 is vertex-distinct iff W_2 is vertex-distinct.

(178) If $G_1 =_G G_2$ and x is a vertex sequence of G_1, then x is a vertex sequence of G_2.

(179) If $G_1 =_G G_2$ and x is a edge sequence of G_1, then x is a edge sequence of G_2.

(180) If $G_1 =_G G_2$ and x is a walk of G_1, then x is a walk of G_2.

(181) If $G_1 =_G G_2$, then G_1.walkOf$(x, e, y) = G_2$.walkOf(x, e, y).

(182) Let W_1 be a walk of G_1 and W_2 be a walk of G_2 such that $G_1 =_G G_2$ and $W_1 = W_2$. Then

(i) W_1 is closed iff W_2 is closed,
(ii) W_1 is directed iff W_2 is directed,
(iii) W_1 is trivial iff W_2 is trivial,
(iv) W_1 is trail-like iff W_2 is trail-like,
(v) W_1 is path-like iff W_2 is path-like, and
(vi) W_1 is vertex-distinct iff W_2 is vertex-distinct.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Some properties of restrictions of finite sequences. *Formalized Mathematics*, 5(**2**):241–245, 1996.
[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[9] Gilbert Lee and Piotr Rudnicki. Alternative graph structures. *Formalized Mathematics*, 13(**2**):235–252, 2005.
[10] Yatsuka Nakamura and Piotr Rudnicki. Vertex sequences induced by chains. *Formalized Mathematics*, 5(**3**):297–304, 1996.
[11] Piotr Rudnicki and Andrzej Trybulec. Abian's fixed point theorem. *Formalized Mathematics*, 6(**3**):335–338, 1997.
[12] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[13] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[15] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[17] Josef Urban. Basic facts about inaccessible and measurable cardinals. *Formalized Mathematics*, 9(**2**):323–329, 2001.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[19] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 22, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 271–277
University of Białystok, 2005

Trees and Graph Components[1]

Gilbert Lee[2]
University of Victoria, Victoria, Canada

Summary. In the graph framework of [11] we define connected and acyclic graphs, components of a graph, and define the notion of cut-vertex (articulation point).

MML identifier: GLIB_002, version: 7.5.01 4.39.921

The articles [15], [8], [14], [17], [12], [18], [6], [1], [16], [7], [3], [4], [5], [9], [2], [11], [10], and [13] provide the terminology and notation for this paper.

1. Preliminaries

Let X be a finite set. Observe that 2^X is finite.

The following proposition is true

(1) For every finite set X such that $1 < \operatorname{card} X$ there exist sets x_1, x_2 such that $x_1 \in X$ and $x_2 \in X$ and $x_1 \neq x_2$.

2. Definitions

Let G be a graph. We say that G is connected if and only if:

(Def. 1) For all vertices u, v of G holds there exists a walk of G which is walk from u to v.

Let G be a graph. We say that G is acyclic if and only if:

(Def. 2) There exists no walk of G which is cycle-like.

Let G be a graph. We say that G is tree-like if and only if:

[1]This work has been partially supported by NSERC, Alberta Ingenuity Fund and iCORE.
[2]Part of author's MSc work.

ISSN 1426–2630

(Def. 3) G is acyclic and connected.

One can verify that every graph which is trivial is also connected.

Let us note that every graph which is trivial and loopless is also tree-like.

Let us note that every graph which is acyclic is also simple.

Let us observe that every graph which is tree-like is also acyclic and connected.

Let us observe that every graph which is acyclic and connected is also tree-like.

Let G be a graph and let v be a vertex of G. Observe that every subgraph of G induced by $\{v\}$ and $\emptyset$ is tree-like.

Let G be a graph and let v be a set. We say that G is dtree rooted at v if and only if:

(Def. 4) G is tree-like and for every vertex x of G holds there exists a dwalk of G which is walk from v to x.

Let us observe that there exists a graph which is trivial, finite, and tree-like and there exists a graph which is non trivial, finite, and tree-like.

Let G be a graph. Note that there exists a subgraph of G which is trivial, finite, and tree-like.

Let G be an acyclic graph. Observe that every subgraph of G is acyclic.

Let G be a graph and let v be a vertex of G. The functor G.reachableFrom(v) yields a non empty subset of the vertices of G and is defined as follows:

(Def. 5) For every set x holds $x \in G$.reachableFrom(v) iff there exists a walk of G which is walk from v to x.

Let G be a graph and let v be a vertex of G. The functor G.reachableDFrom(v) yielding a non empty subset of the vertices of G is defined by:

(Def. 6) For every set x holds $x \in G$.reachableDFrom(v) iff there exists a dwalk of G which is walk from v to x.

Let G_1 be a graph and let G_2 be a subgraph of G_1. We say that G_2 is component-like if and only if:

(Def. 7) G_2 is connected and it is not true that there exists a connected subgraph G_3 of G_1 such that $G_2 \subset G_3$.

Let G be a graph. Note that every subgraph of G which is component-like is also connected.

Let G be a graph and let v be a vertex of G. Note that every subgraph of G induced by G.reachableFrom(v) is component-like.

Let G be a graph. Observe that there exists a subgraph of G which is component-like.

Let G be a graph. A component of G is a component-like subgraph of G.

Let G be a graph. The functor G.componentSet() yielding a non empty family of subsets of the vertices of G is defined as follows:

(Def. 8) For every set x holds $x \in G$.componentSet() iff there exists a vertex v of G such that $x = G$.reachableFrom(v).

Let G be a graph and let X be an element of G.componentSet(). Observe that every subgraph of G induced by X is component-like.

Let G be a graph. The functor G.numComponents() yielding a cardinal number is defined by:

(Def. 9) $G.\mathrm{numComponents}() = \overline{\overline{G.\mathrm{componentSet}()}}$.

Let G be a finite graph. Then G.numComponents() is a non empty natural number.

Let G be a graph and let v be a vertex of G. We say that v is cut-vertex if and only if:

(Def. 10) For every subgraph G_2 of G with vertex v removed holds $G.\mathrm{numComponents}() < G_2.\mathrm{numComponents}()$.

Let G be a finite graph and let v be a vertex of G. Let us observe that v is cut-vertex if and only if:

(Def. 11) For every subgraph G_2 of G with vertex v removed holds $G.\mathrm{numComponents}() < G_2.\mathrm{numComponents}()$.

Let G be a non trivial finite connected graph. Observe that there exists a vertex of G which is non cut-vertex.

Let G be a non trivial finite tree-like graph. One can check that there exists a vertex of G which is endvertex.

Let G be a non trivial finite tree-like graph and let v be an endvertex vertex of G. Observe that every subgraph of G with vertex v removed is tree-like.

Let G_4 be a graph sequence. We say that G_4 is connected if and only if:

(Def. 12) For every natural number n holds $G_4 \rightarrow n$ is connected.

We say that G_4 is acyclic if and only if:

(Def. 13) For every natural number n holds $G_4 \rightarrow n$ is acyclic.

We say that G_4 is tree-like if and only if:

(Def. 14) For every natural number n holds $G_4 \rightarrow n$ is tree-like.

One can check the following observations:

* every graph sequence which is trivial is also connected,
* every graph sequence which is trivial and loopless is also tree-like,
* every graph sequence which is acyclic is also simple,
* every graph sequence which is tree-like is also acyclic and connected, and
* every graph sequence which is acyclic and connected is also tree-like.

Let us note that there exists a graph sequence which is halting, finite, and tree-like.

Let G_4 be a connected graph sequence and let n be a natural number. Note that $G_4 \rightarrow n$ is connected.

Let G_4 be an acyclic graph sequence and let n be a natural number. Observe that $G_4{\rightarrow}n$ is acyclic.

Let G_4 be a tree-like graph sequence and let n be a natural number. Note that $G_4{\rightarrow}n$ is tree-like.

3. Theorems

For simplicity, we use the following convention: G, G_1, G_2 are graphs, e, x, y are sets, v, v_1, v_2 are vertices of G, and W is a walk of G.

We now state a number of propositions:

(2) For every non trivial connected graph G and for every vertex v of G holds v is not isolated.

(3) Let G_1 be a non trivial graph, v be a vertex of G_1, and G_2 be a subgraph of G_1 with vertex v removed. Suppose G_2 is connected and there exists a set e such that $e \in v$.edgesInOut() and e does not join v and v in G_1. Then G_1 is connected.

(4) Let G_1 be a non trivial connected graph, v be a vertex of G_1, and G_2 be a subgraph of G_1 with vertex v removed. If v is endvertex, then G_2 is connected.

(5) Let G_1 be a connected graph, W be a walk of G_1, e be a set, and G_2 be a subgraph of G_1 with edge e removed. If W is cycle-like and $e \in W$.edges(), then G_2 is connected.

(6) If there exists a vertex v_1 of G such that for every vertex v_2 of G holds there exists a walk of G which is walk from v_1 to v_2, then G is connected.

(7) Every trivial graph is connected.

(8) If $G_1 =_G G_2$ and G_1 is connected, then G_2 is connected.

(9) $v \in G$.reachableFrom(v).

(10) If $x \in G$.reachableFrom(v_1) and e joins x and y in G, then $y \in G$.reachableFrom(v_1).

(11) G.edgesBetween(G.reachableFrom(v)) = G.edgesInOut(G.reachableFrom(v)).

(12) If $v_1 \in G$.reachableFrom(v_2), then G.reachableFrom(v_1) = G.reachableFrom(v_2).

(13) If $v \in W$.vertices(), then W.vertices() $\subseteq G$.reachableFrom(v).

(14) Let G_1 be a graph, G_2 be a subgraph of G_1, v_1 be a vertex of G_1, and v_2 be a vertex of G_2. If $v_1 = v_2$, then G_2.reachableFrom(v_2) $\subseteq$ G_1.reachableFrom(v_1).

(15) If there exists a vertex v of G such that G.reachableFrom(v) = the vertices of G, then G is connected.

(16) If G is connected, then for every vertex v of G holds G.reachableFrom(v) = the vertices of G.

(17) For every vertex v_1 of G_1 and for every vertex v_2 of G_2 such that $G_1 =_G G_2$ and $v_1 = v_2$ holds G_1.reachableFrom(v_1) = G_2.reachableFrom(v_2).

(18) $v \in G$.reachableDFrom(v).

(19) If $x \in G$.reachableDFrom(v_1) and e joins x to y in G, then $y \in G$.reachableDFrom(v_1).

(20) G.reachableDFrom(v) $\subseteq$ G.reachableFrom(v).

(21) Let G_1 be a graph, G_2 be a subgraph of G_1, v_1 be a vertex of G_1, and v_2 be a vertex of G_2. If $v_1 = v_2$, then G_2.reachableDFrom(v_2) $\subseteq$ G_1.reachableDFrom(v_1).

(22) For every vertex v_1 of G_1 and for every vertex v_2 of G_2 such that $G_1 =_G G_2$ and $v_1 = v_2$ holds G_1.reachableDFrom(v_1) = G_2.reachableDFrom(v_2).

(23) For every graph G_1 and for every connected subgraph G_2 of G_1 such that G_2 is spanning holds G_1 is connected.

(24) $\bigcup$(G.componentSet()) = the vertices of G.

(25) G is connected iff G.componentSet() = {the vertices of G}.

(26) If $G_1 =_G G_2$, then G_1.componentSet() = G_2.componentSet().

(27) If $x \in G$.componentSet(), then x is a non empty subset of the vertices of G.

(28) G is connected iff G.numComponents() = 1.

(29) If $G_1 =_G G_2$, then G_1.numComponents() = G_2.numComponents().

(30) G is a component of G iff G is connected.

(31) For every component C of G holds the edges of C = G.edgesBetween(the vertices of C).

(32) For all components C_1, C_2 of G holds the vertices of C_1 = the vertices of C_2 iff $C_1 =_G C_2$.

(33) Let C be a component of G and v be a vertex of G. Then $v \in$ the vertices of C if and only if the vertices of C = G.reachableFrom(v).

(34) Let C_1, C_2 be components of G and v be a set. If $v \in$ the vertices of C_1 and $v \in$ the vertices of C_2, then $C_1 =_G C_2$.

(35) Let G be a connected graph and v be a vertex of G. Then v is non cut-vertex if and only if for every subgraph G_2 of G with vertex v removed holds G_2.numComponents() $\leq$ G.numComponents().

(36) Let G be a connected graph, v be a vertex of G, and G_2 be a subgraph of G with vertex v removed. If v is not cut-vertex, then G_2 is connected.

(37) Let G be a non trivial finite connected graph. Then there exist vertices v_1, v_2 of G such that $v_1 \neq v_2$ and v_1 is not cut-vertex and v_2 is not cut-vertex.

(38) If v is cut-vertex, then G is non trivial.

(39) Let v_1 be a vertex of G_1 and v_2 be a vertex of G_2. If $G_1 =_G G_2$ and $v_1 = v_2$, then if v_1 is cut-vertex, then v_2 is cut-vertex.

(40) For every finite connected graph G holds G.order() $\leq G$.size() $+ 1$.

(41) Every acyclic graph is simple.

(42) Let G be an acyclic graph, W be a path of G, and e be a set. If $e \notin W$.edges() and $e \in W$.last().edgesInOut(), then W.addEdge(e) is path-like.

(43) Let G be a non trivial finite acyclic graph. Suppose the edges of $G \neq \emptyset$. Then there exist vertices v_1, v_2 of G such that $v_1 \neq v_2$ and v_1 is endvertex and v_2 is endvertex and $v_2 \in G$.reachableFrom(v_1).

(44) If $G_1 =_G G_2$ and G_1 is acyclic, then G_2 is acyclic.

(45) Let G be a non trivial finite tree-like graph. Then there exist vertices v_1, v_2 of G such that $v_1 \neq v_2$ and v_1 is endvertex and v_2 is endvertex.

(46) For every finite graph G holds G is tree-like iff G is acyclic and G.order() $= G$.size() $+ 1$.

(47) For every finite graph G holds G is tree-like iff G is connected and G.order() $= G$.size() $+ 1$.

(48) If $G_1 =_G G_2$ and G_1 is tree-like, then G_2 is tree-like.

(49) If G is dtree rooted at x, then x is a vertex of G.

(50) If $G_1 =_G G_2$ and G_1 is dtree rooted at x, then G_2 is dtree rooted at x.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.
[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[10] Gilbert Lee. Walks in Graphs. *Formalized Mathematics*, 13(**2**):253–269, 2005.
[11] Gilbert Lee and Piotr Rudnicki. Alternative graph structures. *Formalized Mathematics*, 13(**2**):235–252, 2005.
[12] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[13] Piotr Rudnicki and Andrzej Trybulec. Abian's fixed point theorem. *Formalized Mathematics*, 6(**3**):335–338, 1997.
[14] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[16] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received February 0, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 279–293
University of Białystok, 2005

Weighted and Labeled Graphs[1]

Gilbert Lee[2]
University of Victoria, Victoria, Canada

Summary. In the graph framework of [17] we introduce new selectors: weights for edges and labels for both edges and vertices. We introduce also a number of tools for accessing and modifying these new fields.

MML identifier: `GLIB_003`, version: `7.5.01 4.39.921`

The articles [20], [19], [22], [14], [23], [9], [6], [15], [1], [18], [21], [7], [12], [10], [11], [3], [24], [4], [13], [2], [5], [8], [17], and [16] provide the notation and terminology for this paper.

1. Preliminaries

Let D be a set, let f_1 be a finite sequence of elements of D, and let f_2 be a FinSubsequence of f_1. Then $\operatorname{Seq} f_2$ is a finite sequence of elements of D.

Let F be a real-yielding binary relation and let X be a set. One can check that $F{\restriction}X$ is real-yielding.

Next we state two propositions:

(1) Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}$ be sets and p be a finite sequence. Suppose $p = \langle x_1\rangle ^\frown \langle x_2\rangle ^\frown \langle x_3\rangle ^\frown \langle x_4\rangle ^\frown \langle x_5\rangle ^\frown \langle x_6\rangle ^\frown \langle x_7\rangle ^\frown \langle x_8\rangle ^\frown \langle x_9\rangle ^\frown \langle x_{10}\rangle$. Then $\operatorname{len} p = 10$ and $p(1) = x_1$ and $p(2) = x_2$ and $p(3) = x_3$ and $p(4) = x_4$ and $p(5) = x_5$ and $p(6) = x_6$ and $p(7) = x_7$ and $p(8) = x_8$ and $p(9) = x_9$ and $p(10) = x_{10}$.

(2) Let f_1 be a finite sequence of elements of $\mathbb{R}$ and f_2 be a FinSubsequence of f_1. If for every natural number i such that $i \in \operatorname{dom} f_1$ holds $0 \leq f_1(i)$, then $\sum \operatorname{Seq} f_2 \leq \sum f_1$.

[1]This work has been partially supported by NSERC, Alberta Ingenuity Fund and iCORE.
[2]Part of author's MSc work.

ISSN 1426–2630

2. Definitions

The natural number WeightSelector is defined by:

(Def. 1) WeightSelector $= 5$.

The natural number ELabelSelector is defined as follows:

(Def. 2) ELabelSelector $= 6$.

The natural number VLabelSelector is defined as follows:

(Def. 3) VLabelSelector $= 7$.

Let G be a graph structure. We say that G is weighted if and only if:

(Def. 4) WeightSelector $\in \operatorname{dom} G$ and G(WeightSelector) is a many sorted set indexed by the edges of G.

We say that G is elabeled if and only if:

(Def. 5) ELabelSelector $\in \operatorname{dom} G$ and there exists a function f such that $G(\text{ELabelSelector}) = f$ and $\operatorname{dom} f \subseteq$ the edges of G.

We say that G is vlabeled if and only if:

(Def. 6) VLabelSelector $\in \operatorname{dom} G$ and there exists a function f such that $G(\text{VLabelSelector}) = f$ and $\operatorname{dom} f \subseteq$ the vertices of G.

Let us mention that there exists a graph structure which is graph-like, weighted, elabeled, and vlabeled.

A w-graph is a weighted graph. A e-graph is a elabeled graph. A v-graph is a vlabeled graph. A we-graph is a weighted elabeled graph. A wv-graph is a weighted vlabeled graph. A ev-graph is a elabeled vlabeled graph. A wev-graph is a weighted elabeled vlabeled graph.

Let G be a w-graph. The weight of G yielding a many sorted set indexed by the edges of G is defined by:

(Def. 7) The weight of $G = G(\text{WeightSelector})$.

Let G be a e-graph. The elabel of G yields a function and is defined by:

(Def. 8) The elabel of $G = G(\text{ELabelSelector})$.

Let G be a v-graph. The vlabel of G yielding a function is defined by:

(Def. 9) The vlabel of $G = G(\text{VLabelSelector})$.

Let G be a graph and let X be a set. One can check the following observations:

* $G.\text{set}(\text{WeightSelector}, X)$ is graph-like,
* $G.\text{set}(\text{ELabelSelector}, X)$ is graph-like, and
* $G.\text{set}(\text{VLabelSelector}, X)$ is graph-like.

Let G be a finite graph and let X be a set. One can check the following observations:

* $G.\text{set}(\text{WeightSelector}, X)$ is finite,

* G.set(ELabelSelector, X) is finite, and
* G.set(VLabelSelector, X) is finite.

Let G be a loopless graph and let X be a set. One can check the following observations:

* G.set(WeightSelector, X) is loopless,
* G.set(ELabelSelector, X) is loopless, and
* G.set(VLabelSelector, X) is loopless.

Let G be a trivial graph and let X be a set. One can check the following observations:

* G.set(WeightSelector, X) is trivial,
* G.set(ELabelSelector, X) is trivial, and
* G.set(VLabelSelector, X) is trivial.

Let G be a non trivial graph and let X be a set. One can verify the following observations:

* G.set(WeightSelector, X) is non trivial,
* G.set(ELabelSelector, X) is non trivial, and
* G.set(VLabelSelector, X) is non trivial.

Let G be a non-multi graph and let X be a set. One can check the following observations:

* G.set(WeightSelector, X) is non-multi,
* G.set(ELabelSelector, X) is non-multi, and
* G.set(VLabelSelector, X) is non-multi.

Let G be a non-directed-multi graph and let X be a set. One can verify the following observations:

* G.set(WeightSelector, X) is non-directed-multi,
* G.set(ELabelSelector, X) is non-directed-multi, and
* G.set(VLabelSelector, X) is non-directed-multi.

Let G be a connected graph and let X be a set. One can check the following observations:

* G.set(WeightSelector, X) is connected,
* G.set(ELabelSelector, X) is connected, and
* G.set(VLabelSelector, X) is connected.

Let G be an acyclic graph and let X be a set. One can verify the following observations:

* G.set(WeightSelector, X) is acyclic,
* G.set(ELabelSelector, X) is acyclic, and
* G.set(VLabelSelector, X) is acyclic.

Let G be a w-graph and let X be a set. Observe that G.set(ELabelSelector, X) is weighted and G.set(VLabelSelector, X) is weighted.

Let G be a graph and let X be a many sorted set indexed by the edges of G. Note that G.set(WeightSelector, X) is weighted.

Let G be a graph, let W_1 be a non empty set, and let W be a function from the edges of G into W_1. Note that G.set(WeightSelector, W) is weighted.

Let G be a e-graph and let X be a set. Note that G.set(WeightSelector, X) is elabeled and G.set(VLabelSelector, X) is elabeled.

Let G be a graph, let Y be a set, and let X be a partial function from the edges of G to Y. One can check that G.set(ELabelSelector, X) is elabeled.

Let G be a graph and let X be a many sorted set indexed by the edges of G. One can verify that G.set(ELabelSelector, X) is elabeled.

Let G be a v-graph and let X be a set. Note that G.set(WeightSelector, X) is vlabeled and G.set(ELabelSelector, X) is vlabeled.

Let G be a graph, let Y be a set, and let X be a partial function from the vertices of G to Y. Note that G.set(VLabelSelector, X) is vlabeled.

Let G be a graph and let X be a many sorted set indexed by the vertices of G. One can verify that G.set(VLabelSelector, X) is vlabeled.

Let G be a graph. Note that G.set(ELabelSelector, $\emptyset$) is elabeled and G.set(VLabelSelector, $\emptyset$) is vlabeled.

Let G be a graph. Note that there exists a subgraph of G which is weighted, elabeled, and vlabeled.

Let G be a w-graph and let G_2 be a weighted subgraph of G. We say that G_2 inherits weight if and only if:

(Def. 10) The weight of G_2 = (the weight of G)$\restriction$(the edges of G_2).

Let G be a e-graph and let G_2 be a elabeled subgraph of G. We say that G_2 inherits elabel if and only if:

(Def. 11) The elabel of G_2 = (the elabel of G)$\restriction$(the edges of G_2).

Let G be a v-graph and let G_2 be a vlabeled subgraph of G. We say that G_2 inherits vlabel if and only if:

(Def. 12) The vlabel of G_2 = (the vlabel of G)$\restriction$(the vertices of G_2).

Let G be a w-graph. Observe that there exists a weighted subgraph of G which inherits weight.

Let G be a e-graph. One can check that there exists a elabeled subgraph of G which inherits elabel.

Let G be a v-graph. One can verify that there exists a vlabeled subgraph of G which inherits vlabel.

Let G be a we-graph. Note that there exists a weighted elabeled subgraph of G which inherits weight and elabel.

Let G be a wv-graph. Observe that there exists a weighted vlabeled subgraph of G which inherits weight and vlabel.

Let G be a ev-graph. Observe that there exists a elabeled vlabeled subgraph of G which inherits elabel and vlabel.

Let G be a wev-graph. One can verify that there exists a weighted elabeled vlabeled subgraph of G which inherits weight, elabel, and vlabel.

Let G be a w-graph. A w-subgraph of G is a weighted subgraph of G inheriting weight.

Let G be a e-graph. A e-subgraph of G is a elabeled subgraph of G inheriting elabel.

Let G be a v-graph. A v-subgraph of G is a vlabeled subgraph of G inheriting vlabel.

Let G be a we-graph. A we-subgraph of G is a weighted elabeled subgraph of G inheriting weight and elabel.

Let G be a wv-graph. A wv-subgraph of G is a weighted vlabeled subgraph of G inheriting weight and vlabel.

Let G be a ev-graph. A ev-subgraph of G is a elabeled vlabeled subgraph of G inheriting elabel and vlabel.

Let G be a wev-graph. A wev-subgraph of G is a weighted elabeled vlabeled subgraph of G inheriting weight, elabel, and vlabel.

Let G be a graph and let V, E be sets. One can verify that there exists a subgraph of G induced by V and E which is weighted, elabeled, and vlabeled.

Let G be a w-graph and let V, E be sets. One can verify that there exists a weighted subgraph of G induced by V and E which inherits weight.

Let G be a e-graph and let V, E be sets. One can verify that there exists a elabeled subgraph of G induced by V and E which inherits elabel.

Let G be a v-graph and let V, E be sets. One can verify that there exists a vlabeled subgraph of G induced by V and E which inherits vlabel.

Let G be a we-graph and let V, E be sets. Note that there exists a weighted elabeled subgraph of G induced by V and E which inherits weight and elabel.

Let G be a wv-graph and let V, E be sets. Observe that there exists a weighted vlabeled subgraph of G induced by V and E which inherits weight and vlabel.

Let G be a ev-graph and let V, E be sets. Note that there exists a elabeled vlabeled subgraph of G induced by V and E which inherits elabel and vlabel.

Let G be a wev-graph and let V, E be sets. Observe that there exists a weighted elabeled vlabeled subgraph of G induced by V and E which inherits weight, elabel, and vlabel.

Let G be a w-graph and let V, E be sets. A induced w-subgraph of G, V, E is a weighted subgraph of G induced by V and E inheriting weight.

Let G be a e-graph and let V, E be sets. A induced e-subgraph of G, V, E is a elabeled subgraph of G induced by V and E inheriting elabel.

Let G be a v-graph and let V, E be sets. A induced v-subgraph of G, V, E is a vlabeled subgraph of G induced by V and E inheriting vlabel.

Let G be a we-graph and let V, E be sets. A induced we-subgraph of G, V, E is a weighted elabeled subgraph of G induced by V and E inheriting weight and elabel.

Let G be a wv-graph and let V, E be sets. A induced wv-subgraph of G, V, E is a weighted vlabeled subgraph of G induced by V and E inheriting weight and vlabel.

Let G be a ev-graph and let V, E be sets. A induced ev-subgraph of G, V, E is a elabeled vlabeled subgraph of G induced by V and E inheriting elabel and vlabel.

Let G be a wev-graph and let V, E be sets. A induced wev-subgraph of G, V, E is a weighted elabeled vlabeled subgraph of G induced by V and E inheriting weight, elabel, and vlabel.

Let G be a w-graph and let V be a set. A induced w-subgraph of G, V is a induced w-subgraph of G, V, G.edgesBetween(V).

Let G be a e-graph and let V be a set. A induced e-subgraph of G, V is a induced e-subgraph of G, V, G.edgesBetween(V).

Let G be a v-graph and let V be a set. A induced v-subgraph of G, V is a induced v-subgraph of G, V, G.edgesBetween(V).

Let G be a we-graph and let V be a set. A induced we-subgraph of G, V is a induced we-subgraph of G, V, G.edgesBetween(V).

Let G be a wv-graph and let V be a set. A induced wv-subgraph of G, V is a induced wv-subgraph of G, V, G.edgesBetween(V).

Let G be a ev-graph and let V be a set. A induced ev-subgraph of G, V is a induced ev-subgraph of G, V, G.edgesBetween(V).

Let G be a wev-graph and let V be a set. A induced wev-subgraph of G, V is a induced wev-subgraph of G, V, G.edgesBetween(V).

Let G be a w-graph. We say that G is real-weighted if and only if:

(Def. 13) The weight of G is real-yielding.

Let G be a w-graph. We say that G is nonnegative-weighted if and only if:

(Def. 14) $\operatorname{rng}(\text{the weight of } G) \subseteq \mathbb{R}_{\geq 0}$.

Let us note that every w-graph which is nonnegative-weighted is also real-weighted.

Let G be a e-graph. We say that G is real-elabeled if and only if:

(Def. 15) The elabel of G is real-yielding.

Let G be a v-graph. We say that G is real-vlabeled if and only if:

(Def. 16) The vlabel of G is real-yielding.

Let G be a wev-graph. We say that G is real-wev if and only if:

(Def. 17) G is real-weighted, real-elabeled, and real-vlabeled.

Let us note that every wev-graph which is real-wev is also real-weighted, real-elabeled, and real-vlabeled and every wev-graph which is real-weighted, real-elabeled, and real-vlabeled is also real-wev.

Let G be a graph and let X be a function from the edges of G into $\mathbb{R}$. Note that G.set(WeightSelector, X) is real-weighted.

Let G be a graph and let X be a partial function from the edges of G to $\mathbb{R}$. One can verify that G.set(ELabelSelector, X) is real-elabeled.

Let G be a graph and let X be a real-yielding many sorted set indexed by the edges of G. One can verify that G.set(ELabelSelector, X) is real-elabeled.

Let G be a graph and let X be a partial function from the vertices of G to $\mathbb{R}$. Observe that G.set(VLabelSelector, X) is real-vlabeled.

Let G be a graph and let X be a real-yielding many sorted set indexed by the vertices of G. One can verify that G.set(VLabelSelector, X) is real-vlabeled.

Let G be a graph. Observe that G.set(ELabelSelector, $\emptyset$) is real-elabeled and G.set(VLabelSelector, $\emptyset$) is real-vlabeled.

Let G be a graph, let v be a vertex of G, and let v_1 be a real number. Note that G.set(VLabelSelector, $v \dot{\longmapsto} v_1$) is vlabeled.

Let G be a graph, let v be a vertex of G, and let v_1 be a real number. One can verify that G.set(VLabelSelector, $v \dot{\longmapsto} v_1$) is real-vlabeled.

One can check that there exists a wev-graph which is finite, trivial, tree-like, nonnegative-weighted, and real-wev and there exists a wev-graph which is finite, non trivial, tree-like, nonnegative-weighted, and real-wev.

Let G be a finite w-graph. Note that the weight of G is finite.

Let G be a finite e-graph. Note that the elabel of G is finite.

Let G be a finite v-graph. Note that the vlabel of G is finite.

Let G be a real-weighted w-graph. Observe that the weight of G is real-yielding.

Let G be a real-elabeled e-graph. One can verify that the elabel of G is real-yielding.

Let G be a real-vlabeled v-graph. Observe that the vlabel of G is real-yielding.

Let G be a real-weighted w-graph and let X be a set. Observe that G.set(ELabelSelector, X) is real-weighted and G.set(VLabelSelector, X) is real-weighted.

Let G be a nonnegative-weighted w-graph and let X be a set. One can check that G.set(ELabelSelector, X) is nonnegative-weighted and G.set(VLabelSelector, X) is nonnegative-weighted.

Let G be a real-elabeled e-graph and let X be a set. One can verify that G.set(WeightSelector, X) is real-elabeled and G.set(VLabelSelector, X) is real-elabeled.

Let G be a real-vlabeled v-graph and let X be a set. Observe that G.set(WeightSelector, X) is real-vlabeled and G.set(ELabelSelector, X) is real-vlabeled.

Let G be a w-graph and let W be a walk of G. The functor W.weightSeq() yielding a finite sequence is defined as follows:

(Def. 18) len(W.weightSeq()) = len(W.edgeSeq()) and for every natural number n such that $1 \leq n$ and $n \leq$ len(W.weightSeq()) holds W.weightSeq()(n) = (the weight of G)(W.edgeSeq()(n)).

Let G be a real-weighted w-graph and let W be a walk of G. Then W.weightSeq() is a finite sequence of elements of $\mathbb{R}$.

Let G be a real-weighted w-graph and let W be a walk of G. The functor W.cost() yielding a real number is defined as follows:

(Def. 19) W.cost() = $\sum$(W.weightSeq()).

Let G be a e-graph. The functor G.labeledE() yields a subset of the edges of G and is defined as follows:

(Def. 20) G.labeledE() = dom (the elabel of G).

Let G be a e-graph and let e, x be sets. The functor G.labelEdge(e, x) yielding a e-graph is defined as follows:

(Def. 21) $G.\mathrm{labelEdge}(e, x) = \begin{cases} G.\mathrm{set}(\mathrm{ELabelSelector}, (\text{the elabel of } G) {+\cdot} (e \dot{\longmapsto} x)), \\ \quad \text{if } e \in \text{the edges of } G, \\ G, \text{ otherwise.} \end{cases}$

Let G be a finite e-graph and let e, x be sets. Note that G.labelEdge(e, x) is finite.

Let G be a loopless e-graph and let e, x be sets. Observe that G.labelEdge(e, x) is loopless.

Let G be a trivial e-graph and let e, x be sets. One can check that G.labelEdge(e, x) is trivial.

Let G be a non trivial e-graph and let e, x be sets. One can verify that G.labelEdge(e, x) is non trivial.

Let G be a non-multi e-graph and let e, x be sets. Observe that G.labelEdge(e, x) is non-multi.

Let G be a non-directed-multi e-graph and let e, x be sets. One can check that G.labelEdge(e, x) is non-directed-multi.

Let G be a connected e-graph and let e, x be sets. Observe that G.labelEdge(e, x) is connected.

Let G be an acyclic e-graph and let e, x be sets. Observe that G.labelEdge(e, x) is acyclic.

Let G be a we-graph and let e, x be sets. Observe that G.labelEdge(e, x) is weighted.

Let G be a ev-graph and let e, x be sets. Note that G.labelEdge(e, x) is vlabeled.

Let G be a real-weighted we-graph and let e, x be sets. Observe that G.labelEdge(e, x) is real-weighted.

Let G be a nonnegative-weighted we-graph and let e, x be sets. Observe that G.labelEdge(e, x) is nonnegative-weighted.

Let G be a real-elabeled e-graph, let e be a set, and let x be a real number. Observe that G.labelEdge(e, x) is real-elabeled.

Let G be a real-vlabeled ev-graph and let e, x be sets. Note that G.labelEdge(e, x) is real-vlabeled.

Let G be a v-graph and let v, x be sets. The functor G.labelVertex(v, x) yielding a v-graph is defined as follows:

(Def. 22) $$G.\mathrm{labelVertex}(v,x) = \begin{cases} G.\mathrm{set}(\mathrm{VLabelSelector}, \\ \quad (\text{the vlabel of } G) {+\cdot} (v \dot{\longmapsto} x)), \\ \quad \text{if } v \in \text{the vertices of } G, \\ G, \text{ otherwise.} \end{cases}$$

Let G be a v-graph. The functor G.labeledV() yielding a subset of the vertices of G is defined as follows:

(Def. 23) G.labeledV() = dom (the vlabel of G).

Let G be a finite v-graph and let v, x be sets. One can check that G.labelVertex(v, x) is finite.

Let G be a loopless v-graph and let v, x be sets. One can check that G.labelVertex(v, x) is loopless.

Let G be a trivial v-graph and let v, x be sets. One can check that G.labelVertex(v, x) is trivial.

Let G be a non trivial v-graph and let v, x be sets. Observe that G.labelVertex(v, x) is non trivial.

Let G be a non-multi v-graph and let v, x be sets. Note that G.labelVertex(v, x) is non-multi.

Let G be a non-directed-multi v-graph and let v, x be sets. One can verify that G.labelVertex(v, x) is non-directed-multi.

Let G be a connected v-graph and let v, x be sets. Observe that G.labelVertex(v, x) is connected.

Let G be an acyclic v-graph and let v, x be sets. Note that G.labelVertex(v, x) is acyclic.

Let G be a wv-graph and let v, x be sets. One can check that G.labelVertex(v, x) is weighted.

Let G be a ev-graph and let v, x be sets. Observe that G.labelVertex(v, x) is elabeled.

Let G be a real-weighted wv-graph and let v, x be sets. Observe that G.labelVertex(v, x) is real-weighted.

Let G be a nonnegative-weighted wv-graph and let v, x be sets. Note that G.labelVertex(v, x) is nonnegative-weighted.

Let G be a real-elabeled ev-graph and let v, x be sets. Observe that G.labelVertex(v, x) is real-elabeled.

Let G be a real-vlabeled v-graph, let v be a set, and let x be a real number. Note that G.labelVertex(v, x) is real-vlabeled.

Let G be a real-weighted w-graph. Observe that every w-subgraph of G is real-weighted.

Let G be a nonnegative-weighted w-graph. Observe that every w-subgraph of G is nonnegative-weighted.

Let G be a real-elabeled e-graph. Observe that every e-subgraph of G is real-elabeled.

Let G be a real-vlabeled v-graph. Observe that every v-subgraph of G is real-vlabeled.

Let G_1 be a graph sequence. We say that G_1 is weighted if and only if:

(Def. 24) For every natural number x holds $G_1 \rightarrow x$ is weighted.

We say that G_1 is elabeled if and only if:

(Def. 25) For every natural number x holds $G_1 \rightarrow x$ is elabeled.

We say that G_1 is vlabeled if and only if:

(Def. 26) For every natural number x holds $G_1 \rightarrow x$ is vlabeled.

Let us mention that there exists a graph sequence which is weighted, elabeled, and vlabeled.

A w-graph sequence is a weighted graph sequence. A e-graph sequence is a elabeled graph sequence. A v-graph sequence is a vlabeled graph sequence. A we-graph sequence is a weighted elabeled graph sequence. A wv-graph sequence is a weighted vlabeled graph sequence. A ev-graph sequence is a elabeled vlabeled graph sequence. A wev-graph sequence is a weighted elabeled vlabeled graph sequence.

Let G_1 be a w-graph sequence and let x be a natural number. One can check that $G_1 \rightarrow x$ is weighted.

Let G_1 be a e-graph sequence and let x be a natural number. One can check that $G_1 \rightarrow x$ is elabeled.

Let G_1 be a v-graph sequence and let x be a natural number. Observe that $G_1 \rightarrow x$ is vlabeled.

Let G_1 be a w-graph sequence. We say that G_1 is real-weighted if and only if:

(Def. 27) For every natural number x holds $G_1 \rightarrow x$ is real-weighted.

We say that G_1 is nonnegative-weighted if and only if:

(Def. 28) For every natural number x holds $G_1 \rightarrow x$ is nonnegative-weighted.

Let G_1 be a e-graph sequence. We say that G_1 is real-elabeled if and only if:

(Def. 29) For every natural number x holds $G_1 \rightarrow x$ is real-elabeled.

Let G_1 be a v-graph sequence. We say that G_1 is real-vlabeled if and only if:

(Def. 30) For every natural number x holds $G_1 \rightarrow x$ is real-vlabeled.

Let G_1 be a wev-graph sequence. We say that G_1 is real-wev if and only if:

(Def. 31) For every natural number x holds $G_1 \rightarrow x$ is real-wev.

Let us note that every wev-graph sequence which is real-wev is also real-weighted, real-elabeled, and real-vlabeled and every wev-graph sequence which is real-weighted, real-elabeled, and real-vlabeled is also real-wev.

Let us observe that there exists a wev-graph sequence which is halting, finite, loopless, trivial, non-multi, simple, real-wev, nonnegative-weighted, and tree-like.

Let G_1 be a real-weighted w-graph sequence and let x be a natural number. One can check that $G_1 \rightarrow x$ is real-weighted.

Let G_1 be a nonnegative-weighted w-graph sequence and let x be a natural number. Observe that $G_1 \rightarrow x$ is nonnegative-weighted.

Let G_1 be a real-elabeled e-graph sequence and let x be a natural number. Note that $G_1 \rightarrow x$ is real-elabeled.

Let G_1 be a real-vlabeled v-graph sequence and let x be a natural number. One can verify that $G_1 \rightarrow x$ is real-vlabeled.

3. Theorems

The following propositions are true:

(3) WeightSelector $= 5$ and ELabelSelector $= 6$ and VLabelSelector $= 7$.

(4)(i) For every w-graph G holds the weight of $G = G$(WeightSelector),

(ii) for every e-graph G holds the elabel of $G = G$(ELabelSelector), and

(iii) for every v-graph G holds the vlabel of $G = G$(VLabelSelector).

(6)[3] For every e-graph G holds dom (the elabel of G) $\subseteq$ the edges of G.

(7) For every v-graph G holds dom (the vlabel of G) $\subseteq$ the vertices of G.

(8) For every graph G and for every set X holds $G =_G G$.set(WeightSelector, X) and $G =_G G$.set(ELabelSelector, X) and $G =_G G$.set(VLabelSelector, X).

(9) For every e-graph G and for every set X holds the elabel of $G =$ the elabel of G.set(WeightSelector, X).

(10) For every v-graph G and for every set X holds the vlabel of $G =$ the vlabel of G.set(WeightSelector, X).

(11) For every w-graph G and for every set X holds the weight of $G =$ the weight of G.set(ELabelSelector, X).

(12) For every v-graph G and for every set X holds the vlabel of $G =$ the vlabel of G.set(ELabelSelector, X).

[3]The proposition (5) has been removed.

(13) For every w-graph G and for every set X holds the weight of $G =$ the weight of G.set(VLabelSelector, X).

(14) For every e-graph G and for every set X holds the elabel of $G =$ the elabel of G.set(VLabelSelector, X).

(15) Let G_3, G_2 be w-graphs and G_4 be a w-graph. Suppose $G_3 =_G G_2$ and the weight of $G_3 =$ the weight of G_2 and G_3 is a w-subgraph of G_4. Then G_2 is a w-subgraph of G_4.

(16) For every w-graph G_3 and for every w-subgraph G_2 of G_3 holds every w-subgraph of G_2 is a w-subgraph of G_3.

(17) Let G_3, G_2 be w-graphs and G_4 be a w-subgraph of G_3. Suppose $G_3 =_G G_2$ and the weight of $G_3 =$ the weight of G_2. Then G_4 is a w-subgraph of G_2.

(18) Let G_3 be a w-graph, G_2 be a w-subgraph of G_3, and x be a set. If $x \in$ the edges of G_2, then (the weight of G_2)$(x) =$ (the weight of G_3)(x).

(19) For every w-graph G and for every walk W of G such that W is trivial holds W.weightSeq() $= \emptyset$.

(20) For every w-graph G and for every walk W of G holds len(W.weightSeq()) $= W$.length().

(21) For every w-graph G and for all sets x, y, e such that e joins x and y in G holds (G.walkOf(x, e, y)).weightSeq() $= \langle$(the weight of G)$(e)\rangle$.

(22) For every w-graph G and for every walk W of G holds W.reverse().weightSeq() $=$ Rev(W.weightSeq()).

(23) For every w-graph G and for all walks W_2, W_3 of G such that W_2.last() $=$ W_3.first() holds (W_2.append(W_3)).weightSeq() $=$ W_2.weightSeq() $\frown$ W_3.weightSeq().

(24) Let G be a w-graph, W be a walk of G, and e be a set. If $e \in W$.last().edgesInOut(), then (W.addEdge(e)).weightSeq() $=$ W.weightSeq() $\frown$ $\langle$(the weight of G)$(e)\rangle$.

(25) Let G be a real-weighted w-graph, W_2 be a walk of G, and W_3 be a subwalk of W_2. Then there exists a FinSubsequence w_1 of W_2.weightSeq() such that W_3.weightSeq() $=$ Seq w_1.

(26) Let G_3, G_2 be w-graphs, W_2 be a walk of G_3, and W_3 be a walk of G_2. If $W_2 = W_3$ and the weight of $G_3 =$ the weight of G_2, then W_2.weightSeq() $=$ W_3.weightSeq().

(27) Let G_3 be a w-graph, G_2 be a w-subgraph of G_3, W_2 be a walk of G_3, and W_3 be a walk of G_2. If $W_2 = W_3$, then W_2.weightSeq() $=$ W_3.weightSeq().

(28) For every real-weighted w-graph G and for every walk W of G such that W is trivial holds W.cost() $= 0$.

(29) Let G be a real-weighted w-graph, v_2, v_3 be vertices of G, and e be a set.

If e joins v_2 and v_3 in G, then $(G.\mathrm{walkOf}(v_2, e, v_3)).\mathrm{cost}() = $ (the weight of $G)(e)$.

(30) For every real-weighted w-graph G and for every walk W of G holds $W.\mathrm{cost}() = W.\mathrm{reverse}().\mathrm{cost}()$.

(31) For every real-weighted w-graph G and for all walks W_2, W_3 of G such that $W_2.\mathrm{last}() = W_3.\mathrm{first}()$ holds $(W_2.\mathrm{append}(W_3)).\mathrm{cost}() = W_2.\mathrm{cost}() + W_3.\mathrm{cost}()$.

(32) Let G be a real-weighted w-graph, W be a walk of G, and e be a set. If $e \in W.\mathrm{last}().\mathrm{edgesInOut}()$, then $(W.\mathrm{addEdge}(e)).\mathrm{cost}() = W.\mathrm{cost}() +$ (the weight of $G)(e)$.

(33) Let G_3, G_2 be real-weighted w-graphs, W_2 be a walk of G_3, and W_3 be a walk of G_2. If $W_2 = W_3$ and the weight of $G_3 =$ the weight of G_2, then $W_2.\mathrm{cost}() = W_3.\mathrm{cost}()$.

(34) Let G_3 be a real-weighted w-graph, G_2 be a w-subgraph of G_3, W_2 be a walk of G_3, and W_3 be a walk of G_2. If $W_2 = W_3$, then $W_2.\mathrm{cost}() = W_3.\mathrm{cost}()$.

(35) Let G be a nonnegative-weighted w-graph, W be a walk of G, and n be a natural number. If $n \in \mathrm{dom}(W.\mathrm{weightSeq}())$, then $0 \leq W.\mathrm{weightSeq}()(n)$.

(36) For every nonnegative-weighted w-graph G and for every walk W of G holds $0 \leq W.\mathrm{cost}()$.

(37) For every nonnegative-weighted w-graph G and for every walk W_2 of G and for every subwalk W_3 of W_2 holds $W_3.\mathrm{cost}() \leq W_2.\mathrm{cost}()$.

(38) Let G be a nonnegative-weighted w-graph and e be a set. If $e \in$ the edges of G, then $0 \leq$ (the weight of $G)(e)$.

(39) Let G be a e-graph and e, x be sets. Suppose $e \in$ the edges of G. Then the elabel of $G.\mathrm{labelEdge}(e, x) =$ (the elabel of $G) {+\cdot} (e \dot{\longmapsto} x)$.

(40) For every e-graph G and for all sets e, x such that $e \in$ the edges of G holds (the elabel of $G.\mathrm{labelEdge}(e, x))(e) = x$.

(41) For every e-graph G and for all sets e, x holds $G =_G G.\mathrm{labelEdge}(e, x)$.

(42) For every we-graph G and for all sets e, x holds the weight of $G =$ the weight of $G.\mathrm{labelEdge}(e, x)$.

(43) For every ev-graph G and for all sets e, x holds the vlabel of $G =$ the vlabel of $G.\mathrm{labelEdge}(e, x)$.

(44) For every e-graph G and for all sets e_1, e_2, x such that $e_1 \neq e_2$ holds (the elabel of $G.\mathrm{labelEdge}(e_1, x))(e_2) =$ (the elabel of $G)(e_2)$.

(45) Let G be a v-graph and v, x be sets. Suppose $v \in$ the vertices of G. Then the vlabel of $G.\mathrm{labelVertex}(v, x) =$ (the vlabel of $G) {+\cdot} (v \dot{\longmapsto} x)$.

(46) For every v-graph G and for all sets v, x such that $v \in$ the vertices of G holds (the vlabel of $G.\mathrm{labelVertex}(v, x))(v) = x$.

(47) For every v-graph G and for all sets v, x holds $G =_G G$.labelVertex(v, x).

(48) For every wv-graph G and for all sets v, x holds the weight of G = the weight of G.labelVertex(v, x).

(49) For every ev-graph G and for all sets v, x holds the elabel of G = the elabel of G.labelVertex(v, x).

(50) For every v-graph G and for all sets v_2, v_3, x such that $v_2 \neq v_3$ holds (the vlabel of G.labelVertex(v_2, x))(v_3) = (the vlabel of G)(v_3).

(51) For all e-graphs G_3, G_2 such that the elabel of G_3 = the elabel of G_2 holds G_3.labeledE() = G_2.labeledE().

(52) For every e-graph G and for all sets e, x such that $e \in$ the edges of G holds (G.labelEdge(e, x)).labeledE() = G.labeledE() $\cup \{e\}$.

(53) For every e-graph G and for all sets e, x such that $e \in$ the edges of G holds G.labeledE() $\subseteq$ (G.labelEdge(e, x)).labeledE().

(54) For every finite e-graph G and for all sets e, x such that $e \in$ the edges of G and $e \notin G$.labeledE() holds card((G.labelEdge(e, x)).labeledE()) = card(G.labeledE()) + 1.

(55) For every e-graph G and for all sets e_1, e_2, x such that $e_2 \notin G$.labeledE() and $e_2 \in$ (G.labelEdge(e_1, x)).labeledE() holds $e_1 = e_2$ and $e_1 \in$ the edges of G.

(56) For every ev-graph G and for all sets v, x holds G.labeledE() = (G.labelVertex(v, x)).labeledE().

(57) For every e-graph G and for all sets e, x such that $e \in$ the edges of G holds $e \in$ (G.labelEdge(e, x)).labeledE().

(58) For all v-graphs G_3, G_2 such that the vlabel of G_3 = the vlabel of G_2 holds G_3.labeledV() = G_2.labeledV().

(59) For every v-graph G and for all sets v, x such that $v \in$ the vertices of G holds (G.labelVertex(v, x)).labeledV() = G.labeledV() $\cup \{v\}$.

(60) For every v-graph G and for all sets v, x such that $v \in$ the vertices of G holds G.labeledV() $\subseteq$ (G.labelVertex(v, x)).labeledV().

(61) For every finite v-graph G and for all sets v, x such that $v \in$ the vertices of G and $v \notin G$.labeledV() holds card((G.labelVertex(v, x)).labeledV()) = card(G.labeledV()) + 1.

(62) For every v-graph G and for all sets v_2, v_3, x such that $v_3 \notin G$.labeledV() and $v_3 \in$ (G.labelVertex(v_2, x)).labeledV() holds $v_2 = v_3$ and $v_2 \in$ the vertices of G.

(63) For every ev-graph G and for all sets e, x holds G.labeledV() = (G.labelEdge(e, x)).labeledV().

(64) For every v-graph G and for every vertex v of G and for every set x holds $v \in$ (G.labelVertex(v, x)).labeledV().

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[10] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[11] Czesław Byliński. Some properties of restrictions of finite sequences. *Formalized Mathematics*, 5(**2**):241–245, 1996.
[12] Jing-Chao Chen and Yatsuka Nakamura. The underlying principle of Dijkstra's shortest path algorithm. *Formalized Mathematics*, 11(**2**):143–152, 2003.
[13] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[14] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[15] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[16] Gilbert Lee. Walks in Graphs. *Formalized Mathematics*, 13(**2**):253–269, 2005.
[17] Gilbert Lee and Piotr Rudnicki. Alternative graph structures. *Formalized Mathematics*, 13(**2**):235–252, 2005.
[18] Yatsuka Nakamura and Piotr Rudnicki. Vertex sequences induced by chains. *Formalized Mathematics*, 5(**3**):297–304, 1996.
[19] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[21] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[24] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 8, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 295–304
University of Białystok, 2005

Correctness of Dijkstra's Shortest Path and Prim's Minimum Spanning Tree Algorithms[1]

Gilbert Lee[2]
University of Victoria
Victoria, Canada

Piotr Rudnicki
University of Alberta
Edmonton, Canada

Summary. We prove correctness for Dijkstra's shortest path algorithm and Prim's minimum weight spanning tree algorithm at the level of graph manipulations.

MML identifier: `GLIB_004`, version: `7.5.01 4.39.921`

The notation and terminology used in this paper are introduced in the following articles: [25], [11], [24], [22], [28], [23], [13], [30], [10], [7], [4], [6], [14], [1], [26], [29], [8], [3], [27], [21], [19], [12], [2], [5], [9], [18], [16], [15], [20], and [17].

1. Preliminaries

One can prove the following propositions:

(1) For all functions f, g holds $\mathrm{support}(f{+}\cdot g) \subseteq \mathrm{support}\, f \cup \mathrm{support}\, g$.

(2) For every function f and for all sets x, y holds $\mathrm{support}(f{+}\cdot(x \dot{\longmapsto} y)) \subseteq \mathrm{support}\, f \cup \{x\}$.

(3) Let A, B be sets, b be a real bag over A, b_1 be a real bag over B, and b_2 be a real bag over $A \setminus B$. If $b = b_1{+}\cdot b_2$, then $\sum b = \sum b_1 + \sum b_2$.

(4) For all sets X, x and for every real bag b over X such that $\mathrm{dom}\, b = \{x\}$ holds $\sum b = b(x)$.

[1]This work has been partially supported by NSERC, Alberta Ingenuity Fund and iCORE.
[2]Part of author's MSc work.

ISSN 1426–2630

(5) For every set A and for all real bags b_1, b_2 over A such that for every set x such that $x \in A$ holds $b_1(x) \leq b_2(x)$ holds $\sum b_1 \leq \sum b_2$.

(6) For every set A and for all real bags b_1, b_2 over A such that for every set x such that $x \in A$ holds $b_1(x) = b_2(x)$ holds $\sum b_1 = \sum b_2$.

(7) For all sets A_1, A_2 and for every real bag b_1 over A_1 and for every real bag b_2 over A_2 such that $b_1 = b_2$ holds $\sum b_1 = \sum b_2$.

(8) For all sets X, x and for every real bag b over X and for every real number y such that $b = \text{EmptyBag}\, X +\cdot (x \mapsto y)$ holds $\sum b = y$.

(9) Let X, x be sets, b_1, b_2 be real bags over X, and y be a real number. If $b_2 = b_1 +\cdot (x \mapsto y)$, then $\sum b_2 = (\sum b_1 + y) - b_1(x)$.

2. Dijkstra's Shortest Path Algorithm: definitions

Let G_1 be a real-weighted w-graph, let G_2 be a w-subgraph of G_1, and let v be a set. We say that G_2 is mincost d-tree rooted at v if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) G_2 is tree-like, and

(ii) for every vertex x of G_2 there exists a dpath W_2 of G_2 such that W_2 is walk from v to x and for every dpath W_1 of G_1 such that W_1 is walk from v to x holds $W_2.\text{cost}() \leq W_1.\text{cost}()$.

Let G be a real-weighted w-graph, let W be a dpath of G, and let x, y be sets. We say that W is mincost d-path from x to y if and only if:

(Def. 2) W is walk from x to y and for every dpath W_2 of G such that W_2 is walk from x to y holds $W.\text{cost}() \leq W_2.\text{cost}()$.

Let G be a finite real-weighted w-graph and let x, y be sets. The G .mincost-d-path(x, y) yielding a real number is defined as follows:

(Def. 3)(i) There exists a dpath W of G such that W is mincost d-path from x to y and the G.mincost-d-path(x, y) $= W.\text{cost}()$ if there exists a dwalk of G which is walk from x to y,

(ii) the G.mincost-d-path(x, y) $= 0$, otherwise.

Let G be a real-wev wev-graph. The functor DIJK : NextBestEdges(G) yielding a subset of the edges of G is defined by the condition (Def. 4).

(Def. 4) Let e_1 be a set. Then $e_1 \in$ DIJK : NextBestEdges(G) if and only if the following conditions are satisfied:

(i) e_1 joins a vertex from G.labeledV() to a vertex from (the vertices of G) $\setminus$ G.labeledV() in G, and

(ii) for every set e_2 such that e_2 joins a vertex from G.labeledV() to a vertex from (the vertices of G) $\setminus$ G.labeledV() in G holds (the vlabel of G)((the source of G)(e_1)) + (the weight of G)(e_1) $\leq$ (the vlabel of G)((the source of G)(e_2)) + (the weight of G)(e_2).

Let G be a real-wev wev-graph. The functor DIJK : Step(G) yields a real-wev wev-graph and is defined by:

$$\text{(Def. 5)}\quad \text{DIJK} : \text{Step}(G) = \begin{cases} G, \text{ if DIJK} : \text{NextBestEdges}(G) = \emptyset, \\ (G.\text{labelEdge}(e, 1)).\text{labelVertex}((\text{the target of } G)(e), \\ \quad (\text{the vlabel of } G)((\text{the source of } G)(e)) + \\ \quad (\text{the weight of } G)(e)), \text{ otherwise.} \end{cases}$$

Let G be a finite real-wev wev-graph. One can verify that DIJK : Step(G) is finite.

Let G be a nonnegative-weighted real-wev wev-graph. Observe that DIJK : Step(G) is nonnegative-weighted.

Let G be a real-weighted w-graph and let s_1 be a vertex of G. The functor DIJK : Init(G, s_1) yielding a real-wev wev-graph is defined by:

(Def. 6) DIJK : Init(G, s_1) = G.set(ELabelSelector, $\emptyset$).set(VLabelSelector, $s_1 \dot{\longmapsto} 0$).

Let G be a real-weighted w-graph and let s_1 be a vertex of G. The functor DIJK : CompSeq(G, s_1) yielding a real-wev wev-graph sequence is defined as follows:

(Def. 7) DIJK : CompSeq(G, s_1)$\rightarrow$0 = DIJK : Init(G, s_1) and for every natural number n holds DIJK : CompSeq(G, s_1)$\rightarrow$($n + 1$) = DIJK : Step((DIJK : CompSeq(G, s_1)$\rightarrow n$)).

Let G be a finite real-weighted w-graph and let s_1 be a vertex of G. Observe that DIJK : CompSeq(G, s_1) is finite.

Let G be a nonnegative-weighted w-graph and let s_1 be a vertex of G. One can verify that DIJK : CompSeq(G, s_1) is nonnegative-weighted.

Let G be a real-weighted w-graph and let s_1 be a vertex of G. The functor DIJK : SSSP(G, s_1) yields a real-wev wev-graph and is defined by:

(Def. 8) DIJK : SSSP(G, s_1) = (DIJK : CompSeq(G, s_1)).Result().

Let G be a finite real-weighted w-graph and let s_1 be a vertex of G. One can check that DIJK : SSSP(G, s_1) is finite.

3. Dijkstra's Shortest Path Algorithm: theorems

The following propositions are true:

(10) Let G be a finite nonnegative-weighted w-graph, W be a dpath of G, x, y be sets, and m, n be natural numbers. Suppose W is mincost d-path from x to y. Then W.cut(m, n) is mincost d-path from (W.cut(m, n)).first() to (W.cut(m, n)).last().

(11) Let G be a finite real-weighted w-graph, W_1, W_2 be dpaths of G, and x, y be sets. Suppose W_1 is mincost d-path from x to y and W_2 is mincost d-path from x to y. Then W_1.cost() = W_2.cost().

(12) Let G be a finite real-weighted w-graph, W be a dpath of G, and x, y be sets. Suppose W is mincost d-path from x to y. Then the G.mincost-d-path(x, y) = W.cost().

(13) Let G be a finite real-wev wev-graph. Then

(i) card((DIJK : Step(G)).labeledV()) = card(G.labeledV()) iff DIJK : NextBestEdges(G) = $\emptyset$, and

(ii) card((DIJK : Step(G)).labeledV()) = card(G.labeledV()) + 1 iff DIJK : NextBestEdges(G) $\neq \emptyset$.

(14) For every real-wev wev-graph G holds $G =_G$ DIJK : Step(G) and the weight of G = the weight of DIJK : Step(G) and G.labeledE() $\subseteq$ (DIJK : Step(G)).labeledE() and G.labeledV() $\subseteq$ (DIJK : Step(G)).labeledV().

(15) For every real-weighted w-graph G and for every vertex s_1 of G holds (DIJK : Init(G, s_1)).labeledV() = $\{s_1\}$.

(16) Let G be a real-weighted w-graph, s_1 be a vertex of G, and i, j be natural numbers. If $i \leq j$, then (DIJK : CompSeq(G, s_1)$\rightarrow i$).labeledV() $\subseteq$ (DIJK : CompSeq(G, s_1)$\rightarrow j$).labeledV() and (DIJK : CompSeq(G, s_1)$\rightarrow i$) .labeledE() $\subseteq$ (DIJK : CompSeq(G, s_1)$\rightarrow j$).labeledE().

(17) Let G be a real-weighted w-graph, s_1 be a vertex of G, and n be a natural number. Then $G =_G$ DIJK : CompSeq(G, s_1)$\rightarrow n$ and the weight of G = the weight of DIJK : CompSeq(G, s_1)$\rightarrow n$.

(18) Let G be a finite real-weighted w-graph, s_1 be a vertex of G, and n be a natural number. Then (DIJK : CompSeq(G, s_1)$\rightarrow n$).labeledV() $\subseteq$ G.reachableDFrom(s_1).

(19) Let G be a finite real-weighted w-graph, s_1 be a vertex of G, and n be a natural number.
Then DIJK : NextBestEdges((DIJK : CompSeq(G, s_1)$\rightarrow n$)) = $\emptyset$ if and only if (DIJK : CompSeq(G, s_1)$\rightarrow n$).labeledV() = G.reachableDFrom(s_1).

(20) Let G be a finite real-weighted w-graph, s_1 be a vertex of G, and n be a natural number. Then $\overline{\overline{\text{(DIJK : CompSeq}(G, s_1)\rightarrow n\text{).labeledV()}}}$ = $\min(n + 1, \text{card}(G\text{.reachableDFrom}(s_1)))$.

(21) Let G be a finite real-weighted w-graph, s_1 be a vertex of G, and n be a natural number. Then (DIJK : CompSeq(G, s_1)$\rightarrow n$).labeledE() $\subseteq$ (DIJK : CompSeq(G, s_1)$\rightarrow n$).edgesBetween((DIJK : CompSeq(G, s_1)$\rightarrow n$) .labeledV()).

(22) Let G be a finite nonnegative-weighted w-graph, s_1 be a vertex of G, n be a natural number, and G_2 be a induced w-subgraph of G, (DIJK : CompSeq(G, s_1)$\rightarrow n$).labeledV(), (DIJK : CompSeq(G, s_1)$\rightarrow n$) .labeledE(). Then

(i) G_2 is mincost d-tree rooted at s_1, and

(ii) for every vertex v of G such that $v \in$ (DIJK : CompSeq(G, s_1).→n) .labeledV() holds the G.mincost-d-path(s_1, v) = (the vlabel of DIJK : CompSeq(G, s_1).→n)(v).

(23) For every finite real-weighted w-graph G and for every vertex s_1 of G holds DIJK : CompSeq(G, s_1) is halting.

Let G be a finite real-weighted w-graph and let s_1 be a vertex of G. Observe that DIJK : CompSeq(G, s_1) is halting.

One can prove the following three propositions:

(24) For every finite real-weighted w-graph G and for every vertex s_1 of G holds (DIJK : CompSeq(G, s_1)).Lifespan() + 1 = card(G.reachableDFrom(s_1)).

(25) For every finite real-weighted w-graph G and for every vertex s_1 of G holds (DIJK : SSSP(G, s_1)).labeledV() = G.reachableDFrom(s_1).

(26) Let G be a finite nonnegative-weighted w-graph, s_1 be a vertex of G, and G_2 be a induced w-subgraph of G, (DIJK : SSSP(G, s_1)).labeledV(), (DIJK : SSSP(G, s_1)).labeledE(). Then

(i) G_2 is mincost d-tree rooted at s_1, and

(ii) for every vertex v of G such that $v \in G$.reachableDFrom(s_1) holds $v \in$ the vertices of G_2 and the G.mincost-d-path(s_1, v) = (the vlabel of DIJK : SSSP(G, s_1))(v).

4. Prim's Algorithm: preliminaries

The non empty finite subset WGraphSelectors of $\mathbb{N}$ is defined as follows:

(Def. 9) WGraphSelectors = {VertexSelector, EdgeSelector, SourceSelector, TargetSelector, WeightSelector}.

Let G be a w-graph. One can check that G.strict(WGraphSelectors) is graph-like and weighted.

Let G be a w-graph. The functor G.allWSubgraphs() yields a non empty set and is defined as follows:

(Def. 10) For every set x holds $x \in G$.allWSubgraphs() iff there exists a w-subgraph G_2 of G such that $x = G_2$ and $\operatorname{dom} G_2$ = WGraphSelectors.

Let G be a finite w-graph. One can check that G.allWSubgraphs() is finite.

Let G be a w-graph and let X be a non empty subset of G.allWSubgraphs(). We see that the element of X is a w-subgraph of G.

Let G be a finite real-weighted w-graph. The functor G.cost() yields a real number and is defined by:

(Def. 11) G.cost() $= \sum$(the weight of G).

The following propositions are true:

(27) For every set x holds $x \in$ WGraphSelectors iff $x =$ VertexSelector or $x =$ EdgeSelector or $x =$ SourceSelector or $x =$ TargetSelector or $x =$ WeightSelector.

(28) For every w-graph G holds WGraphSelectors $\subseteq \operatorname{dom} G$.

(29) For every w-graph G holds $G =_G G$.strict(WGraphSelectors) and the weight of $G =$ the weight of G.strict(WGraphSelectors).

(30) For every w-graph G holds dom(G.strict(WGraphSelectors)) $=$ WGraphSelectors.

(31) For every finite real-weighted w-graph G such that the edges of $G = \emptyset$ holds G.cost() $= 0$.

(32) Let G_1, G_2 be finite real-weighted w-graphs. Suppose the edges of $G_1 =$ the edges of G_2 and the weight of $G_1 =$ the weight of G_2. Then G_1.cost() $=$ G_2.cost().

(33) Let G_1 be a finite real-weighted w-graph, e be a set, and G_2 be a weighted subgraph of G_1 with edge e removed inheriting weight. If $e \in$ the edges of G_1, then G_1.cost() $= G_2$.cost() $+$ (the weight of G_1)(e).

(34) Let G be a finite real-weighted w-graph, V_1 be a non empty subset of the vertices of G, E_1 be a subset of G.edgesBetween(V_1), G_1 be a induced w-subgraph of G, V_1, E_1, e be a set, and G_2 be a induced w-subgraph of G, V_1, $E_1 \cup \{e\}$. If $e \notin E_1$ and $e \in G$.edgesBetween(V_1), then G_1.cost() $+$ (the weight of G)(e) $= G_2$.cost().

5. Prim's Minimum Weight Spanning Tree Algorithm: definitions

Let G be a real-weighted wv-graph. The functor PRIM : NextBestEdges(G) yields a subset of the edges of G and is defined by the condition (Def. 12).

(Def. 12) Let e_1 be a set. Then $e_1 \in$ PRIM : NextBestEdges(G) if and only if the following conditions are satisfied:

(i) e_1 joins a vertex from G.labeledV() and a vertex from (the vertices of G) $\setminus$ G.labeledV() in G, and

(ii) for every set e_2 such that e_2 joins a vertex from G.labeledV() and a vertex from (the vertices of G) $\setminus$ G.labeledV() in G holds (the weight of G)(e_1) $\leq$ (the weight of G)(e_2).

Let G be a real-weighted w-graph. The functor PRIM : Init(G) yields a real-wev wev-graph and is defined by:

(Def. 13) PRIM : Init(G) $= G$.set(VLabelSelector, choose(the vertices of G) $\dot{\longmapsto}$ 1).set(ELabelSelector, $\emptyset$).

Let G be a real-wev wev-graph. The functor PRIM : Step(G) yielding a real-wev wev-graph is defined by:

(Def. 14) $\text{PRIM} : \text{Step}(G) = \begin{cases} G, \text{ if PRIM : NextBestEdges}(G) = \emptyset, \\ (G.\text{labelEdge}(e, 1)).\text{labelVertex}((\text{the target of } G) \\ \quad (e), 1), \text{ if PRIM : NextBestEdges}(G) \neq \emptyset \text{ and} \\ \quad (\text{the source of } G)(e) \in G.\text{labeledV}(), \\ (G.\text{labelEdge}(e, 1)).\text{labelVertex}((\text{the source of } G) \\ \quad (e), 1), \text{ otherwise.} \end{cases}$

Let G be a real-weighted w-graph. The functor PRIM : CompSeq(G) yields a real-wev wev-graph sequence and is defined by:

(Def. 15) PRIM : CompSeq(G)$.\!\!\to\!0$ = PRIM : Init(G) and for every natural number n holds PRIM : CompSeq(G)$.\!\!\to\!(n + 1)$ = PRIM : Step((PRIM : CompSeq(G)$.\!\!\to\! n$)).

Let G be a finite real-weighted w-graph. One can check that PRIM : CompSeq(G) is finite.

Let G be a real-weighted w-graph. The functor PRIM : MST(G) yielding a real-wev wev-graph is defined as follows:

(Def. 16) PRIM : MST(G) = (PRIM : CompSeq(G)).Result().

Let G be a finite real-weighted w-graph. Observe that PRIM : MST(G) is finite.

Let G_1 be a finite real-weighted w-graph and let n be a natural number. Observe that every subgraph of G_1 induced by (PRIM : CompSeq(G_1)$.\!\!\to\! n$).labeledV() is connected.

Let G_1 be a finite real-weighted w-graph and let n be a natural number. Note that every subgraph of G_1 induced by (PRIM : CompSeq(G_1)$.\!\!\to\! n$).labeledV() and (PRIM : CompSeq(G_1)$.\!\!\to\! n$).labeledE() is connected.

Let G be a finite connected real-weighted w-graph. Observe that there exists a w-subgraph of G which is spanning and tree-like.

Let G_1 be a finite connected real-weighted w-graph and let G_2 be a spanning tree-like w-subgraph of G_1. We say that G_2 is min-cost if and only if:

(Def. 17) For every spanning tree-like w-subgraph G_3 of G_1 holds G_2.cost() $\leq$ G_3.cost().

Let G_1 be a finite connected real-weighted w-graph. One can check that there exists a spanning tree-like w-subgraph of G_1 which is min-cost.

Let G be a finite connected real-weighted w-graph. A minimum spanning tree of G is a min-cost spanning tree-like w-subgraph of G.

6. Prim's Minimum Weight Spanning Tree Algorithm: theorems

One can prove the following propositions:

(35) Let G_1, G_2 be finite connected real-weighted w-graphs and G_3 be a w-subgraph of G_1. Suppose G_3 is a minimum spanning tree of G_1 and

$G_1 =_G G_2$ and the weight of G_1 = the weight of G_2. Then G_3 is a minimum spanning tree of G_2.

(36) Let G be a finite connected real-weighted w-graph, G_1 be a minimum spanning tree of G, and G_2 be a w-graph. Suppose $G_1 =_G G_2$ and the weight of G_1 = the weight of G_2. Then G_2 is a minimum spanning tree of G.

(37) Let G be a real-weighted w-graph. Then

(i) $G =_G$ PRIM : Init(G),

(ii) the weight of G = the weight of PRIM : Init(G),

(iii) the elabel of PRIM : Init(G) = $\emptyset$, and

(iv) the vlabel of PRIM : Init(G) = choose(the vertices of G)$\dot{\longmapsto}$1.

(38) For every real-weighted w-graph G holds (PRIM : Init(G)).labeledV() = {choose(the vertices of G)} and (PRIM : Init(G)).labeledE() = $\emptyset$.

(39) For every real-wev wev-graph G such that PRIM : NextBestEdges(G) $\neq \emptyset$ there exists a vertex v of G such that $v \notin G$.labeledV() and PRIM : Step(G) = (G.labelEdge(choose(PRIM : NextBestEdges(G)), 1)).labelVertex(v, 1).

(40) For every real-wev wev-graph G holds $G =_G$ PRIM : Step(G) and the weight of G = the weight of PRIM : Step(G) and G.labeledE() $\subseteq$ (PRIM : Step(G)).labeledE() and G.labeledV() $\subseteq$ (PRIM : Step(G)).labeledV().

(41) Let G be a finite real-weighted w-graph and n be a natural number. Then $G =_G$ PRIM : CompSeq(G)$\to n$ and the weight of PRIM : CompSeq(G)$\to n$ = the weight of G.

(42) Let G be a finite real-weighted w-graph and n be a natural number. Then (PRIM : CompSeq(G)$\to n$).labeledV() is a non empty subset of the vertices of G and (PRIM : CompSeq(G)$\to n$).labeledE() $\subseteq$ G.edgesBetween((PRIM : CompSeq(G)$\to n$).labeledV()).

(43) For every finite real-weighted w-graph G_1 and for every natural number n holds every subgraph of G_1 induced by PRIM : CompSeq(G_1)$\to n$.labeledV() and PRIM : CompSeq(G_1)$\to n$.labeledE() is connected.

(44) For every finite real-weighted w-graph G_1 and for every natural number n holds every subgraph of G_1 induced by PRIM : CompSeq(G_1)$\to n$.labeledV() is connected.

(45) For every finite real-weighted w-graph G and for every natural number n holds (PRIM : CompSeq(G)$\to n$).labeledV() $\subseteq$ G.reachableFrom(choose(the vertices of G)).

(46) Let G be a finite real-weighted w-graph and i, j be natural numbers. If $i \leq j$, then (PRIM : CompSeq(G)$\to i$).labeledV() $\subseteq$ (PRIM : CompSeq(G)$\to j$).labeledV() and (PRIM : CompSeq(G)$\to i$)

.labeledE() $\subseteq$ (PRIM : CompSeq(G)$\rightarrow j$).labeledE().

(47) Let G be a finite real-weighted w-graph and n be a natural number. Then PRIM : NextBestEdges((PRIM : CompSeq(G)$\rightarrow n$)) = $\emptyset$ if and only if (PRIM : CompSeq(G)$\rightarrow n$).labeledV() = G.reachableFrom(choose(the vertices of G)).

(48) Let G be a finite real-weighted w-graph and n be a natural number. Then card((PRIM : CompSeq(G)$\rightarrow n$).labeledV()) = min(n + 1, card(G.reachableFrom(choose(the vertices of G)))).

(49) For every finite real-weighted w-graph G holds PRIM : CompSeq(G) is halting and (PRIM : CompSeq(G)).Lifespan() + 1 = card(G.reachableFrom(choose(the vertices of G))).

(50) For every finite real-weighted w-graph G_1 and for every natural number n holds every subgraph of G_1 induced by PRIM : CompSeq(G_1)$\rightarrow n$.labeledV() and PRIM : CompSeq(G_1)$\rightarrow n$.labeledE() is tree-like.

(51) For every finite connected real-weighted w-graph G holds (PRIM : MST(G)).labeledV() = the vertices of G.

(52) For every finite connected real-weighted w-graph G and for every natural number n holds (PRIM : CompSeq(G)$\rightarrow n$).labeledE() $\subseteq$ (PRIM : MST(G)).labeledE().

(53) For every finite connected real-weighted w-graph G_1 holds every induced w-subgraph of G_1, PRIM : MST(G_1).labeledV(), PRIM : MST(G_1).labeledE() is a minimum spanning tree of G_1.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.
[6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[10] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[11] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[14] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[15] Gilbert Lee. Trees and Graph Components. *Formalized Mathematics*, 13(**2**):271–277, 2005.
[16] Gilbert Lee. Walks in Graphs. *Formalized Mathematics*, 13(**2**):253–269, 2005.
[17] Gilbert Lee. Weighted and Labeled Graphs. *Formalized Mathematics*, 13(**2**):279–293, 2005.
[18] Gilbert Lee and Piotr Rudnicki. Alternative graph structures. *Formalized Mathematics*, 13(**2**):235–252, 2005.
[19] Piotr Rudnicki. Little Bezout theorem (factor theorem). *Formalized Mathematics*, 12(**1**):49–58, 2004.
[20] Piotr Rudnicki and Andrzej Trybulec. Abian's fixed point theorem. *Formalized Mathematics*, 6(**3**):335–338, 1997.
[21] Piotr Rudnicki and Andrzej Trybulec. Multivariate polynomials with arbitrary number of variables. *Formalized Mathematics*, 9(**1**):95–110, 2001.
[22] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[23] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[24] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[25] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[26] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[27] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[28] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[29] Josef Urban. Basic facts about inaccessible and measurable cardinals. *Formalized Mathematics*, 9(**2**):323–329, 2001.
[30] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received February 22, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 305–314
University of Białystok, 2005

Correctnesss of Ford-Fulkerson's Maximum Flow Algorithm[1]

Gilbert Lee[2]
University of Victoria, Victoria, Canada

Summary. We define and prove correctness of Ford-Fulkerson's maximum network flow algorithm at the level of graph manipulations.

MML identifier: `GLIB_005`, version: `7.5.01 4.39.921`

The articles [23], [21], [25], [22], [11], [27], [9], [7], [5], [13], [1], [24], [26], [8], [3], [4], [20], [18], [28], [10], [2], [6], [17], [12], [16], [14], [19], and [15] provide the notation and terminology for this paper.

1. Preliminary Theorems

Let x be a set and let y be a real number. One can verify that $x \dot{\longmapsto} y$ is real-yielding.

Let x be a set and let y be a natural number. One can verify that $x \longmapsto y$ is natural-yielding.

Let f, g be real-yielding functions. Observe that $f {+}{\cdot} g$ is real-yielding.

2. Preliminary Defintions for Ford-Fulkerson Flow Algorithm

Let G be a e-graph. We say that G is complete-elabeled if and only if:

(Def. 1) dom (the elabel of G) = the edges of G.

[1]This work has been partially supported by NSERC, Alberta Ingenuity Fund and iCORE.
[2]Part of author's MSc work.

ISSN 1426–2630

Let G be a graph and let X be a many sorted set indexed by the edges of G. Observe that G.set(ELabelSelector, X) is complete-elabeled.

Let G be a graph, let Y be a non empty set, and let X be a function from the edges of G into Y. One can check that G.set(ELabelSelector, X) is complete-elabeled.

Let G_1 be a e-graph sequence. We say that G_1 is complete-elabeled if and only if:

(Def. 2) For every natural number x holds $G_1 \rightarrow x$ is complete-elabeled.

Let G be a w-graph. We say that G is natural-weighted if and only if:

(Def. 3) The weight of G is natural-yielding.

Let G be a e-graph. We say that G is natural-elabeled if and only if:

(Def. 4) The elabel of G is natural-yielding.

Let G_1 be a w-graph sequence. We say that G_1 is natural-weighted if and only if:

(Def. 5) For every natural number x holds $G_1 \rightarrow x$ is natural-weighted.

Let G_1 be a e-graph sequence. We say that G_1 is natural-elabeled if and only if:

(Def. 6) For every natural number x holds $G_1 \rightarrow x$ is natural-elabeled.

One can verify that every w-graph which is natural-weighted is also nonnegative-weighted.

Let us observe that every e-graph which is natural-elabeled is also real-elabeled.

One can verify that there exists a wev-graph which is finite, trivial, tree-like, natural-weighted, natural-elabeled, complete-elabeled, and real-vlabeled.

One can verify that there exists a wev-graph sequence which is finite, natural-weighted, real-wev, natural-elabeled, and complete-elabeled.

Let G_1 be a complete-elabeled e-graph sequence and let x be a natural number. Note that $G_1 \rightarrow x$ is complete-elabeled.

Let G_1 be a natural-elabeled e-graph sequence and let x be a natural number. One can verify that $G_1 \rightarrow x$ is natural-elabeled.

Let G_1 be a natural-weighted w-graph sequence and let x be a natural number. One can verify that $G_1 \rightarrow x$ is natural-weighted.

Let G be a natural-weighted w-graph. One can check that the weight of G is natural-yielding.

Let G be a natural-elabeled e-graph. Note that the elabel of G is natural-yielding.

Let G be a complete-elabeled e-graph. Then the elabel of G is a many sorted set indexed by the edges of G.

Let G be a natural-weighted w-graph and let X be a set. Note that G.set(ELabelSelector, X) is natural-weighted and G.set(VLabelSelector, X) is natural-weighted.

Let G be a graph and let X be a natural-yielding many sorted set indexed by the edges of G. Observe that G.set(ELabelSelector, X) is natural-elabeled.

Let G be a finite real-weighted real-elabeled complete-elabeled we-graph and let s_1, s_2 be sets. We say that G has valid flow from s_1 to s_2 if and only if the conditions (Def. 7) are satisfied.

(Def. 7)(i) s_1 is a vertex of G,

(ii) s_2 is a vertex of G,

(iii) for every set e such that $e \in$ the edges of G holds $0 \leq$ (the elabel of G)(e) and (the elabel of G)(e) $\leq$ (the weight of G)(e), and

(iv) for every vertex v of G such that $v \neq s_1$ and $v \neq s_2$ holds $\sum$((the elabel of G)$\restriction v$.edgesIn()) $= \sum$((the elabel of G)$\restriction v$.edgesOut()).

Let G be a finite real-weighted real-elabeled complete-elabeled we-graph and let s_1, s_2 be sets. Let us assume that G has valid flow from s_1 to s_2. The functor G.flow(s_1, s_2) yields a real number and is defined as follows:

(Def. 8) G.flow(s_1, s_2) $= \sum$((the elabel of G)$\restriction G$.edgesInto($\{s_2\}$))$-\sum$((the elabel of G)$\restriction G$.edgesOutOf($\{s_2\}$)).

Let G be a finite real-weighted real-elabeled complete-elabeled we-graph and let s_1, s_2 be sets. We say that G has maximum flow from s_1 to s_2 if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) G has valid flow from s_1 to s_2, and

(ii) for every finite real-weighted real-elabeled complete-elabeled we-graph G_2 such that $G_2 =_G G$ and the weight of $G =$ the weight of G_2 and G_2 has valid flow from s_1 to s_2 holds G_2.flow(s_1, s_2) $\leq G$.flow(s_1, s_2).

Let G be a real-weighted real-elabeled wev-graph and let e be a set. We say that e is forward labeling in G if and only if the conditions (Def. 10) are satisfied.

(Def. 10)(i) $e \in$ the edges of G,

(ii) (the source of G)(e) $\in G$.labeledV(),

(iii) (the target of G)(e) $\notin G$.labeledV(), and

(iv) (the elabel of G)(e) $<$ (the weight of G)(e).

Let G be a real-elabeled ev-graph and let e be a set. We say that e is backward labeling in G if and only if:

(Def. 11) $e \in$ the edges of G and (the target of G)(e) $\in G$.labeledV() and (the source of G)(e) $\notin G$.labeledV() and $0 <$ (the elabel of G)(e).

Let G be a real-weighted real-elabeled we-graph and let W be a walk of G. We say that W is augmenting if and only if the condition (Def. 12) is satisfied.

(Def. 12) Let n be an odd natural number such that $n < \operatorname{len} W$. Then

(i) if $W(n+1)$ joins $W(n)$ to $W(n+2)$ in G, then (the elabel of G)($W(n+1)$) $<$ (the weight of G)($W(n+1)$), and

(ii) if $W(n+1)$ does not join $W(n)$ to $W(n+2)$ in G, then $0 <$ (the elabel of G)$(W(n+1))$.

Let G be a real-weighted real-elabeled we-graph. One can check that every walk of G which is trivial is also augmenting.

Let G be a real-weighted real-elabeled we-graph. Note that there exists a path of G which is vertex-distinct and augmenting.

Let G be a real-weighted real-elabeled we-graph, let W be an augmenting walk of G, and let m, n be natural numbers. Note that W.cut(m, n) is augmenting.

Next we state two propositions:

(1) Let G_3, G_2 be real-weighted real-elabeled we-graphs, W_1 be a walk of G_3, and W_2 be a walk of G_2. Suppose that

(i) W_1 is augmenting,
(ii) $G_3 =_G G_2$,
(iii) the weight of G_3 = the weight of G_2,
(iv) the elabel of G_3 = the elabel of G_2, and
(v) $W_1 = W_2$.

Then W_2 is augmenting.

(2) Let G be a real-weighted real-elabeled we-graph, W be an augmenting walk of G, and e, v be sets. Suppose that

(i) $v \notin W$.vertices(), and
(ii) e joins W.last() to v in G and (the elabel of G)$(e) <$ (the weight of G)(e) or e joins v to W.last() in G and $0 <$ (the elabel of G)(e).

Then W.addEdge(e) is augmenting.

3. Algorithm for Finding Augmenting Path in a Graph

Let G be a real-weighted real-elabeled wev-graph. The functor AP : NextBestEdges(G) yielding a subset of the edges of G is defined as follows:

(Def. 13) For every set e holds $e \in$ AP : NextBestEdges(G) iff e is forward labeling in G or backward labeling in G.

Let G be a real-weighted real-elabeled wev-graph. The functor AP : Step(G) yields a real-weighted real-elabeled wev-graph and is defined by:

(Def. 14)
$$\text{AP : Step}(G) = \begin{cases} G, \text{ if AP : NextBestEdges}(G) = \emptyset, \\ G.\text{labelVertex}((\text{the source of } G)(e), e), \\ \quad \text{if AP : NextBestEdges}(G) \neq \emptyset \text{ and (the source of } G) \\ \quad (e) \notin G.\text{labeledV}(), \\ G.\text{labelVertex}((\text{the target of } G)(e), e), \text{ otherwise.} \end{cases}$$

Let G be a finite real-weighted real-elabeled wev-graph. One can check that AP : Step(G) is finite.

Let G be a real-weighted real-elabeled we-graph and let s_1 be a vertex of G. The functor $\mathrm{AP:CompSeq}(G, s_1)$ yielding a real-weighted real-elabeled wev-graph sequence is defined as follows:

(Def. 15) $\mathrm{AP:CompSeq}(G, s_1)\rightarrow 0 = G.\mathrm{set}(\mathrm{VLabelSelector}, s_1 \dot{\longmapsto} 1)$ and for every natural number n holds $\mathrm{AP:CompSeq}(G, s_1)\rightarrow(n+1) = \mathrm{AP:Step}((\mathrm{AP:CompSeq}(G, s_1)\rightarrow n))$.

Let G be a finite real-weighted real-elabeled we-graph and let s_1 be a vertex of G. One can check that $\mathrm{AP:CompSeq}(G, s_1)$ is finite.

The following three propositions are true:

(3) Let G be a real-weighted real-elabeled we-graph and s_1 be a vertex of G. Then

(i) $G =_G \mathrm{AP:CompSeq}(G, s_1)\rightarrow 0$,

(ii) the weight of G = the weight of $\mathrm{AP:CompSeq}(G, s_1)\rightarrow 0$,

(iii) the elabel of G = the elabel of $\mathrm{AP:CompSeq}(G, s_1)\rightarrow 0$, and

(iv) $(\mathrm{AP:CompSeq}(G, s_1)\rightarrow 0).\mathrm{labeledV}() = \{s_1\}$.

(4) Let G be a real-weighted real-elabeled we-graph, s_1 be a vertex of G, and i, j be natural numbers. If $i \leq j$, then $(\mathrm{AP:CompSeq}(G, s_1)\rightarrow i).\mathrm{labeledV}() \subseteq$ $(\mathrm{AP:CompSeq}(G, s_1)\rightarrow j).\mathrm{labeledV}()$.

(5) Let G be a real-weighted real-elabeled we-graph, s_1 be a vertex of G, and n be a natural number. Then $G =_G \mathrm{AP:CompSeq}(G, s_1)\rightarrow n$ and the weight of G = the weight of $\mathrm{AP:CompSeq}(G, s_1)\rightarrow n$ and the elabel of G = the elabel of $\mathrm{AP:CompSeq}(G, s_1)\rightarrow n$.

Let G be a real-weighted real-elabeled we-graph and let s_1 be a vertex of G. The functor $\mathrm{AP:FindAugPath}(G, s_1)$ yielding a real-weighted real-elabeled wev-graph is defined as follows:

(Def. 16) $\mathrm{AP:FindAugPath}(G, s_1) = (\mathrm{AP:CompSeq}(G, s_1)).\mathrm{Result}()$.

We now state two propositions:

(6) For every finite real-weighted real-elabeled we-graph G and for every vertex s_1 of G holds $\mathrm{AP:CompSeq}(G, s_1)$ is halting.

(7) Let G be a finite real-weighted real-elabeled we-graph, s_1 be a vertex of G, n be a natural number, and v be a set. If $v \in (\mathrm{AP:CompSeq}(G, s_1)\rightarrow n).\mathrm{labeledV}()$, then (the vlabel of $\mathrm{AP:CompSeq}(G, s_1)\rightarrow n)(v)$ = (the vlabel of $\mathrm{AP:FindAugPath}(G, s_1))(v)$.

Let G be a finite real-weighted real-elabeled we-graph and let s_1, s_2 be vertices of G. The functor $\mathrm{AP:GetAugPath}(G, s_1, s_2)$ yielding a vertex-distinct augmenting path of G is defined by:

(Def. 17)(i) $\mathrm{AP:GetAugPath}(G, s_1, s_2)$ is walk from s_1 to s_2 and for every even natural number n such that $n \in \mathrm{dom}\,\mathrm{AP:GetAugPath}(G, s_1, s_2)$ holds $(\mathrm{AP:GetAugPath}(G, s_1, s_2))(n)$ = (the vlabel of $\mathrm{AP:FindAugPath}(G, s_1))$

((AP : GetAugPath(G, s_1, s_2))($n + 1$)) if $s_2 \in$ (AP : FindAugPath(G, s_1)).labeledV(),

(ii) AP : GetAugPath(G, s_1, s_2) = G.walkOf(s_1), otherwise.

Next we state three propositions:

(8) Let G be a real-weighted real-elabeled we-graph, s_1 be a vertex of G, n be a natural number, and v be a set. Suppose $v \in$ (AP : CompSeq(G, s_1)$\rightarrow n$).labeledV(). Then there exists a path P of G such that P is augmenting and walk from s_1 to v and P.vertices() $\subseteq$ (AP : CompSeq(G, s_1)$\rightarrow n$).labeledV().

(9) Let G be a finite real-weighted real-elabeled we-graph, s_1 be a vertex of G, and v be a set. Then $v \in$ (AP : FindAugPath(G, s_1)).labeledV() if and only if there exists a path of G which is augmenting and walk from s_1 to v.

(10) Let G be a finite real-weighted real-elabeled we-graph and s_1 be a vertex of G. Then $s_1 \in$ (AP : FindAugPath(G, s_1)).labeledV() and $G =_G$ AP : FindAugPath(G, s_1) and the weight of G = the weight of AP : FindAugPath(G, s_1) and the elabel of G = the elabel of AP : FindAugPath(G, s_1).

4. Definition of Ford-Fulkerson Maximum Flow Algorithm

Let G be a real-weighted real-elabeled we-graph and let W be an augmenting walk of G. The functor W.flowSeq() yields a finite sequence of elements of $\mathbb{R}$ and is defined by the conditions (Def. 18).

(Def. 18)(i) dom(W.flowSeq()) = dom(W.edgeSeq()), and

(ii) for every natural number n such that $n \in$ dom(W.flowSeq()) holds if $W(2\cdot n)$ joins $W(2\cdot n-1)$ to $W(2\cdot n+1)$ in G, then W.flowSeq()(n) = (the weight of G)($W(2 \cdot n)$) − (the elabel of G)($W(2 \cdot n)$) and if $W(2 \cdot n)$ does not join $W(2 \cdot n - 1)$ to $W(2 \cdot n + 1)$ in G, then W.flowSeq()(n) = (the elabel of G)($W(2 \cdot n)$).

Let G be a real-weighted real-elabeled we-graph and let W be an augmenting walk of G. The functor W.tolerance() yielding a real number is defined as follows:

(Def. 19)(i) W.tolerance() $\in$ rng(W.flowSeq()) and for every real number k such that $k \in$ rng(W.flowSeq()) holds W.tolerance() $\leq k$ if W is non trivial,

(ii) W.tolerance() = 0, otherwise.

Let G be a natural-weighted natural-elabeled we-graph and let W be an augmenting walk of G. Then W.tolerance() is a natural number.

Let G be a real-weighted real-elabeled we-graph and let P be an augmenting path of G. The functor FF : PushFlow(G, P) yielding a many sorted set indexed by the edges of G is defined by the conditions (Def. 20).

(Def. 20)(i) For every set e such that $e \in$ the edges of G and $e \notin P$.edges() holds (FF : PushFlow$(G, P))(e) =$ (the elabel of $G)(e)$, and

(ii) for every odd natural number n such that $n < \operatorname{len} P$ holds if $P(n+1)$ joins $P(n)$ to $P(n+2)$ in G, then (FF : PushFlow$(G, P))(P(n+1)) =$ (the elabel of $G)(P(n+1)) + P$.tolerance() and if $P(n+1)$ does not join $P(n)$ to $P(n+2)$ in G, then (FF : PushFlow$(G, P))(P(n+1)) =$ (the elabel of $G)(P(n+1)) - P$.tolerance().

Let G be a real-weighted real-elabeled we-graph and let P be an augmenting path of G. Observe that FF : PushFlow(G, P) is real-yielding.

Let G be a natural-weighted natural-elabeled we-graph and let P be an augmenting path of G. Note that FF : PushFlow(G, P) is natural-yielding.

Let G be a real-weighted real-elabeled we-graph and let P be an augmenting path of G. The functor FF : AugmentPath(G, P) yielding a real-weighted real-elabeled complete-elabeled we-graph is defined as follows:

(Def. 21) FF : AugmentPath$(G, P) = G$.set(ELabelSelector, FF : PushFlow(G, P)).

Let G be a finite real-weighted real-elabeled we-graph and let P be an augmenting path of G. Observe that FF : AugmentPath(G, P) is finite.

Let G be a finite nonnegative-weighted real-elabeled we-graph and let P be an augmenting path of G. Note that FF : AugmentPath(G, P) is nonnegative-weighted.

Let G be a finite natural-weighted natural-elabeled we-graph and let P be an augmenting path of G. Note that FF : AugmentPath(G, P) is natural-weighted and natural-elabeled.

Let G be a finite real-weighted real-elabeled complete-elabeled we-graph and let s_2, s_1 be vertices of G. The functor FF : Step(G, s_1, s_2) yields a finite real-weighted real-elabeled complete-elabeled we-graph and is defined by:

(Def. 22) $$\text{FF : Step}(G, s_1, s_2) = \begin{cases} \text{FF : AugmentPath}(G, \text{AP : GetAugPath}(G, s_1, \\ \quad s_2)), \text{ if } s_2 \in (\text{AP : FindAugPath}(G, s_1)) \\ \quad .\text{labeledV}(), \\ G, \text{ otherwise.} \end{cases}$$

Let G be a finite nonnegative-weighted real-elabeled complete-elabeled we-graph and let s_1, s_2 be vertices of G. One can check that FF : Step(G, s_1, s_2) is nonnegative-weighted.

Let G be a finite natural-weighted natural-elabeled complete-elabeled we-graph and let s_1, s_2 be vertices of G. One can verify that FF : Step(G, s_1, s_2) is natural-weighted and natural-elabeled.

Let G be a finite real-weighted w-graph and let s_1, s_2 be vertices of G. The functor FF : CompSeq(G, s_1, s_2) yields a finite real-weighted real-elabeled complete-elabeled we-graph sequence and is defined by the conditions (Def. 23).

(Def. 23)(i) FF : CompSeq$(G, s_1, s_2){\rightarrow}0 = G$.set(ELabelSelector, (the edges of $G) \longmapsto 0$), and

(ii) for every natural number n there exist vertices s'_1, s'_2 of FF : CompSeq$(G, s_1, s_2) \rightarrow n$ such that $s'_1 = s_1$ and $s'_2 = s_2$ and FF : CompSeq$(G, s_1, s_2) \rightarrow (n + 1) =$ FF : Step(FF : CompSeq$(G, s_1, s_2) \rightarrow n, s'_1, s'_2)$.

Let G be a finite nonnegative-weighted w-graph and let s_2, s_1 be vertices of G. One can verify that FF : CompSeq(G, s_1, s_2) is nonnegative-weighted.

Let G be a finite natural-weighted w-graph and let s_2, s_1 be vertices of G. One can check that FF : CompSeq(G, s_1, s_2) is natural-weighted and natural-elabeled.

Let G be a finite real-weighted w-graph and let s_2, s_1 be vertices of G. The functor FF : MaxFlow(G, s_1, s_2) yields a finite real-weighted real-elabeled complete-elabeled we-graph and is defined by:

(Def. 24) FF : MaxFlow$(G, s_1, s_2) =$ (FF : CompSeq(G, s_1, s_2)).Result().

5. Theorems for Ford-Fulkerson Maximum Flow Algorithm

One can prove the following propositions:

(11) Let G be a finite real-weighted real-elabeled complete-elabeled we-graph, s_1, s_2 be sets, and V be a subset of the vertices of G. Suppose G has valid flow from s_1 to s_2 and $s_1 \in V$ and $s_2 \notin V$. Then G.flow$(s_1, s_2) = \sum$((the elabel of G)$\upharpoonright G$.edgesDBetween(V, (the vertices of G) $\setminus V$)) $-\sum$((the elabel of G)$\upharpoonright G$.edgesDBetween((the vertices of G) $\setminus V, V$)).

(12) Let G be a finite real-weighted real-elabeled complete-elabeled we-graph, s_1, s_2 be sets, and V be a subset of the vertices of G. Suppose G has valid flow from s_1 to s_2 and $s_1 \in V$ and $s_2 \notin V$. Then G.flow$(s_1, s_2) \leq \sum$((the weight of G)$\upharpoonright G$.edgesDBetween(V, (the vertices of G) $\setminus V$)).

(13) Let G be a real-weighted real-elabeled we-graph and P be an augmenting path of G. Then $G =_G$ FF : AugmentPath(G, P) and the weight of $G =$ the weight of FF : AugmentPath(G, P).

(14) Let G be a finite real-weighted real-elabeled we-graph and W be an augmenting walk of G. If W is non trivial, then $0 < W$.tolerance().

(15) Let G be a finite real-weighted real-elabeled complete-elabeled we-graph, s_1, s_2 be sets, and P be an augmenting path of G. Suppose $s_1 \neq s_2$ and G has valid flow from s_1 to s_2 and P is walk from s_1 to s_2. Then FF : AugmentPath(G, P) has valid flow from s_1 to s_2.

(16) Let G be a finite real-weighted real-elabeled complete-elabeled we-graph, s_1, s_2 be sets, and P be an augmenting path of G. Suppose $s_1 \neq s_2$ and G has valid flow from s_1 to s_2 and P is walk from s_1 to s_2. Then (G.flow(s_1, s_2)) + P.tolerance() = FF : AugmentPath(G, P).flow(s_1, s_2).

(17) Let G be a finite real-weighted w-graph, s_1, s_2 be vertices of G, and n be a natural number. Then $\mathrm{FF:CompSeq}(G, s_1, s_2) \to n =_G G$ and the weight of $G =$ the weight of $\mathrm{FF:CompSeq}(G, s_1, s_2) \to n$.

(18) Let G be a finite nonnegative-weighted w-graph, s_1, s_2 be vertices of G, and n be a natural number. If $s_1 \neq s_2$, then $\mathrm{FF:CompSeq}(G, s_1, s_2) \to n$ has valid flow from s_1 to s_2.

(19) For every finite natural-weighted w-graph G and for all vertices s_1, s_2 of G such that $s_1 \neq s_2$ holds $\mathrm{FF:CompSeq}(G, s_1, s_2)$ is halting.

(20) Let G be a finite real-weighted real-elabeled complete-elabeled we-graph and s_1, s_2 be sets such that $s_1 \neq s_2$ and G has valid flow from s_1 to s_2 and there exists no augmenting path of G which is walk from s_1 to s_2. Then G has maximum flow from s_1 to s_2.

(21) Let G be a finite real-weighted w-graph and s_1, s_2 be vertices of G. Then $G =_G \mathrm{FF:MaxFlow}(G, s_1, s_2)$ and the weight of $G =$ the weight of $\mathrm{FF:MaxFlow}(G, s_1, s_2)$.

(22) Let G be a finite natural-weighted w-graph and s_1, s_2 be vertices of G. If $s_2 \neq s_1$, then $\mathrm{FF:MaxFlow}(G, s_1, s_2)$ has maximum flow from s_1 to s_2.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.
[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[6] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[10] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[11] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[12] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[13] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[14] Gilbert Lee. Walks in Graphs. *Formalized Mathematics*, 13(**2**):253–269, 2005.
[15] Gilbert Lee. Weighted and Labeled Graphs. *Formalized Mathematics*, 13(**2**):279–293, 2005.
[16] Gilbert Lee and Piotr Rudnicki. Alternative graph structures. *Formalized Mathematics*, 13(**2**):235–252, 2005.
[17] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Formalized Mathematics*, 5(**2**):167–172, 1996.
[18] Piotr Rudnicki. Little Bezout theorem (factor theorem). *Formalized Mathematics*, 12(**1**):49–58, 2004.

[19] Piotr Rudnicki and Andrzej Trybulec. Abian's fixed point theorem. *Formalized Mathematics*, 6(**3**):335–338, 1997.
[20] Piotr Rudnicki and Andrzej Trybulec. Multivariate polynomials with arbitrary number of variables. *Formalized Mathematics*, 9(**1**):95–110, 2001.
[21] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
[22] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[23] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[24] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[25] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[26] Josef Urban. Basic facts about inaccessible and measurable cardinals. *Formalized Mathematics*, 9(**2**):323–329, 2001.
[27] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[28] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 22, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 315–323
University of Białystok, 2005

Properties of Connected Subsets of the Real Line

Artur Korniłowicz[1]
Institute of Computer Science
University of Białystok
Sosnowa 64, 15-887 Białystok, Poland

MML identifier: RCOMP_3, version: 7.5.01 4.39.921

The papers [31], [36], [3], [37], [27], [18], [9], [38], [10], [22], [14], [4], [34], [5], [39], [1], [33], [30], [2], [23], [21], [6], [20], [35], [29], [24], [28], [40], [17], [13], [12], [26], [15], [8], [11], [16], [19], [25], [32], and [7] provide the notation and terminology for this paper.

1. Preliminaries

Let X be a non empty set. Observe that Ω_X is non empty.

Let us observe that every subspace of the metric space of real numbers is real-membered.

Let S be a real-membered 1-sorted structure. One can check that the carrier of S is real-membered.

One can check that there exists a real-membered set which is non empty, finite, lower bounded, and upper bounded.

We now state three propositions:

(1) For every non empty lower bounded real-membered set X and for every closed subset Y of $\mathbb{R}$ such that $X \subseteq Y$ holds $\inf X \in Y$.

(2) For every non empty upper bounded real-membered set X and for every closed subset Y of $\mathbb{R}$ such that $X \subseteq Y$ holds $\sup X \in Y$.

(3) For all subsets X, Y of $\mathbb{R}$ holds $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$.

[1]The paper was written during the author's post-doctoral fellowship granted by Shinshu University, Japan.

ISSN 1426–2630

2. Intervals

In the sequel a, b, r, s are real numbers.

Let us consider r, s. One can check the following observations:

* $[r,s[$ is bounded,
* $]r,s]$ is bounded, and
* $]r,s[$ is bounded.

Let us consider r, s. One can verify the following observations:

* $[r,s]$ is connected,
* $[r,s[$ is connected,
* $]r,s]$ is connected, and
* $]r,s[$ is connected.

Let us observe that there exists a subset of $\mathbb{R}$ which is open, bounded, connected, and non empty.

One can prove the following propositions:

(4) If $r < s$, then $\inf[r,s[= r$.

(5) If $r < s$, then $\sup[r,s[= s$.

(6) If $r < s$, then $\inf]r,s] = r$.

(7) If $r < s$, then $\sup]r,s] = s$.

(8) If $a \leq b$ or $r \leq s$ and if $[a,b] = [r,s]$, then $a = r$ and $b = s$.

(9) If $a < b$ or $r < s$ and if $]a,b[=]r,s[$, then $a = r$ and $b = s$.

(10) If $a < b$ or $r < s$ and if $]a,b] =]r,s]$, then $a = r$ and $b = s$.

(11) If $a < b$ or $r < s$ and if $[a,b[= [r,s[$, then $a = r$ and $b = s$.

(12) If $a < b$ and $[a,b[\subseteq [r,s]$, then $r \leq a$ and $b \leq s$.

(13) If $a < b$ and $[a,b[\subseteq [r,s[$, then $r \leq a$ and $b \leq s$.

(14) If $a < b$ and $]a,b] \subseteq [r,s]$, then $r \leq a$ and $b \leq s$.

(15) If $a < b$ and $]a,b] \subseteq]r,s]$, then $r \leq a$ and $b \leq s$.

3. Halflines

One can prove the following propositions:

(16) $[a,b]^{\mathrm{c}} =]-\infty,a[\cup]b,+\infty[$.

(17) $]a,b[^{\mathrm{c}} =]-\infty,a] \cup [b,+\infty[$.

(18) $[a,b[^{\mathrm{c}} =]-\infty,a[\cup [b,+\infty[$.

(19) $]a,b]^{\mathrm{c}} =]-\infty,a] \cup]b,+\infty[$.

(20) If $a \leq b$, then $[a,b] \cap (]-\infty,a] \cup [b,+\infty[) = \{a,b\}$.

Let us consider a. One can verify the following observations:

* $]-\infty, a]$ is non lower bounded, upper bounded, and connected,
* $]-\infty, a[$ is non lower bounded, upper bounded, and connected,
* $[a, +\infty[$ is lower bounded, non upper bounded, and connected, and
* $]a, +\infty[$ is lower bounded, non upper bounded, and connected.

The following propositions are true:

(21) $\sup]-\infty, a] = a$.

(22) $\sup]-\infty, a[= a$.

(23) $\inf[a, +\infty[= a$.

(24) $\inf]a, +\infty[= a$.

4. Connectedness

Let us observe that $\Omega_{\mathbb{R}}$ is connected, non lower bounded, and non upper bounded.

One can prove the following propositions:

(25) For every bounded connected subset X of $\mathbb{R}$ such that $\inf X \in X$ and $\sup X \in X$ holds $X = [\inf X, \sup X]$.

(26) For every bounded subset X of $\mathbb{R}$ such that $\inf X \notin X$ holds $X \subseteq]\inf X, \sup X]$.

(27) For every bounded connected subset X of $\mathbb{R}$ such that $\inf X \notin X$ and $\sup X \in X$ holds $X =]\inf X, \sup X]$.

(28) For every bounded subset X of $\mathbb{R}$ such that $\sup X \notin X$ holds $X \subseteq [\inf X, \sup X[$.

(29) For every bounded connected subset X of $\mathbb{R}$ such that $\inf X \in X$ and $\sup X \notin X$ holds $X = [\inf X, \sup X[$.

(30) For every bounded subset X of $\mathbb{R}$ such that $\inf X \notin X$ and $\sup X \notin X$ holds $X \subseteq]\inf X, \sup X[$.

(31) For every non empty bounded connected subset X of $\mathbb{R}$ such that $\inf X \notin X$ and $\sup X \notin X$ holds $X =]\inf X, \sup X[$.

(32) For every subset X of $\mathbb{R}$ such that X is upper bounded holds $X \subseteq]-\infty, \sup X]$.

(33) For every connected subset X of $\mathbb{R}$ such that X is not lower bounded and X is upper bounded and $\sup X \in X$ holds $X =]-\infty, \sup X]$.

(34) For every subset X of $\mathbb{R}$ such that X is upper bounded and $\sup X \notin X$ holds $X \subseteq]-\infty, \sup X[$.

(35) For every non empty connected subset X of $\mathbb{R}$ such that X is not lower bounded and X is upper bounded and $\sup X \notin X$ holds $X =]-\infty, \sup X[$.

(36) For every subset X of $\mathbb{R}$ such that X is lower bounded holds $X \subseteq [\inf X, +\infty[$.

(37) For every connected subset X of $\mathbb{R}$ such that X is lower bounded and X is not upper bounded and $\inf X \in X$ holds $X = [\inf X, +\infty[$.

(38) For every subset X of $\mathbb{R}$ such that X is lower bounded and $\inf X \notin X$ holds $X \subseteq]\inf X, +\infty[$.

(39) For every non empty connected subset X of $\mathbb{R}$ such that X is lower bounded and X is not upper bounded and $\inf X \notin X$ holds $X =]\inf X, +\infty[$.

(40) For every connected subset X of $\mathbb{R}$ such that X is not upper bounded and X is not lower bounded holds $X = \mathbb{R}$.

(41) Let X be a connected subset of $\mathbb{R}$. Then X is empty or $X = \mathbb{R}$ or there exists a such that $X =]-\infty, a]$ or there exists a such that $X =]-\infty, a[$ or there exists a such that $X = [a, +\infty[$ or there exists a such that $X =]a, +\infty[$ or there exist a, b such that $a \leq b$ and $X = [a, b]$ or there exist a, b such that $a < b$ and $X = [a, b[$ or there exist a, b such that $a < b$ and $X =]a, b]$ or there exist a, b such that $a < b$ and $X =]a, b[$.

(42) For every non empty connected subset X of $\mathbb{R}$ such that $r \notin X$ holds $r \leq \inf X$ or $\sup X \leq r$.

(43) Let X, Y be non empty bounded connected subsets of $\mathbb{R}$. Suppose $\inf X \leq \inf Y$ and $\sup Y \leq \sup X$ and if $\inf X = \inf Y$ and $\inf Y \in Y$, then $\inf X \in X$ and if $\sup X = \sup Y$ and $\sup Y \in Y$, then $\sup X \in X$. Then $Y \subseteq X$.

Let us observe that there exists a subset of $\mathbb{R}$ which is open, closed, connected, non empty, and non bounded.

5. $\mathbb{R}^1$

Next we state several propositions:

(44) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $a \leq b$ and $X = [a, b]$ holds $\mathrm{Fr}\, X = \{a, b\}$.

(45) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $a < b$ and $X =]a, b[$ holds $\mathrm{Fr}\, X = \{a, b\}$.

(46) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $a < b$ and $X = [a, b[$ holds $\mathrm{Fr}\, X = \{a, b\}$.

(47) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $a < b$ and $X =]a, b]$ holds $\mathrm{Fr}\, X = \{a, b\}$.

(48) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $X = [a, b]$ holds $\mathrm{Int}\, X =]a, b[$.

(49) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $X =]a, b[$ holds $\mathrm{Int}\, X =]a, b[$.

(50) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $X = [a, b[$ holds $\mathrm{Int}\, X =]a, b[$.

(51) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $X =]a, b]$ holds $\mathrm{Int}\, X =]a, b[$.

Let X be a convex subset of $\mathbb{R}^{\mathbf{1}}$. Observe that $\mathbb{R}^{\mathbf{1}}\restriction X$ is convex.

Let A be a connected subset of $\mathbb{R}$. One can check that $R^1 A$ is convex.

We now state the proposition

(52) Let X be a subset of $\mathbb{R}^{\mathbf{1}}$ and Y be a subset of $\mathbb{R}$. If $X = Y$, then X is connected iff Y is connected.

6. Topology of Closed Intervals

Let us consider r. Note that $[r, r]_{\mathrm{T}}$ is trivial.

The following four propositions are true:

(53) If $r \leq s$, then every subset of $[r, s]_{\mathrm{T}}$ is a bounded subset of $\mathbb{R}$.

(54) If $r \leq s$, then for every subset X of $[r, s]_{\mathrm{T}}$ such that $X = [a, b[$ and $r < a$ and $b \leq s$ holds $\operatorname{Int} X =]a, b[$.

(55) If $r \leq s$, then for every subset X of $[r, s]_{\mathrm{T}}$ such that $X =]a, b]$ and $r \leq a$ and $b < s$ holds $\operatorname{Int} X =]a, b[$.

(56) Let X be a subset of $[r, s]_{\mathrm{T}}$ and Y be a subset of $\mathbb{R}$. If $X = Y$, then X is connected iff Y is connected.

Let T be a topological space. Observe that there exists a subset of T which is open, closed, and connected.

Let T be a non empty connected topological space. Observe that there exists a subset of T which is non empty, open, closed, and connected.

We now state the proposition

(57) Suppose $r \leq s$. Let X be an open connected subset of $[r, s]_{\mathrm{T}}$. Then
(i) X is empty, or
(ii) $X = [r, s]$, or
(iii) there exists a real number a such that $r < a$ and $a \leq s$ and $X = [r, a[$, or
(iv) there exists a real number a such that $r \leq a$ and $a < s$ and $X =]a, s]$, or
(v) there exist real numbers a, b such that $r \leq a$ and $a < b$ and $b \leq s$ and $X =]a, b[$.

7. Minimal Cover of Intervals

Next we state three propositions:

(58) Let T be a 1-sorted structure and F be a family of subsets of T. Then F is a cover of T if and only if F is a cover of Ω_T.

(59) Let T be a 1-sorted structure, F be a finite family of subsets of T, and F_1 be a family of subsets of T. Suppose F is a cover of T and $F_1 = F \setminus \{X; X$

ranges over subsets of T: $X \in F \wedge \bigvee_{Y : \text{subset of } T} (Y \in F \wedge X \subseteq Y \wedge X \neq Y)\}$. Then F_1 is a cover of T.

(60) Let S be a trivial non empty 1-sorted structure, s be a point of S, and F be a family of subsets of S. If F is a cover of S, then $\{s\} \in F$.

Let T be a topological structure and let F be a family of subsets of T. We say that F is connected if and only if:

(Def. 1) For every subset X of T such that $X \in F$ holds X is connected.

Let T be a topological space. Note that there exists a family of subsets of T which is non empty, open, closed, and connected.

In the sequel n, m are natural numbers and F is a family of subsets of $[r, s]_{\mathrm{T}}$.

The following two propositions are true:

(61) Let L be a topological space and G, G_1 be families of subsets of L. Suppose G is a cover of L and finite. Let A_1 be a set such that $G_1 = G \setminus \{X; X$ ranges over subsets of L: $X \in G \wedge \bigvee_{Y : \text{subset of } L} (Y \in G \wedge X \subseteq Y \wedge X \neq Y)\}$ and $A_1 = \{C; C$ ranges over families of subsets of L: C is a cover of $L \wedge C \subseteq G_1\}$. Then A_1 has the lower Zorn property w.r.t. $\subseteq_{(A_1)}$.

(62) Let L be a topological space and G, A_1 be sets. Suppose $A_1 = \{C; C$ ranges over families of subsets of L: C is a cover of $L \wedge C \subseteq G\}$. Let M be a set. Suppose M is minimal in $\subseteq_{(A_1)}$ and $M \in \mathrm{field}(\subseteq_{(A_1)})$. Let A_4 be a subset of L. Suppose $A_4 \in M$. Then it is not true that there exist subsets A_2, A_3 of L such that $A_2 \in M$ and $A_3 \in M$ and $A_4 \subseteq A_2 \cup A_3$ and $A_4 \neq A_2$ and $A_4 \neq A_3$.

Let r, s be real numbers and let F be a family of subsets of $[r, s]_{\mathrm{T}}$. Let us assume that F is a cover of $[r, s]_{\mathrm{T}}$ F is open F is connected and $r \leq s$. A finite sequence of elements of $2^{\mathbb{R}}$ is said to be an interval cover of F if it satisfies the conditions (Def. 2).

(Def. 2)(i) $\mathrm{rng}\,\mathrm{it} \subseteq F$,

(ii) $\bigcup \mathrm{rng}\,\mathrm{it} = [r, s]$,

(iii) for every natural number n such that $1 \leq n$ holds if $n \leq \mathrm{len}\,\mathrm{it}$, then it_n is non empty and if $n + 1 \leq \mathrm{len}\,\mathrm{it}$, then $\inf(\mathrm{it}_n) \leq \inf(\mathrm{it}_{n+1})$ and $\sup(\mathrm{it}_n) \leq \sup(\mathrm{it}_{n+1})$ and $\inf(\mathrm{it}_{n+1}) < \sup(\mathrm{it}_n)$ and if $n + 2 \leq \mathrm{len}\,\mathrm{it}$, then $\sup(\mathrm{it}_n) \leq \inf(\mathrm{it}_{n+2})$,

(iv) if $[r, s] \in F$, then $\mathrm{it} = \langle [r, s] \rangle$, and

(v) if $[r, s] \notin F$, then there exists a real number p such that $r < p$ and $p \leq s$ and $\mathrm{it}(1) = [r, p[$ and there exists a real number p such that $r \leq p$ and $p < s$ and $\mathrm{it}(\mathrm{len}\,\mathrm{it}) =]p, s]$ and for every natural number n such that $1 < n$ and $n < \mathrm{len}\,\mathrm{it}$ there exist real numbers p, q such that $r \leq p$ and $p < q$ and $q \leq s$ and $\mathrm{it}(n) =]p, q[$.

We now state the proposition

(63) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$ and $[r, s] \in F$, then $\langle [r, s] \rangle$ is an interval cover of F.

In the sequel C denotes an interval cover of F.

One can prove the following propositions:

(64) Let F be a family of subsets of $[r, r]_{\mathrm{T}}$ and C be an interval cover of F. If F is a cover of $[r, r]_{\mathrm{T}}$, open, and connected, then $C = \langle [r, r] \rangle$.

(65) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$, then $1 \leq \operatorname{len} C$.

(66) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$ and $\operatorname{len} C = 1$, then $C = \langle [r, s] \rangle$.

(67) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$ and $n \in \operatorname{dom} C$ and $m \in \operatorname{dom} C$ and $n < m$, then $\inf(C_n) \leq \inf(C_m)$.

(68) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$ and $n \in \operatorname{dom} C$ and $m \in \operatorname{dom} C$ and $n < m$, then $\sup(C_n) \leq \sup(C_m)$.

(69) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$ and $1 \leq n$ and $n + 1 \leq \operatorname{len} C$, then $]\inf(C_{n+1}), \sup(C_n)[$ is non empty.

(70) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$, then $\inf(C_1) = r$.

(71) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$, then $r \in C_1$.

(72) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$, then $\sup(C_{\operatorname{len} C}) = s$.

(73) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$, then $s \in C_{\operatorname{len} C}$.

Let r, s be real numbers, let F be a family of subsets of $[r, s]_{\mathrm{T}}$, and let C be an interval cover of F. Let us assume that F is a cover of $[r, s]_{\mathrm{T}}$ F is open F is connected and $r \leq s$. A finite sequence of elements of $\mathbb{R}$ is said to be a chain of rivets in interval cover C if it satisfies the conditions (Def. 3).

(Def. 3)(i) $\operatorname{len} \mathrm{it} = \operatorname{len} C + 1$,

(ii) $\mathrm{it}(1) = r$,

(iii) $\mathrm{it}(\operatorname{len} \mathrm{it}) = s$, and

(iv) for every natural number n such that $1 \leq n$ and $n + 1 < \operatorname{len} \mathrm{it}$ holds $\mathrm{it}(n + 1) \in]\inf(C_{n+1}), \sup(C_n)[$.

In the sequel G denotes a chain of rivets in interval cover C.

One can prove the following propositions:

(74) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$, then $2 \leq \operatorname{len} G$.

(75) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$ and $\operatorname{len} C = 1$, then $G = \langle r, s \rangle$.

(76) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$ and $1 \leq n$ and $n + 1 < \operatorname{len} G$, then $G(n + 1) < \sup(C_n)$.

(77) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$ and $1 < n$ and $n \leq \operatorname{len} C$, then $\inf(C_n) < G(n)$.

(78) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$ and $1 \leq n$ and $n < \operatorname{len} C$, then $G(n) \leq \inf(C_{n+1})$.

(79) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r < s$, then G is increasing.

(80) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$ and $1 \leq n$ and $n < \operatorname{len} G$, then $[G(n), G(n+1)] \subseteq C(n)$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[4] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.

[5] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(**2**):265–267, 1990.

[6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[7] Józef Białas. Properties of fields. *Formalized Mathematics*, 1(**5**):807–812, 1990.

[8] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[11] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.

[12] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[13] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[14] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[15] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.

[16] Adam Grabowski and Artur Korniłowicz. Algebraic properties of homotopies. *Formalized Mathematics*, 12(**3**):251–260, 2004.

[17] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.

[18] Zbigniew Karno. On discrete and almost discrete topological spaces. *Formalized Mathematics*, 3(**2**):305–310, 1992.

[19] Artur Korniłowicz. The fundamental group of convex subspaces of $\mathcal{E}^n_{\mathrm{T}}$. *Formalized Mathematics*, 12(**3**):295–299, 2004.

[20] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.

[21] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.

[22] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[23] Jarosław Kotowicz. The limit of a real function at infinity. *Formalized Mathematics*, 2(**1**):17–28, 1991.

[24] Yatsuka Nakamura. Half open intervals in real numbers. *Formalized Mathematics*, 10(**1**):21–22, 2002.

[25] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.

[26] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.

[27] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[28] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[29] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[30] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[31] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[32] Andrzej Trybulec. Some lemmas for the Jordan curve theorem. *Formalized Mathematics*, 9(**3**):481–484, 2001.
[33] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(**4**):341–347, 2003.
[34] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[35] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[36] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[37] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[38] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[39] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.
[40] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received February 22, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 325–331
University of Białystok, 2005

The Fundamental Group of the Circle

Artur Korniłowicz[1]
Institute of Computer Science
University of Białystok
Sosnowa 64, 15-887 Białystok, Poland

Summary. The article formalizes a proof of the theorem counting the fundamental group of a circle taken from [18]. The last result describes an isomorphism between the additive group of integers and the fundamental group of a simple closed curve.

MML identifier: `TOPALG_5`, version: `7.5.01 4.39.921`

The notation and terminology used in this paper have been introduced in the following articles: [38], [10], [44], [2], [45], [33], [7], [1], [46], [9], [27], [8], [6], [40], [12], [3], [37], [19], [41], [26], [4], [34], [28], [32], [42], [36], [43], [20], [35], [39], [11], [30], [31], [29], [22], [21], [14], [13], [5], [15], [47], [16], [17], [25], [23], and [24].

1. Preliminaries

Let us observe that every element of $\mathbb{Z}^+$ is integer.

Let us note that $\mathbb{Z}^+$ is infinite.

Let S be an infinite 1-sorted structure. Note that the carrier of S is infinite.

In the sequel a, r, s denote real numbers.

One can prove the following propositions:

(1) If $r \leq s$ and $0 < a$, then for every point p of $[r, s]_{\rm M}$ holds $\mathrm{Ball}(p, a) = [r, s]$ or $\mathrm{Ball}(p, a) = [r, p+a[$ or $\mathrm{Ball}(p, a) =]p-a, s]$ or $\mathrm{Ball}(p, a) =]p-a, p+a[$.

[1]The paper was written during author's post-doctoral fellowship granted by Shinshu University, Japan.

ISSN 1426–2630

(2) Suppose $r \leq s$. Then there exists a basis B of $[r, s]_{\mathrm{T}}$ such that

(i) there exists a many sorted set f indexed by $[r, s]_{\mathrm{T}}$ such that for every point y of $[r, s]_{\mathrm{M}}$ holds $f(y) = \{\mathrm{Ball}(y, \frac{1}{n}); n$ ranges over natural numbers: $n \neq 0\}$ and $B = \bigcup f$, and

(ii) for every subset X of $[r, s]_{\mathrm{T}}$ such that $X \in B$ holds X is connected.

(3) For every topological structure T and for every subset A of T and for every point t of T such that $t \in A$ holds $\mathrm{skl}(t, A) \subseteq A$.

Let T be a topological space and let A be an open subset of T. Observe that $T \restriction A$ is open.

Next we state several propositions:

(4) Let T be a topological space, S be a subspace of T, A be a subset of T, and B be a subset of S. If $A = B$, then $T \restriction A = S \restriction B$.

(5) Let S, T be topological spaces, A, B be subsets of T, and C, D be subsets of S. Suppose that

(i) the topological structure of S = the topological structure of T,

(ii) $A = C$,

(iii) $B = D$, and

(iv) A and B are separated.

Then C and D are separated.

(6) Let S, T be topological spaces. Suppose the topological structure of S = the topological structure of T and S is connected. Then T is connected.

(7) Let S, T be topological spaces, A be a subset of S, and B be a subset of T. Suppose the topological structure of S = the topological structure of T and $A = B$ and A is connected. Then B is connected.

(8) Let S, T be non empty topological spaces, s be a point of S, t be a point of T, and A be a neighbourhood of s. Suppose the topological structure of S = the topological structure of T and $s = t$. Then A is a neighbourhood of t.

(9) Let S, T be non empty topological spaces, A be a subset of S, B be a subset of T, and N be a neighbourhood of A. Suppose the topological structure of S = the topological structure of T and $A = B$. Then N is a neighbourhood of B.

(10) Let S, T be non empty topological spaces, A, B be subsets of T, and f be a map from S into T. Suppose f is a homeomorphism and A is a component of B. Then $f^{-1}(A)$ is a component of $f^{-1}(B)$.

2. Local Connectedness

The following propositions are true:

(11) Let T be a non empty topological space, S be a non empty subspace of T, A be a non empty subset of T, and B be a non empty subset of S. If $A = B$ and A is locally connected, then B is locally connected.

(12) Let S, T be non empty topological spaces. Suppose the topological structure of $S =$ the topological structure of T and S is locally connected. Then T is locally connected.

(13) For every non empty topological space T holds T is locally connected iff Ω_T is locally connected.

(14) Let T be a non empty topological space and S be a non empty open subspace of T. If T is locally connected, then S is locally connected.

(15) Let S, T be non empty topological spaces. Suppose S and T are homeomorphic and S is locally connected. Then T is locally connected.

(16) Let T be a non empty topological space. Given a basis B of T such that let X be a subset of T. If $X \in B$, then X is connected. Then T is locally connected.

(17) If $r \leq s$, then $[r, s]_{\mathrm{T}}$ is locally connected.

Let us mention that $\mathbb{I}$ is locally connected.

Let A be a non empty open subset of $\mathbb{I}$. Observe that $\mathbb{I} \restriction A$ is locally connected.

3. Some Useful Functions

Let r be a real number. The functor $\mathrm{ExtendInt}\, r$ yielding a map from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def. 1) For every point x of $\mathbb{I}$ holds $(\mathrm{ExtendInt}\, r)(x) = r \cdot x$.

Let r be a real number. One can check that $\mathrm{ExtendInt}\, r$ is continuous.

Let r be a real number. Then $\mathrm{ExtendInt}\, r$ is a path from $R^1 0$ to $R^1 r$.

Let S, T, Y be non empty topological spaces, let H be a map from $[\!: S, T :\!]$ into Y, and let t be a point of T. The functor $\mathrm{Prj1}(t, H)$ yields a map from S into Y and is defined by:

(Def. 2) For every point s of S holds $(\mathrm{Prj1}(t, H))(s) = H(s, t)$.

Let S, T, Y be non empty topological spaces, let H be a map from $[\!: S, T :\!]$ into Y, and let s be a point of S. The functor $\mathrm{Prj2}(s, H)$ yields a map from T into Y and is defined as follows:

(Def. 3) For every point t of T holds $(\mathrm{Prj2}(s, H))(t) = H(s, t)$.

Let S, T, Y be non empty topological spaces, let H be a continuous map from $[\!: S, T :\!]$ into Y, and let t be a point of T. Note that $\mathrm{Prj1}(t, H)$ is continuous.

Let S, T, Y be non empty topological spaces, let H be a continuous map from $[\!: S, T :\!]$ into Y, and let s be a point of S. One can check that $\mathrm{Prj2}(s, H)$ is continuous.

One can prove the following two propositions:

(18) Let T be a non empty topological space, a, b be points of T, P, Q be paths from a to b, H be a homotopy between P and Q, and t be a point of $\mathbb{I}$. If H is continuous, then $\mathrm{Prj1}(t, H)$ is continuous.

(19) Let T be a non empty topological space, a, b be points of T, P, Q be paths from a to b, H be a homotopy between P and Q, and s be a point of $\mathbb{I}$. If H is continuous, then $\mathrm{Prj2}(s, H)$ is continuous.

Let r be a real number. The functor $\mathrm{cLoop}\, r$ yielding a map from $\mathbb{I}$ into TopUnitCircle 2 is defined as follows:

(Def. 4) For every point x of $\mathbb{I}$ holds $(\mathrm{cLoop}\, r)(x) = [\cos(2\cdot\pi\cdot r\cdot x), \sin(2\cdot\pi\cdot r\cdot x)]$.

The following proposition is true

(20) $\mathrm{cLoop}\, r = \mathrm{CircleMap}\cdot\mathrm{ExtendInt}\, r$.

Let n be an integer. Then $\mathrm{cLoop}\, n$ is a loop of $c[10]$.

4. Main Theorems

Next we state four propositions:

(21) Let U_1 be a family of subsets of TopUnitCircle 2. Suppose U_1 is a cover of TopUnitCircle 2 and open. Let Y be a non empty topological space, F be a continuous map from $[\!: Y, \mathbb{I} :\!]$ into TopUnitCircle 2, and y be a point of Y. Then there exists a non empty finite sequence T of elements of $\mathbb{R}$ such that

(i) $T(1) = 0$,

(ii) $T(\mathrm{len}\, T) = 1$,

(iii) T is increasing, and

(iv) there exists an open subset N of Y such that $y \in N$ and for every natural number i such that $i \in \mathrm{dom}\, T$ and $i+1 \in \mathrm{dom}\, T$ there exists a non empty subset U_2 of TopUnitCircle 2 such that $U_2 \in U_1$ and $F^{\circ}[\!: N, [T(i), T(i+1)] :\!] \subseteq U_2$.

(22) Let Y be a non empty topological space, F be a map from $[\!: Y, \mathbb{I} :\!]$ into TopUnitCircle 2, and F_1 be a map from $[\!: Y, \mathrm{Sspace}(0_{\mathbb{I}}) :\!]$ into $\mathbb{R}^{\mathbf{1}}$. Suppose F is continuous and F_1 is continuous and $F\restriction[\!:\text{the carrier of } Y,\ \{0\} :\!] = \mathrm{CircleMap}\cdot F_1$. Then there exists a map G from $[\!: Y, \mathbb{I} :\!]$ into $\mathbb{R}^{\mathbf{1}}$ such that

(i) G is continuous,

(ii) $F = \mathrm{CircleMap}\cdot G$,

(iii) $G\restriction[\!:\text{the carrier of } Y,\ \{0\} :\!] = F_1$, and

(iv) for every map H from $[\!: Y, \mathbb{I} :\!]$ into $\mathbb{R}^{\mathbf{1}}$ such that H is continuous and $F = \mathrm{CircleMap}\cdot H$ and $H\restriction[\!:\text{the carrier of } Y,\ \{0\} :\!] = F_1$ holds $G = H$.

(23) Let x_0, y_0 be points of TopUnitCircle 2, x_1 be a point of $\mathbb{R}^{\mathbf{1}}$, and f be a path from x_0 to y_0. Suppose $x_1 \in \mathrm{CircleMap}^{-1}(\{x_0\})$. Then there exists a map f_1 from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that

(i) $f_1(0) = x_1$,

(ii) $f = \mathrm{CircleMap} \cdot f_1$,

(iii) f_1 is continuous, and

(iv) for every map f_2 from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that f_2 is continuous and $f = \mathrm{CircleMap} \cdot f_2$ and $f_2(0) = x_1$ holds $f_1 = f_2$.

(24) Let x_0, y_0 be points of TopUnitCircle 2, P, Q be paths from x_0 to y_0, F be a homotopy between P and Q, and x_1 be a point of $\mathbb{R}^{\mathbf{1}}$. Suppose P, Q are homotopic and $x_1 \in \mathrm{CircleMap}^{-1}(\{x_0\})$. Then there exists a point y_1 of $\mathbb{R}^{\mathbf{1}}$ and there exist paths P_1, Q_1 from x_1 to y_1 and there exists a homotopy F_1 between P_1 and Q_1 such that P_1, Q_1 are homotopic and $F = \mathrm{CircleMap} \cdot F_1$ and $y_1 \in \mathrm{CircleMap}^{-1}(\{y_0\})$ and for every homotopy F_2 between P_1 and Q_1 such that $F = \mathrm{CircleMap} \cdot F_2$ holds $F_1 = F_2$.

The map Ciso from $\mathbb{Z}^+$ into $\pi_1(\mathrm{TopUnitCircle}\,2, c[10])$ is defined by:

(Def. 5) For every integer n holds $(\mathrm{Ciso})(n) = [\mathrm{cLoop}\,n]_{\mathrm{EqRel}(\mathrm{TopUnitCircle}\,2, c[10])}$.

One can prove the following proposition

(25) For every integer i and for every path f from $R^1 0$ to $R^1 i$ holds $(\mathrm{Ciso})(i) = [\mathrm{CircleMap} \cdot f]_{\mathrm{EqRel}(\mathrm{TopUnitCircle}\,2, c[10])}$.

Ciso is a homomorphism from $\mathbb{Z}^+$ to $\pi_1(\mathrm{TopUnitCircle}\,2, c[10])$.

Let us mention that Ciso is one-to-one and onto.

We now state two propositions:

(26) Ciso is isomorphism.

(27) Let S be a subspace of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve and x be a point of S. Then $\mathbb{Z}^+$ and $\pi_1(S, x)$ are isomorphic.

Let S be a subspace of $\mathcal{E}^2_{\mathrm{T}}$ satisfying conditions of simple closed curve and let x be a point of S. Note that $\pi_1(S, x)$ is infinite.

References

[1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[6] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[11] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[13] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[14] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[15] Adam Grabowski. Introduction to the homotopy theory. *Formalized Mathematics*, 6(**4**):449–454, 1997.
[16] Adam Grabowski. Properties of the product of compact topological spaces. *Formalized Mathematics*, 8(**1**):55–59, 1999.
[17] Adam Grabowski and Artur Korniłowicz. Algebraic properties of homotopies. *Formalized Mathematics*, 12(**3**):251–260, 2004.
[18] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.
[19] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[20] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[21] Zbigniew Karno. Separated and weakly separated subspaces of topological spaces. *Formalized Mathematics*, 2(**5**):665–674, 1991.
[22] Zbigniew Karno. Maximal discrete subspaces of almost discrete topological spaces. *Formalized Mathematics*, 4(**1**):125–135, 1993.
[23] Artur Korniłowicz. The fundamental group of convex subspaces of $\mathcal{E}^n_{\mathrm{T}}$. *Formalized Mathematics*, 12(**3**):295–299, 2004.
[24] Artur Korniłowicz and Yasunari Shidama. Some properties of circles on the plane. *Formalized Mathematics*, 13(**1**):117–124, 2005.
[25] Artur Korniłowicz, Yasunari Shidama, and Adam Grabowski. The fundamental group. *Formalized Mathematics*, 12(**3**):261–268, 2004.
[26] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[27] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[28] Yatsuka Nakamura. Half open intervals in real numbers. *Formalized Mathematics*, 10(**1**):21–22, 2002.
[29] Yatsuka Nakamura and Andrzej Trybulec. Components and unions of components. *Formalized Mathematics*, 5(**4**):513–517, 1996.
[30] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.
[31] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.
[32] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[33] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.
[34] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[35] Alexander Yu. Shibakov and Andrzej Trybulec. The Cantor set. *Formalized Mathematics*, 5(**2**):233–236, 1996.
[36] Dariusz Surowik. Cyclic groups and some of their properties – part I. *Formalized Mathematics*, 2(**5**):623–627, 1991.
[37] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[38] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[39] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[40] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[41] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[42] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[43] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(**4**):573–578, 1991.

[44] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[45] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[46] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[47] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

Received February 22, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 333–336
University of Białystok, 2005

Brouwer Fixed Point Theorem for Disks on the Plane

Artur Korniłowicz[1]
Institute of Computer Science
University of Białystok
Sosnowa 64, 15-887 Białystok
Poland

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. The article formalizes the proof of Brouwer's Fixed Point Theorem for 2-dimensional disks. Assuming, on the contrary, that the theorem is false, we show that a circle is a retract of a disk. Next, using the retraction, we prove that any loop in the circle is homotopic to the constant loop what contradicts with infiniteness of the fundamental group of a circle, see [15].

MML identifier: `BROUWER`, version: `7.5.01 4.39.921`

The terminology and notation used in this paper are introduced in the following papers: [26], [9], [29], [2], [22], [28], [30], [6], [8], [7], [5], [4], [12], [3], [25], [16], [23], [21], [20], [27], [11], [13], [14], [18], [17], [19], [10], [1], and [24].

In this paper n is a natural number, a, r are real numbers, and x is a point of $\mathcal{E}^n_{\rm T}$.

Let S, T be non empty topological spaces. The functor $\mathrm{DiffElems}(S,T)$ yielding a subset of $[\!:S, T:\!]$ is defined by:

(Def. 1) $\mathrm{DiffElems}(S,T) = \{\langle s, t\rangle; s$ ranges over points of S, t ranges over points of T: $s \neq t\}$.

One can prove the following proposition

(1) Let S, T be non empty topological spaces and x be a set. Then $x \in \mathrm{DiffElems}(S,T)$ if and only if there exists a point s of S and there exists a point t of T such that $x = \langle s, t\rangle$ and $s \neq t$.

[1]The paper was written during the first author's post-doctoral fellowship granted by Shinshu University, Japan.

ISSN 1426–2630

Let S be a non trivial non empty topological space and let T be a non empty topological space. One can check that DiffElems(S,T) is non empty.

Let S be a non empty topological space and let T be a non trivial non empty topological space. Note that DiffElems(S,T) is non empty.

We now state the proposition

(2) $\overline{\mathrm{Ball}}(x,0)=\{x\}$.

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a real number. The functor Tdisk(x,r) yields a subspace of $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 2) $\mathrm{Tdisk}(x,r)=(\mathcal{E}^n_{\mathrm{T}})\restriction\overline{\mathrm{Ball}}(x,r)$.

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a non negative real number. Note that Tdisk(x,r) is non empty.

We now state the proposition

(3) The carrier of $\mathrm{Tdisk}(x,r)=\overline{\mathrm{Ball}}(x,r)$.

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a real number. Note that Tdisk(x,r) is convex.

We adopt the following convention: n denotes a natural number, r denotes a non negative real number, and s, t, x denote points of $\mathcal{E}^n_{\mathrm{T}}$.

One can prove the following two propositions:

(4) If $s\neq t$ and s is a point of Tdisk(x,r) and s is not a point of Tcircle(x,r), then there exists a point e of Tcircle(x,r) such that $\{e\}=\mathrm{halfline}(s,t)\cap\mathrm{Sphere}(x,r)$.

(5) Suppose $s\neq t$ and $s\in$ the carrier of Tcircle(x,r) and t is a point of Tdisk(x,r). Then there exists a point e of Tcircle(x,r) such that $e\neq s$ and $\{s,e\}=\mathrm{halfline}(s,t)\cap\mathrm{Sphere}(x,r)$.

Let n be a non empty natural number, let o be a point of $\mathcal{E}^n_{\mathrm{T}}$, let s, t be points of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a non negative real number. Let us assume that s is a point of Tdisk(o,r), and t is a point of Tdisk(o,r) and $s\neq t$. The functor HC(s,t,o,r) yields a point of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 3) $\mathrm{HC}(s,t,o,r)\in\mathrm{halfline}(s,t)\cap\mathrm{Sphere}(o,r)$ and $\mathrm{HC}(s,t,o,r)\neq s$.

In the sequel n is a non empty natural number and s, t, o are points of $\mathcal{E}^n_{\mathrm{T}}$.

We now state three propositions:

(6) If s is a point of Tdisk(o,r) and t is a point of Tdisk(o,r) and $s\neq t$, then HC(s,t,o,r) is a point of Tcircle(o,r).

(7) Let S, T, O be elements of $\mathcal{R}^n$. Suppose $S=s$ and $T=t$ and $O=o$. Suppose s is a point of Tdisk(o,r) and t is a point of Tdisk(o,r) and $s\neq t$ and $a=\frac{-|(t-s,s-o)|+\sqrt{|(t-s,s-o)|^{\mathbf{2}}-\sum^2(T-S)\cdot(\sum^2(S-O)-r^{\mathbf{2}})}}{\sum^2(T-S)}$. Then $\mathrm{HC}(s,t,o,r)=(1-a)\cdot s+a\cdot t$.

(8) Let r_1, r_2, s_1, s_2 be real numbers and s, t, o be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that s is a point of Tdisk(o,r) and t is a point of Tdisk(o,r) and

$s \neq t$ and $r_1 = t_{\mathbf{1}} - s_{\mathbf{1}}$ and $r_2 = t_{\mathbf{2}} - s_{\mathbf{2}}$ and $s_1 = s_{\mathbf{1}} - o_{\mathbf{1}}$ and $s_2 = s_{\mathbf{2}} - o_{\mathbf{2}}$ and $a = \frac{-(s_1 \cdot r_1 + s_2 \cdot r_2) + \sqrt{(s_1 \cdot r_1 + s_2 \cdot r_2)^{\mathbf{2}} - (r_1{}^{\mathbf{2}} + r_2{}^{\mathbf{2}}) \cdot ((s_1{}^{\mathbf{2}} + s_2{}^{\mathbf{2}}) - r^{\mathbf{2}})}}{r_1{}^{\mathbf{2}} + r_2{}^{\mathbf{2}}}$. Then $\mathrm{HC}(s, t, o, r) = [s_{\mathbf{1}} + a \cdot r_1, s_{\mathbf{2}} + a \cdot r_2]$.

Let n be a non empty natural number, let o be a point of $\mathcal{E}^n_{\mathrm{T}}$, let r be a non negative real number, let x be a point of $\mathrm{Tdisk}(o, r)$, and let f be a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. Let us assume that x is not a fixpoint of f. The functor $\mathrm{HC}(x, f)$ yielding a point of $\mathrm{Tcircle}(o, r)$ is defined as follows:

(Def. 4) There exist points y, z of $\mathcal{E}^n_{\mathrm{T}}$ such that $y = x$ and $z = f(x)$ and $\mathrm{HC}(x, f) = \mathrm{HC}(z, y, o, r)$.

The following two propositions are true:

(9) Let x be a point of $\mathrm{Tdisk}(o, r)$ and f be a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. If x is not a fixpoint of f and x is a point of $\mathrm{Tcircle}(o, r)$, then $\mathrm{HC}(x, f) = x$.

(10) Let r be a positive real number, o be a point of $\mathcal{E}^2_{\mathrm{T}}$, and Y be a non empty subspace of $\mathrm{Tdisk}(o, r)$. If $Y = \mathrm{Tcircle}(o, r)$, then Y is not a retract of $\mathrm{Tdisk}(o, r)$.

Let n be a non empty natural number, let r be a non negative real number, let o be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let f be a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. The functor BR-map f yielding a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tcircle}(o, r)$ is defined as follows:

(Def. 5) For every point x of $\mathrm{Tdisk}(o, r)$ holds $(\text{BR-map}\, f)(x) = \mathrm{HC}(x, f)$.

The following propositions are true:

(11) Let o be a point of $\mathcal{E}^n_{\mathrm{T}}$, x be a point of $\mathrm{Tdisk}(o, r)$, and f be a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. If x is not a fixpoint of f and x is a point of $\mathrm{Tcircle}(o, r)$, then $(\text{BR-map}\, f)(x) = x$.

(12) For every continuous map f from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$ such that f has no fixpoint holds $\text{BR-map}\, f \restriction \mathrm{Sphere}(o, r) = \mathrm{id}_{\mathrm{Tcircle}(o, r)}$.

(13) Let r be a positive real number, o be a point of $\mathcal{E}^2_{\mathrm{T}}$, and f be a continuous map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. If f has no fixpoint, then BR-map f is continuous.

(14) For every non negative real number r and for every point o of $\mathcal{E}^2_{\mathrm{T}}$ holds every continuous map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$ has a fixpoint.

(15) Let r be a non negative real number, o be a point of $\mathcal{E}^2_{\mathrm{T}}$, and f be a continuous map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. Then there exists a point x of $\mathrm{Tdisk}(o, r)$ such that $f(x) = x$.

References

[1] Kanchun and Yatsuka Nakamura. The inner product of finite sequences and of points of n-dimensional topological space. *Formalized Mathematics*, 11(**2**):179–183, 2003.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[5] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[10] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[11] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[13] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[14] Adam Grabowski. Introduction to the homotopy theory. *Formalized Mathematics*, 6(**4**):449–454, 1997.
[15] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.
[16] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[17] Artur Korniłowicz. The fundamental group of convex subspaces of $\mathcal{E}^n_{\rm T}$. *Formalized Mathematics*, 12(**3**):295–299, 2004.
[18] Artur Korniłowicz and Yasunari Shidama. Intersections of intervals and balls in $\mathcal{E}^n_{\rm T}$. *Formalized Mathematics*, 12(**3**):301–306, 2004.
[19] Artur Korniłowicz and Yasunari Shidama. Some properties of circles on the plane. *Formalized Mathematics*, 13(**1**):117–124, 2005.
[20] Michał Muzalewski. Categories of groups. *Formalized Mathematics*, 2(**4**):563–571, 1991.
[21] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[22] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.
[23] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[24] Piotr Rudnicki and Andrzej Trybulec. Abian's fixed point theorem. *Formalized Mathematics*, 6(**3**):335–338, 1997.
[25] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[26] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[27] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[28] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[29] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[30] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 22, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 337–345
University of Białystok, 2005

Stirling Numbers of the Second Kind

Karol Pąk
Institute of Mathematics
University of Białystok
Akademicka 2, 15-267 Białystok, Poland

Summary. In this paper we define Stirling numbers of the second kind by cardinality of certain functional classes so that

$$S(n,k) = \{f \text{ where } f \text{ is function of } n,k : f \text{ is onto increasing}\}$$

After that we show basic properties of this number in order to prove recursive dependence of Stirling number of the second kind. Consecutive theorems are introduced to prove formula

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k-1}(-1)^i\binom{k}{i}(k-i)^n$$

where $k \leq n$.

MML identifier: `STIRL2_1`, version: `7.5.01 4.39.921`

The papers [18], [9], [21], [14], [23], [6], [24], [2], [3], [8], [10], [1], [22], [7], [11], [20], [16], [19], [4], [5], [13], [12], [17], and [15] provide the terminology and notation for this paper.

For simplicity, we adopt the following convention: k, l, m, n, i, j denote natural numbers, K, N denote non empty subsets of $\mathbb{N}$, K_1, N_1, M_1 denote subsets of $\mathbb{N}$, and X, Y denote sets.

Let us consider k. Then $\{k\}$ is a subset of $\mathbb{N}$. Let us consider l. Then $\{k,l\}$ is a subset of $\mathbb{N}$. Let us consider m. Then $\{k,l,m\}$ is a non empty subset of $\mathbb{N}$.

The following propositions are true:

(1) $\min N = \min^* N$.

(2) $\min(\min K, \min N) = \min(K \cup N)$.

(3) $\min(\min^* K_1, \min^* N_1) \leq \min^*(K_1 \cup N_1)$.

ISSN 1426–2630

(4) If $\min^* N_1 \notin N_1 \cap K_1$, then $\min^* N_1 = \min^*(N_1 \setminus K_1)$.

(5) $\min^*\{n\} = n$ and $\min\{n\} = n$.

(6) $\min^*\{n, k\} = \min(n, k)$ and $\min\{n, k\} = \min(n, k)$.

(7) $\min^*\{n, k, l\} = \min(n, \min(k, l))$.

(8) n is a subset of $\mathbb{N}$.

Let us consider n. One can verify that every element of n is natural.

We now state several propositions:

(9) If $N \subseteq n$, then $n - 1$ is a natural number.

(10) If $k \in n$, then $k \leq n - 1$ and $n - 1$ is a natural number.

(11) $\min^* n = 0$.

(12) If $N \subseteq n$, then $\min^* N \leq n - 1$.

(13) If $N \subseteq n$ and $N \neq \{n - 1\}$, then $\min^* N < n - 1$.

(14) If $N_1 \subseteq n$ and $n > 0$, then $\min^* N_1 \leq n - 1$.

In the sequel f, g are functions from n into k.

Let us consider n, X, let f be a function from n into X, and let x be a set. Then $f^{-1}(x)$ is a subset of $\mathbb{N}$.

Let us consider X, k, let f be a function from X into k, and let x be a set. Then $f(x)$ is an element of k.

Let us consider X, N_1, let f be a function from X into N_1, and let x be a set. One can verify that $f(x)$ is natural.

Let us consider n, k and let f be a function from n into k. We say that f is increasing if and only if:

(Def. 1) $n = 0$ iff $k = 0$ and for all l, m such that $l \in \operatorname{rng} f$ and $m \in \operatorname{rng} f$ and $l < m$ holds $\min^*(f^{-1}(\{l\})) < \min^*(f^{-1}(\{m\}))$.

We now state several propositions:

(15) If $n = 0$ and $k = 0$, then f is onto and increasing.

(16) If $n > 0$, then $\min^*(f^{-1}(\{m\})) \leq n - 1$.

(17) If f is onto, then $n \geq k$.

(18) If f is onto and increasing, then for every m such that $m < k$ holds $m \leq \min^*(f^{-1}(\{m\}))$.

(19) If f is onto and increasing, then for every m such that $m < k$ holds $\min^*(f^{-1}(\{m\})) \leq (n - k) + m$.

(20) If f is onto and increasing and $n = k$, then $f = \mathrm{id}_n$.

(21) If $f = \mathrm{id}_n$ and $n > 0$, then f is increasing.

(22) If $n = 0$ iff $k = 0$, then there exists a function from n into k which is increasing.

(23) If $n = 0$ iff $k = 0$ and $n \geq k$, then there exists a function from n into k which is onto and increasing.

The scheme *Sch1* deals with natural numbers $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

$\{f; f$ ranges over functions from $\mathcal{A}$ into $\mathcal{B} : \mathcal{P}[f]\}$ is finite

for all values of the parameters.

In the sequel f is a function from n into k.

One can prove the following propositions:

(24) For all n, k holds $\{f : f$ is onto and increasing$\}$ is finite.

(25) For all n, k holds $\overline{\overline{\{f : f \text{ is onto and increasing}\}}}$ is a natural number.

Let us consider n, k. The functor $n\,\mathrm{block}\,k$ yields a natural number and is defined by:

(Def. 2) $n\,\mathrm{block}\,k = \overline{\overline{\{f : f \text{ is onto and increasing}\}}}$.

Next we state several propositions:

(26) $n\,\mathrm{block}\,n = \mathbf{1}$.

(27) If $k \neq 0$, then $0\,\mathrm{block}\,k = 0$.

(28) $0\,\mathrm{block}\,k = \mathbf{1}$ iff $k = 0$.

(29) If $n < k$, then $n\,\mathrm{block}\,k = 0$.

(30) $n\,\mathrm{block}\,0 = \mathbf{1}$ iff $n = 0$.

(31) If $n \neq 0$, then $n\,\mathrm{block}\,0 = 0$.

(32) If $n \neq 0$, then $n\,\mathrm{block}\,1 = \mathbf{1}$.

(33) $1 \leq k$ and $k \leq n$ or $k = n$ iff $n\,\mathrm{block}\,k > 0$.

In the sequel x, y denote sets.

Now we present three schemes. The scheme *Sch2* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, a function $\mathcal{E}$ from $\mathcal{A}$ into $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function h from $\mathcal{C}$ into $\mathcal{D}$ such that $h{\restriction}\mathcal{A} = \mathcal{E}$ and for every x such that $x \in \mathcal{C} \setminus \mathcal{A}$ holds $h(x) = \mathcal{F}(x)$

provided the parameters satisfy the following conditions:

- For every x such that $x \in \mathcal{C} \setminus \mathcal{A}$ holds $\mathcal{F}(x) \in \mathcal{D}$,
- $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{D}$, and
- If $\mathcal{B}$ is empty, then $\mathcal{A}$ is empty.

The scheme *Sch3* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary predicate $\mathcal{P}$, and states that:

$$\overline{\overline{\{f; f \text{ ranges over functions from } \mathcal{A} \text{ into } \mathcal{B} : \mathcal{P}[f, \mathcal{A}, \mathcal{B}]\}}} = \overline{\overline{\{f; f \text{ ranges over functions from } \mathcal{C} \text{ into } \mathcal{D} : \mathcal{P}[f, \mathcal{C}, \mathcal{D}] \wedge \mathrm{rng}(f{\restriction}\mathcal{A}) \subseteq \mathcal{B} \wedge \bigwedge_x (x \in \mathcal{C} \setminus \mathcal{A} \Rightarrow f(x) = \mathcal{F}(x))\}}}$$

provided the following requirements are met:

- For every x such that $x \in \mathcal{C} \setminus \mathcal{A}$ holds $\mathcal{F}(x) \in \mathcal{D}$,
- $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{D}$,
- If $\mathcal{B}$ is empty, then $\mathcal{A}$ is empty, and

- Let f be a function from $\mathcal{C}$ into $\mathcal{D}$. Suppose that for every x such that $x \in \mathcal{C} \setminus \mathcal{A}$ holds $\mathcal{F}(x) = f(x)$. Then $\mathcal{P}[f, \mathcal{C}, \mathcal{D}]$ if and only if $\mathcal{P}[f{\restriction}\mathcal{A}, \mathcal{A}, \mathcal{B}]$.

The scheme *Sch4* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and a ternary predicate $\mathcal{P}$, and states that:

$$\overline{\overline{\{f; f \text{ ranges over functions from } \mathcal{A} \text{ into } \mathcal{B} : \mathcal{P}[f, \mathcal{A}, \mathcal{B}]\}}} = \overline{\overline{\{f; f \text{ ranges over functions from } \mathcal{A} \cup \{\mathcal{C}\} \text{ into } \mathcal{B} \cup \{\mathcal{D}\} : \mathcal{P}[f, \mathcal{A} \cup \{\mathcal{C}\}, \mathcal{B} \cup \{\mathcal{D}\}] \wedge \operatorname{rng}(f{\restriction}\mathcal{A}) \subseteq \mathcal{B} \wedge f(\mathcal{C}) = \mathcal{D}\}}}$$

provided the parameters meet the following conditions:

- If $\mathcal{B}$ is empty, then $\mathcal{A}$ is empty,
- $\mathcal{C} \notin \mathcal{A}$, and
- For every function f from $\mathcal{A} \cup \{\mathcal{C}\}$ into $\mathcal{B} \cup \{\mathcal{D}\}$ such that $f(\mathcal{C}) = \mathcal{D}$ holds $\mathcal{P}[f, \mathcal{A} \cup \{\mathcal{C}\}, \mathcal{B} \cup \{\mathcal{D}\}]$ iff $\mathcal{P}[f{\restriction}\mathcal{A}, \mathcal{A}, \mathcal{B}]$.

We now state several propositions:

(34) For every function f from $n + 1$ into $k + 1$ such that f is onto and increasing and $f^{-1}(\{f(n)\}) = \{n\}$ holds $f(n) = k$.

(35) For every function f from $n+1$ into k such that $k \neq 0$ and $f^{-1}(\{f(n)\}) \neq \{n\}$ there exists m such that $m \in f^{-1}(\{f(n)\})$ and $m \neq n$.

(36) Let f be a function from n into k and g be a function from $n + m$ into $k + l$. Suppose g is increasing and $f = g{\restriction}n$. Let given i, j. If $i \in \operatorname{rng} f$ and $j \in \operatorname{rng} f$ and $i < j$, then $\min^*(f^{-1}(\{i\})) < \min^*(f^{-1}(\{j\}))$.

(37) Let f be a function from $n+1$ into $k+1$. Suppose f is onto and increasing and $f^{-1}(\{f(n)\}) = \{n\}$. Then $\operatorname{rng}(f{\restriction}n) \subseteq k$ and for every function g from n into k such that $g = f{\restriction}n$ holds g is onto and increasing.

(38) Let f be a function from $n + 1$ into k and g be a function from n into k. Suppose f is onto and increasing and $f^{-1}(\{f(n)\}) \neq \{n\}$ and $f{\restriction}n = g$. Then g is onto and increasing.

(39) Let f be a function from n into k and g be a function from $n + 1$ into $k + m$. Suppose f is onto and increasing and $f = g{\restriction}n$. Let given i, j. If $i \in \operatorname{rng} g$ and $j \in \operatorname{rng} g$ and $i < j$, then $\min^*(g^{-1}(\{i\})) < \min^*(g^{-1}(\{j\}))$.

(40) Let f be a function from n into k and g be a function from $n + 1$ into $k + 1$. Suppose f is onto and increasing and $f = g{\restriction}n$ and $g(n) = k$. Then g is onto and increasing and $g^{-1}(\{g(n)\}) = \{n\}$.

(41) Let f be a function from n into k and g be a function from $n + 1$ into k. Suppose f is onto and increasing and $f = g{\restriction}n$ and $g(n) < k$. Then g is onto and increasing and $g^{-1}(\{g(n)\}) \neq \{n\}$.

In the sequel f_1 denotes a function from $n + 1$ into $k + 1$ and f denotes a function from n into k.

We now state the proposition

(42) $\overline{\overline{\{f_1 : f_1 \text{ is onto and increasing} \wedge f_1{}^{-1}(\{f_1(n)\}) = \{n\}\}}} =$

$\overline{\overline{\{f : f \text{ is onto and increasing}\}}}$.

In the sequel f' is a function from $n+1$ into k.

The following proposition is true

(43) For every l such that $l < k$ holds $\overline{\overline{\{f' : f' \text{ is onto and increasing} \ \wedge \ f'^{-1}(\{f'(n)\}) \neq \{n\} \ \wedge \ f'(n) = l\}}} = \overline{\overline{\{f : f \text{ is onto and increasing}\}}}$.

For simplicity, we adopt the following convention: D denotes a non empty set, F, G denote finite 0-sequences of D, F_1 denotes a finite 0-sequence of $\mathbb{N}$, b denotes a binary operation on D, and d, d_1, d_2 denote elements of D.

Let us consider D, F, b. Let us assume that b has a unity or $\operatorname{len} F \geq 1$. The functor $b \odot F$ yielding an element of D is defined as follows:

(Def. 3)(i) $b \odot F = \mathbf{1}_b$ if b has a unity and $\operatorname{len} F = 0$,

(ii) there exists a function f from $\mathbb{N}$ into D such that $f(0) = F(0)$ and for every n such that $n+1 < \operatorname{len} F$ holds $f(n+1) = b(f(n), F(n+1))$ and $b \odot F = f(\operatorname{len} F - 1)$, otherwise.

One can prove the following three propositions:

(44) $b \odot \langle d \rangle = d$.

(45) If b has a unity or $\operatorname{len} F > 0$, then $b \odot F \frown \langle d \rangle = b(b \odot F, d)$.

(46) If $F \neq \langle\rangle_D$, then there exist G, d such that $F = G \frown \langle d \rangle$.

The scheme *Sch5* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every finite 0-sequence F of $\mathcal{A}$ holds $\mathcal{P}[F]$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[\langle\rangle_{\mathcal{A}}]$, and
- For every finite 0-sequence F of $\mathcal{A}$ and for every element d of $\mathcal{A}$ such that $\mathcal{P}[F]$ holds $\mathcal{P}[F \frown \langle d \rangle]$.

Next we state the proposition

(47) If b is associative and if b has a unity or $\operatorname{len} F \geq 1$ and $\operatorname{len} G \geq 1$, then $b \odot F \frown G = b(b \odot F, b \odot G)$.

Let us consider D and let us consider d, d_1. Then $\langle d, d_1 \rangle$ is a finite 0-sequence of D. Let us consider d_2. Then $\langle d, d_1, d_2 \rangle$ is a finite 0-sequence of D.

The following propositions are true:

(48) $b \odot \langle d_1, d_2 \rangle = b(d_1, d_2)$.

(49) $b \odot \langle d, d_1, d_2 \rangle = b(b(d, d_1), d_2)$.

Let us consider F_1. The functor $\sum F_1$ yields a natural number and is defined by:

(Def. 4) $\sum F_1 = +_{\mathbb{N}} \odot F_1$.

Let us consider F_1, x. Then $F_1(x)$ is a natural number.

One can prove the following propositions:

(50) If for every n such that $n \in \operatorname{dom} F_1$ holds $F_1(n) \leq k$, then $\sum F_1 \leq \operatorname{len} F_1 \cdot k$.

(51) If for every n such that $n \in \operatorname{dom} F_1$ holds $F_1(n) \geq k$, then $\sum F_1 \geq \operatorname{len} F_1 \cdot k$.

(52) If $\operatorname{len} F_1 > 0$ and there exists x such that $x \in \operatorname{dom} F_1$ and $F_1(x) = k$, then $\sum F_1 \geq k$.

(53) $\sum F_1 = 0$ iff $\operatorname{len} F_1 = 0$ or for every n such that $n \in \operatorname{dom} F_1$ holds $F_1(n) = 0$.

(54) For every function f and for every n holds $\bigcup \operatorname{rng}(f \restriction n) \cup f(n) = \bigcup \operatorname{rng}(f \restriction (n+1))$.

Now we present three schemes. The scheme *Sch6* deals with a non empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a finite 0-sequence p of $\mathcal{A}$ such that $\operatorname{dom} p = \mathcal{B}$ and for every k such that $k \in \mathcal{B}$ holds $\mathcal{P}[k, p(k)]$

provided the parameters have the following property:

- For every k such that $k \in \mathcal{B}$ there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[k, x]$.

The scheme *Sch7* deals with a non empty set $\mathcal{A}$ and a finite 0-sequence $\mathcal{B}$ of $\mathcal{A}$, and states that:

There exists a finite 0-sequence C_1 of $\mathbb{N}$ such that $\operatorname{dom} C_1 = \operatorname{dom} \mathcal{B}$ and for every i such that $i \in \operatorname{dom} C_1$ holds $C_1(i) = \overline{\overline{\mathcal{B}(i)}}$ and $\overline{\overline{\bigcup \operatorname{rng} \mathcal{B}}} = \sum C_1$

provided the following requirements are met:

- For every i such that $i \in \operatorname{dom} \mathcal{B}$ holds $\mathcal{B}(i)$ is finite, and
- For all i, j such that $i \in \operatorname{dom} \mathcal{B}$ and $j \in \operatorname{dom} \mathcal{B}$ and $i \neq j$ holds $\mathcal{B}(i)$ misses $\mathcal{B}(j)$.

The scheme *Sch8* deals with finite sets $\mathcal{A}$, $\mathcal{B}$, a set $\mathcal{C}$, a function $\mathcal{D}$ from $\operatorname{card} \mathcal{B}$ into $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a finite 0-sequence F of $\mathbb{N}$ such that

(i) $\operatorname{dom} F = \operatorname{card} \mathcal{B}$,

(ii) $\overline{\overline{\{g;\ g \text{ ranges over functions from } \mathcal{A} \text{ into } \mathcal{B} : \mathcal{P}[g]\}}} = \sum F$,

and

(iii) for every i such that $i \in \operatorname{dom} F$ holds $F(i) = \overline{\overline{\{g;\ g \text{ ranges over functions from } \mathcal{A} \text{ into } \mathcal{B} : \mathcal{P}[g] \ \wedge\ g(\mathcal{C}) = \mathcal{D}(i)\}}}$

provided the parameters have the following properties:

- $\mathcal{D}$ is onto and one-to-one,
- $\mathcal{B}$ is non empty, and
- $\mathcal{C} \in \mathcal{A}$.

One can prove the following propositions:

(55) $k \cdot (n \operatorname{block} k) = \overline{\overline{\{f' : f' \text{ is onto and increasing } \wedge\ f'^{-1}(\{f'(n)\}) \neq \{n\}\}}}$.

(56) $(n+1)\,\mathrm{block}(k+1) = (k+1)\cdot(n\,\mathrm{block}(k+1)) + (n\,\mathrm{block}\,k)$.

(57) If $n \geq 1$, then $n\,\mathrm{block}\,2 = \frac{1}{2}\cdot(2^n - 2)$.

(58) If $n \geq 2$, then $n\,\mathrm{block}\,3 = \frac{1}{6}\cdot((3^n - 3\cdot 2^n) + 3)$.

(59) If $n \geq 3$, then $n\,\mathrm{block}\,4 = \frac{1}{24}\cdot(((4^n - 4\cdot 3^n) + 6\cdot 2^n) - 4)$.

(60) $3! = 6$ and $4! = 24$.

(61) $\binom{n}{1} = n$ and $\binom{n}{2} = \frac{n\cdot(n-1)}{2}$ and $\binom{n}{3} = \frac{n\cdot(n-1)\cdot(n-2)}{6}$ and $\binom{n}{4} = \frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)}{24}$.

(62) $(n+1)\,\mathrm{block}\,n = \binom{n+1}{2}$.

(63) $(n+2)\,\mathrm{block}\,n = 3\cdot\binom{n+2}{4} + \binom{n+2}{3}$.

(64) For every function F and for every y holds $\mathrm{rng}(F{\restriction}(\mathrm{dom}\,F \setminus F^{-1}(\{y\}))) = \mathrm{rng}\,F \setminus \{y\}$ and for every x such that $x \neq y$ holds $(F{\restriction}(\mathrm{dom}\,F \setminus F^{-1}(\{y\})))^{-1}(\{x\}) = F^{-1}(\{x\})$.

(65) If $\overline{\overline{X}} = k+1$ and $x \in X$, then $\overline{\overline{X \setminus \{x\}}} = k$.

The scheme *Sch9* concerns a unary predicate $\mathcal{P}$ and a binary predicate $\mathcal{Q}$, and states that:

For every function F such that $\mathrm{rng}\,F$ is finite holds $\mathcal{P}[F]$

provided the following conditions are met:

- $\mathcal{P}[\emptyset]$, and
- For every function F such that for every x such that $x \in \mathrm{rng}\,F$ and $\mathcal{Q}[x, F]$ holds $\mathcal{P}[F{\restriction}(\mathrm{dom}\,F \setminus F^{-1}(\{x\}))]$ holds $\mathcal{P}[F]$.

We now state several propositions:

(66) For every subset N of $\mathbb{N}$ such that N is finite there exists k such that for every n such that $n \in N$ holds $n \leq k$.

(67) Let given X, Y, x, y. Suppose if Y is empty, then X is empty and $x \notin X$. Let F be a function from X into Y. Then there exists a function G from $X \cup \{x\}$ into $Y \cup \{y\}$ such that $G{\restriction}X = F$ and $G(x) = y$.

(68) Let given X, Y, x, y such that if Y is empty, then X is empty. Let F be a function from X into Y and G be a function from $X \cup \{x\}$ into $Y \cup \{y\}$ such that $G{\restriction}X = F$ and $G(x) = y$. Then

(i) if F is onto, then G is onto, and

(ii) if $y \notin Y$ and F is one-to-one, then G is one-to-one.

(69) Let N be a finite subset of $\mathbb{N}$. Then there exists a function O_1 from N into $\mathrm{card}\,N$ such that O_1 is bijective and for all n, k such that $n \in \mathrm{dom}\,O_1$ and $k \in \mathrm{dom}\,O_1$ and $n < k$ holds $O_1(n) < O_1(k)$.

(70) Let X, Y be finite sets and F be a function from X into Y. If $\mathrm{card}\,X = \mathrm{card}\,Y$, then F is onto iff F is one-to-one.

(71) Let F, G be functions and given y. Suppose $y \in \mathrm{rng}(G\cdot F)$ and G is one-to-one. Then there exists x such that $x \in \mathrm{dom}\,G$ and $x \in \mathrm{rng}\,F$ and $G^{-1}(\{y\}) = \{x\}$ and $F^{-1}(\{x\}) = (G\cdot F)^{-1}(\{y\})$.

Let us consider N_1, K_1 and let f be a function from N_1 into K_1. We say that f is increasing if and only if:

(Def. 5) For all l, m such that $l \in \operatorname{rng} f$ and $m \in \operatorname{rng} f$ and $l < m$ holds $\min^*(f^{-1}(\{l\})) < \min^*(f^{-1}(\{m\}))$.

The following four propositions are true:

(72) For every function F from N_1 into K_1 such that F is increasing holds $\min^* \operatorname{rng} F = F(\min^* \operatorname{dom} F)$.

(73) Let F be a function from N_1 into K_1. Suppose $\operatorname{rng} F$ is finite. Then there exists a function I from N_1 into K_1 and there exists a permutation P of $\operatorname{rng} F$ such that $F = P \cdot I$ and $\operatorname{rng} F = \operatorname{rng} I$ and I is increasing.

(74) Let F be a function from N_1 into K_1. Suppose $\operatorname{rng} F$ is finite. Let I_1, I_2 be functions from N_1 into M_1 and P_1, P_2 be functions. Suppose that P_1 is one-to-one and P_2 is one-to-one and $\operatorname{rng} I_1 = \operatorname{rng} I_2$ and $\operatorname{rng} I_1 = \operatorname{dom} P_1$ and $\operatorname{dom} P_1 = \operatorname{dom} P_2$ and $F = P_1 \cdot I_1$ and $F = P_2 \cdot I_2$ and I_1 is increasing and I_2 is increasing. Then $P_1 = P_2$ and $I_1 = I_2$.

(75) Let F be a function from N_1 into K_1. Suppose $\operatorname{rng} F$ is finite. Let I_1, I_2 be functions from N_1 into K_1 and P_1, P_2 be permutations of $\operatorname{rng} F$. Suppose $F = P_1 \cdot I_1$ and $F = P_2 \cdot I_2$ and $\operatorname{rng} F = \operatorname{rng} I_1$ and $\operatorname{rng} F = \operatorname{rng} I_2$ and I_1 is increasing and I_2 is increasing. Then $P_1 = P_2$ and $I_1 = I_2$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.
[4] Patrick Braselmann and Peter Koepke. Equivalences of inconsistency and Henkin models. *Formalized Mathematics*, 13(**1**):45–48, 2005.
[5] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[10] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[11] Agata Darmochwał and Andrzej Trybulec. Similarity of formulae. *Formalized Mathematics*, 2(**5**):635–642, 1991.
[12] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.
[13] Library Committee of the Association of Mizar Users. Binary operations on numbers. *To appear in Formalized Mathematics.*
[14] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[15] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[16] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[17] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.
[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[19] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[20] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[22] Tetsuya Tsunetou, Grzegorz Bancerek, and Yatsuka Nakamura. Zero-based finite sequences. *Formalized Mathematics*, 9(**4**):825–829, 2001.
[23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[24] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received March 15, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 347–352
University of Białystok, 2005

Limit of Sequence of Subsets

Bo Zhang
Shinshu University
Nagano, Japan

Hiroshi Yamazaki
Shinshu University
Nagano, Japan

Yatsuka Nakamura
Shinshu University
Nagano, Japan

Summary. A concept of "limit of sequence of subsets" is defined here. This article contains the following items: 1. definition of the superior sequence and the inferior sequence of sets, 2. definition of the superior limit and the inferior limit of sets, and additional properties for the sigma-field of sets, 3. definition of the limit value of a convergent sequence of sets, and additional properties for the sigma-field of sets.

MML identifier: `SETLIM_1`, version: `7.5.01 4.39.921`

The notation and terminology used here are introduced in the following papers: [9], [1], [13], [2], [10], [6], [11], [4], [12], [14], [8], [7], [3], and [5].

For simplicity, we adopt the following rules: n, m, k, k_1, k_2 denote natural numbers, x, X, Y, Z denote sets, A denotes a subset of X, B, A_1, A_2, A_3 denote sequences of subsets of X, S_1 denotes a σ-field of subsets of X, and S, S_2, S_3, S_4 denote sequences of subsets of S_1.

Next we state a number of propositions:

(1) For every function f from $\mathbb{N}$ into Y and for every n holds $\{f(k) : n \leq k\} \neq \emptyset$.

(2) For every function f from $\mathbb{N}$ into Y holds $f(n+m) \in \{f(k) : n \leq k\}$.

(3) For every function f from $\mathbb{N}$ into Y holds $\{f(k_1) : n \leq k_1\} = \{f(k_2) : n+1 \leq k_2\} \cup \{f(n)\}$.

(4) Let f be a function from $\mathbb{N}$ into Y. Then for every k_1 holds $x \in f(n+k_1)$ if and only if for every Z such that $Z \in \{f(k_2) : n \leq k_2\}$ holds $x \in Z$.

(5) For every non empty set Y and for every function f from $\mathbb{N}$ into Y holds $x \in \operatorname{rng} f$ iff there exists n such that $x = f(n)$.

(6) For every non empty set Y and for every function f from $\mathbb{N}$ into Y holds $\operatorname{rng} f = \{f(k)\}$.

ISSN 1426–2630

(7) For every non empty set Y and for every function f from $\mathbb{N}$ into Y holds $\mathrm{rng}(f \uparrow k) = \{f(n) : k \leq n\}$.

(8) $x \in \bigcap \mathrm{rng}\, B$ iff for every n holds $x \in B(n)$.

(9) $\mathrm{Intersection}\, B = \bigcap \mathrm{rng}\, B$.

(10) $\mathrm{Intersection}\, B \subseteq \bigcup B$.

(11) If for every n holds $B(n) = A$, then $\bigcup B = A$.

(12) If for every n holds $B(n) = A$, then $\mathrm{Intersection}\, B = A$.

(13) If B is constant, then $\bigcup B = \mathrm{Intersection}\, B$.

(14) If B is constant and the value of $B = A$, then for every n holds $\bigcup\{B(k) : n \leq k\} = A$.

(15) If B is constant and the value of $B = A$, then for every n holds $\bigcap\{B(k) : n \leq k\} = A$.

(16) Let given X, B and f be a function. Suppose $\mathrm{dom}\, f = \mathbb{N}$ and for every n holds $f(n) = \bigcap\{B(k) : n \leq k\}$. Then f is a sequence of subsets of X.

(17) Let X be a set, B be a sequence of subsets of X, and f be a function. Suppose $\mathrm{dom}\, f = \mathbb{N}$ and for every n holds $f(n) = \bigcup\{B(k) : n \leq k\}$. Then f is a function from $\mathbb{N}$ into 2^X.

Let us consider X, B. We say that B is monotone if and only if:

(Def. 1) B is non-decreasing or non-increasing.

Let B be a function. The inferior setsequence B yields a function and is defined by the conditions (Def. 2).

(Def. 2)(i) dom (the inferior setsequence B) $= \mathbb{N}$, and

(ii) for every n holds (the inferior setsequence B)$(n) = \bigcap\{B(k) : n \leq k\}$.

Let X be a set and let B be a sequence of subsets of X. Then the inferior setsequence B is a sequence of subsets of X.

Let B be a function. The superior setsequence B yields a function and is defined by the conditions (Def. 3).

(Def. 3)(i) dom (the superior setsequence B) $= \mathbb{N}$, and

(ii) for every n holds (the superior setsequence B)$(n) = \bigcup\{B(k) : n \leq k\}$.

Let X be a set and let B be a sequence of subsets of X. Then the superior setsequence B is a sequence of subsets of X.

Next we state several propositions:

(18) (The inferior setsequence B)$(0) = \mathrm{Intersection}\, B$.

(19) (The superior setsequence B)$(0) = \bigcup B$.

(20) $x \in$ (the inferior setsequence B)(n) iff for every k holds $x \in B(n+k)$.

(21) $x \in$ (the superior setsequence B)(n) iff there exists k such that $x \in B(n+k)$.

(22) (The inferior setsequence B)$(n) =$ (the inferior setsequence B)$(n+1) \cap B(n)$.

(23) (The superior setsequence $B)(n) =$ (the superior setsequence $B)(n+1) \cup B(n)$.

(24) The inferior setsequence B is non-decreasing.

(25) The superior setsequence B is non-increasing.

(26) The inferior setsequence B is monotone and the superior setsequence B is monotone.

Let X be a set and let A be a sequence of subsets of X. Observe that the inferior setsequence A is non-decreasing.

Let X be a set and let A be a sequence of subsets of X. Observe that the superior setsequence A is non-increasing.

The following propositions are true:

(27) Intersection $B \subseteq$ (the inferior setsequence $B)(n)$.

(28) (The superior setsequence $B)(n) \subseteq \bigcup B$.

(29) For all B, n holds $\{B(k) : n \leq k\}$ is a family of subsets of X.

(30) $\bigcup B =$ (Intersection Complement $B)^{\rm c}$.

(31) (The inferior setsequence $B)(n)$ = (the superior setsequence Complement $B)(n)^{\rm c}$.

(32) (The superior setsequence $B)(n)$ = (the inferior setsequence Complement $B)(n)^{\rm c}$.

(33) Complement (the inferior setsequence B) = the superior setsequence Complement B.

(34) Complement (the superior setsequence B) = the inferior setsequence Complement B.

(35) Suppose that for every n holds $A_3(n) = A_1(n) \cup A_2(n)$. Let given n. Then (the inferior setsequence $B)(n) \cup$ (the inferior setsequence $A_2)(n) \subseteq$ (the inferior setsequence $A_3)(n)$.

(36) Suppose that for every n holds $A_3(n) = A_1(n) \cap A_2(n)$. Let given n. Then (the inferior setsequence $A_3)(n) =$ (the inferior setsequence $A_1)(n) \cap$ (the inferior setsequence $A_2)(n)$.

(37) Suppose that for every n holds $A_3(n) = A_1(n) \cup A_2(n)$. Let given n. Then (the superior setsequence $A_3)(n) =$ (the superior setsequence $A_1)(n) \cup$ (the superior setsequence $A_2)(n)$.

(38) Suppose that for every n holds $A_3(n) = A_1(n) \cap A_2(n)$. Let given n. Then (the superior setsequence $A_3)(n) \subseteq$ (the superior setsequence $A_1)(n) \cap$ (the superior setsequence $A_2)(n)$.

(39) If B is constant and the value of $B = A$, then for every n holds (the inferior setsequence $B)(n) = A$.

(40) If B is constant and the value of $B = A$, then for every n holds (the superior setsequence $B)(n) = A$.

(41) If B is non-decreasing, then $B(n) \subseteq$ (the superior setsequence $B)(n+1)$.

(42) If B is non-decreasing, then (the superior setsequence $B)(n)$ = (the superior setsequence $B)(n+1)$.

(43) If B is non-decreasing, then (the superior setsequence $B)(n) = \bigcup B$.

(44) If B is non-decreasing, then Intersection (the superior setsequence B) = $\bigcup B$.

(45) If B is non-decreasing, then $B(n) \subseteq$ (the inferior setsequence $B)(n+1)$.

(46) If B is non-decreasing, then (the inferior setsequence $B)(n) = B(n)$.

(47) If B is non-decreasing, then the inferior setsequence $B = B$.

(48) If B is non-increasing, then (the superior setsequence $B)(n+1) \subseteq B(n)$.

(49) If B is non-increasing, then (the superior setsequence $B)(n) = B(n)$.

(50) If B is non-increasing, then the superior setsequence $B = B$.

(51) If B is non-increasing, then (the inferior setsequence $B)(n+1) \subseteq B(n)$.

(52) If B is non-increasing, then (the inferior setsequence $B)(n)$ = (the inferior setsequence $B)(n+1)$.

(53) If B is non-increasing, then (the inferior setsequence $B)(n)$ = Intersection B.

(54) If B is non-increasing, then $\bigcup$(the inferior setsequence B) = Intersection B.

Let X be a set and let B be a sequence of subsets of X. Then $\liminf B$ can be characterized by the condition:

(Def. 4) $\liminf B = \bigcup$(the inferior setsequence B).

Let X be a set and let B be a sequence of subsets of X. Then $\limsup B$ can be characterized by the condition:

(Def. 5) $\limsup B$ = Intersection (the superior setsequence B).

Let X be a set and let B be a sequence of subsets of X. We introduce $\lim B$ as a synonym of $\limsup B$.

Next we state a number of propositions:

(55) Intersection $B \subseteq \liminf B$.

(56) $\liminf B = \lim$ (the inferior setsequence B).

(57) $\limsup B = \lim$ (the superior setsequence B).

(58) $\limsup B = (\liminf \text{Complement } B)^{\mathrm{c}}$.

(59) If B is constant and the value of $B = A$, then B is convergent and $\lim B = A$ and $\liminf B = A$ and $\limsup B = A$.

(60) If B is non-decreasing, then $\limsup B = \bigcup B$.

(61) If B is non-decreasing, then $\liminf B = \bigcup B$.

(62) If B is non-increasing, then $\limsup B$ = Intersection B.

(63) If B is non-increasing, then $\liminf B$ = Intersection B.

(64) If B is non-decreasing, then B is convergent and $\lim B = \bigcup B$.

(65) If B is non-increasing, then B is convergent and $\lim B = \mathrm{Intersection}\, B$.

(66) If B is monotone, then B is convergent.

Let X be a set, let S_1 be a σ-field of subsets of X, and let S be a sequence of subsets of S_1. Let us observe that S is constant if and only if:

(Def. 6) There exists an element A of S_1 such that for every n holds $S(n) = A$.

Let X be a set, let S_1 be a σ-field of subsets of X, and let S be a sequence of subsets of S_1. Then the inferior setsequence S is a sequence of subsets of S_1.

Let X be a set, let S_1 be a σ-field of subsets of X, and let S be a sequence of subsets of S_1. Then the superior setsequence S is a sequence of subsets of S_1.

The following propositions are true:

(67) $x \in \limsup S$ iff for every n there exists k such that $x \in S(n+k)$.

(68) $x \in \liminf S$ iff there exists n such that for every k holds $x \in S(n+k)$.

(69) $\mathrm{Intersection}\, S \subseteq \liminf S$.

(70) $\limsup S \subseteq \bigcup S$.

(71) $\liminf S \subseteq \limsup S$.

Let X be a set, let S_1 be a σ-field of subsets of X, and let S be a sequence of subsets of S_1. The functor $S^{\mathbf{c}}$ yields a sequence of subsets of S_1 and is defined by:

(Def. 7) $S^{\mathbf{c}} = \mathrm{Complement}\, S$.

Next we state a number of propositions:

(72) $\liminf S = (\limsup(S^{\mathbf{c}}))^{\mathrm{c}}$.

(73) $\limsup S = (\liminf(S^{\mathbf{c}}))^{\mathrm{c}}$.

(74) If for every n holds $S_4(n) = S_2(n) \cup S_3(n)$, then $\liminf S_2 \cup \liminf S_3 \subseteq \liminf S_4$.

(75) If for every n holds $S_4(n) = S_2(n) \cap S_3(n)$, then $\liminf S_4 = \liminf S_2 \cap \liminf S_3$.

(76) If for every n holds $S_4(n) = S_2(n) \cup S_3(n)$, then $\limsup S_4 = \limsup S_2 \cup \limsup S_3$.

(77) If for every n holds $S_4(n) = S_2(n) \cap S_3(n)$, then $\limsup S_4 \subseteq \limsup S_2 \cap \limsup S_3$.

(78) If S is constant and the value of $S = A$, then S is convergent and $\lim S = A$ and $\liminf S = A$ and $\limsup S = A$.

(79) If S is non-decreasing, then $\limsup S = \bigcup S$.

(80) If S is non-decreasing, then $\liminf S = \bigcup S$.

(81) If S is non-decreasing, then S is convergent and $\lim S = \bigcup S$.

(82) If S is non-increasing, then $\limsup S = \mathrm{Intersection}\, S$.

(83) If S is non-increasing, then $\liminf S = \mathrm{Intersection}\, S$.

(84) If S is non-increasing, then S is convergent and $\lim S = \text{Intersection}\, S$.

(85) If S is monotone, then S is convergent.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[5] Adam Grabowski. On the Kuratowski limit operators. *Formalized Mathematics*, 11(**4**):399–409, 2003.
[6] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[7] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.
[8] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[9] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[11] Andrzej Trybulec. Moore-Smith convergence. *Formalized Mathematics*, 6(**2**):213–225, 1997.
[12] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[13] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[14] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received March 15, 2005

FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 353–359
University of Białystok, 2005

The Properties of Supercondensed Sets, Subcondensed Sets and Condensed Sets

Magdalena Jastrzębska
Institute of Mathematics
University of Białystok
Akademicka 2, 15-267 Białystok
Poland

Adam Grabowski[1]
Institute of Mathematics
University of Białystok
Akademicka 2, 15-267 Białystok
Poland

Summary. We formalized the article "New concepts in the theory of topological space – supercondensed set, subcondensed set, and condensed set" by Yoshinori Isomichi [4]. First we defined supercondensed, subcondensed, and condensed sets and then gradually, defining other attributes such as regular open set or regular closed set, we formalized all the theorems and remarks that one can find in Isomichi's article.

In the last section, the classification of subsets of a topological space is given, depending on the inclusion relation between the interior of the closure and the closure of the interior of a given subset.

MML identifier: `ISOMICHI`, version: `7.5.01 4.39.921`

The notation and terminology used in this paper are introduced in the following papers: [10], [11], [1], [6], [8], [9], [7], [12], [2], [3], and [5].

1. Preliminaries

In this paper T denotes a topological space and A, B denote subsets of T.

Let D be a non trivial set. Note that $\mathrm{ADTS}(D)$ is non trivial.

One can check that there exists a topological space which is anti-discrete, non trivial, non empty, and strict.

One can prove the following propositions:

[1]This work has been partially supported by the KBN grant 4 T11C 039 24 and the FP6 IST grant TYPES No. 510996.

ISSN 1426–2630

(1) $\operatorname{Int}\overline{\operatorname{Int} A} \cap \operatorname{Int}\overline{\operatorname{Int} B} = \operatorname{Int}\overline{\operatorname{Int}(A \cap B)}$.

(2) $\overline{\operatorname{Int}\overline{A \cup B}} = \overline{\operatorname{Int}\overline{A}} \cup \overline{\operatorname{Int}\overline{B}}$.

2. Connections between Supercondensed, Condensed, and Subcondensed Sets

Let T be a topological structure and let A be a subset of T. We say that A is supercondensed if and only if:

(Def. 1) $\operatorname{Int}\overline{A} = \operatorname{Int} A$.

We say that A is subcondensed if and only if:

(Def. 2) $\overline{\operatorname{Int} A} = \overline{A}$.

Next we state two propositions:

(3) If A is closed, then A is supercondensed.

(4) If A is open, then A is subcondensed.

Let T be a topological space and let A be a subset of T. Let us observe that A is condensed if and only if:

(Def. 3) $\overline{\operatorname{Int} A} = \overline{A}$ and $\operatorname{Int}\overline{A} = \operatorname{Int} A$.

We now state the proposition

(5) A is condensed iff A is subcondensed and supercondensed.

Let T be a topological space. One can verify that every subset of T which is condensed is also subcondensed and supercondensed and every subset of T which is subcondensed and supercondensed is also condensed.

Let T be a topological space. Observe that there exists a subset of T which is condensed, subcondensed, and supercondensed.

One can prove the following propositions:

(6) If A is supercondensed, then A^{c} is subcondensed.

(7) If A is subcondensed, then A^{c} is supercondensed.

(8) A is supercondensed iff $\operatorname{Int}\overline{A} \subseteq A$.

(9) A is subcondensed iff $A \subseteq \overline{\operatorname{Int} A}$.

Let T be a topological space. Note that every subset of T which is subcondensed is also semi-open and every subset of T which is semi-open is also subcondensed.

We now state the proposition

(10) A is condensed iff $\operatorname{Int}\overline{A} \subseteq A$ and $A \subseteq \overline{\operatorname{Int} A}$.

3. Regular Open and Regular Closed Sets

Let T be a topological structure and let A be a subset of T. We introduce A is regular open as a synonym of A is open condensed.

Let T be a topological structure and let A be a subset of T. We introduce A is regular closed as a synonym of A is closed condensed.

The following proposition is true

(11) For every topological space T holds Ω_T is regular open and Ω_T is regular closed.

Let T be a topological space. Note that Ω_T is regular open and regular closed.

We now state the proposition

(12) For every topological space X holds $\emptyset_X$ is regular open and $\emptyset_X$ is regular closed.

Let T be a topological space. One can verify that $\emptyset_T$ is regular open and regular closed.

The following propositions are true:

(14)[2] $\mathrm{Int}\,\overline{\emptyset_T} = \emptyset_T$.

(15) If A is regular open, then A^{c} is regular closed.

Let T be a topological space. Observe that there exists a subset of T which is regular open and regular closed.

Let T be a topological space and let A be a regular open subset of T. Observe that A^{c} is regular closed.

One can prove the following proposition

(16) If A is regular closed, then A^{c} is regular open.

Let T be a topological space and let A be a regular closed subset of T. One can check that A^{c} is regular open.

Let T be a topological space. Note that every subset of T which is regular open is also open and every subset of T which is regular closed is also closed.

Next we state the proposition

(17) $\mathrm{Int}\,\overline{A}$ is regular open and $\overline{\mathrm{Int}\,A}$ is regular closed.

Let T be a topological space and let A be a subset of T. Observe that $\mathrm{Int}\,\overline{A}$ is regular open and $\overline{\mathrm{Int}\,A}$ is regular closed.

Next we state two propositions:

(18) A is regular open iff A is supercondensed and open.

(19) A is regular closed iff A is subcondensed and closed.

Let T be a topological space. One can check the following observations:

* every subset of T which is regular open is also condensed and open,

[2]The proposition (13) has been removed.

* every subset of T which is condensed and open is also regular open,
* every subset of T which is regular closed is also condensed and closed, and
* every subset of T which is condensed and closed is also regular closed.

One can prove the following two propositions:

(20) A is condensed iff there exists B such that B is regular open and $B \subseteq A$ and $A \subseteq \overline{B}$.

(21) A is condensed iff there exists B such that B is regular closed and $\operatorname{Int} B \subseteq A$ and $A \subseteq B$.

4. Boundaries and Borders

Let T be a topological structure and let A be a subset of T. We introduce $\operatorname{Bound} A$ as a synonym of $\operatorname{Fr} A$.

Let T be a topological structure and let A be a subset of T. Then $\operatorname{Fr} A$ can be characterized by the condition:

(Def. 4) $\operatorname{Fr} A = \overline{A} \setminus \operatorname{Int} A$.

One can prove the following proposition

(22) $\operatorname{Fr} A$ is closed.

Let T be a topological space and let A be a subset of T. Observe that $\operatorname{Fr} A$ is closed.

One can prove the following proposition

(23) A is condensed iff $\operatorname{Fr} A = \overline{\operatorname{Int} A} \setminus \operatorname{Int} \overline{A}$ and $\operatorname{Fr} A = \overline{\operatorname{Int} A} \cap \overline{\operatorname{Int}(A^{c})}$.

Let T be a topological structure and let A be a subset of T. The functor $\operatorname{Border} A$ yields a subset of T and is defined by:

(Def. 5) $\operatorname{Border} A = \operatorname{Int} \operatorname{Fr} A$.

One can prove the following proposition

(24) $\operatorname{Border} A$ is regular open and $\operatorname{Border} A = \operatorname{Int} \overline{A} \setminus \overline{\operatorname{Int} A}$ and $\operatorname{Border} A = \operatorname{Int} \overline{A} \cap \operatorname{Int} \overline{A^{c}}$.

Let T be a topological space and let A be a subset of T. One can verify that $\operatorname{Border} A$ is regular open.

One can prove the following two propositions:

(25) A is supercondensed iff $\operatorname{Int} A$ is regular open and $\operatorname{Border} A$ is empty.

(26) A is subcondensed iff $\overline{A}$ is regular closed and $\operatorname{Border} A$ is empty.

Let T be a topological space and let A be a subset of T. One can verify that $\operatorname{Border} \operatorname{Border} A$ is empty.

The following proposition is true

(27) A is condensed iff $\operatorname{Int} A$ is regular open and $\overline{A}$ is regular closed and $\operatorname{Border} A$ is empty.

5. Auxiliary Theorems about Intervals

Next we state a number of propositions:

(28) For every subset A of $\mathbb{R}^1$ and for every real number a such that $A =]-\infty, a]$ holds $\mathrm{Int}\, A =]-\infty, a[$.

(29) For every subset A of $\mathbb{R}^1$ and for every real number a such that $A = [a, +\infty[$ holds $\mathrm{Int}\, A =]a, +\infty[$.

(30) For every subset A of $\mathbb{R}^1$ and for all real numbers a, b such that $A =]-\infty, a] \cup]a, b[_{\mathbb{IQ}} \cup [b, +\infty[$ holds $\overline{A} =$ the carrier of $\mathbb{R}^1$.

(31) For every subset A of $\mathbb{R}^1$ and for all real numbers a, b such that $A =]a, b[_{\mathbb{Q}}$ holds $\mathrm{Int}\, A = \emptyset$.

(32) For every subset A of $\mathbb{R}^1$ and for all real numbers a, b such that $A =]a, b[_{\mathbb{IQ}}$ holds $\mathrm{Int}\, A = \emptyset$.

(33) For all real numbers a, b holds $]-\infty, a] \setminus]-\infty, b[= [b, a]$.

(34) For all real numbers a, b such that $a < b$ holds $[b, +\infty[$ misses $]-\infty, a[$.

(35) For all real numbers a, b such that $a \geq b$ holds $]a, b[_{\mathbb{IQ}} = \emptyset$.

(36) For all real numbers a, b holds $]a, b[_{\mathbb{IQ}} \subseteq [a, +\infty[$.

(37) For every subset A of $\mathbb{R}^1$ and for all real numbers a, b, c such that $A =]-\infty, a[\cup]b, c[_{\mathbb{Q}}$ and $a < b$ and $b < c$ holds $\mathrm{Int}\, A =]-\infty, a[$.

(38) For all real numbers a, b holds $[a, b]$ misses $]b, +\infty[$.

(39) For every real number b holds $[b, +\infty[\setminus]b, +\infty[= \{b\}$.

(40) For all real numbers a, b such that $a < b$ holds $[a, b] = [a, +\infty[\setminus]b, +\infty[$.

(41) For all real numbers a, b such that $a < b$ holds $\mathbb{R} =]-\infty, a[\cup [a, b] \cup]b, +\infty[$.

(42) For all real numbers a, b holds $]a, b[=]a, +\infty[\setminus [b, +\infty[$.

(43) For all real numbers a, b, c such that $b < c$ and $c < a$ holds $]-\infty, a[\setminus [b, c] =]-\infty, b[\cup]c, a[$.

(44) For every subset A of $\mathbb{R}^1$ and for all real numbers a, b, c such that $A =]-\infty, a] \cup [b, c]$ and $a < b$ and $b < c$ holds $\mathrm{Int}\, A =]-\infty, a[\cup]b, c[$.

6. Classification of Subsets

Let A, B be sets. We introduce A and B are $\subseteq$-incomparable as an antonym of A and B are $\subseteq$-comparable.

We now state the proposition

(45) For all sets A, B holds A and B are $\subseteq$-incomparable or $A \subseteq B$ or $B \subset A$.

Let us consider T, A. We say that A is of the 1^{st} class if and only if:

(Def. 6) $\mathrm{Int}\, \overline{A} \subseteq \overline{\mathrm{Int}\, A}$.

We say that A is of the $2^{\rm nd}$ class if and only if:

(Def. 7) $\overline{\operatorname{Int} A} \subset \operatorname{Int} \overline{A}$.

We say that A is of the $3^{\rm rd}$ class if and only if:

(Def. 8) $\overline{\operatorname{Int} A}$ and $\operatorname{Int} \overline{A}$ are $\subseteq$-incomparable.

The following proposition is true

(46) A is of the $1^{\rm st}$ class, or of the $2^{\rm nd}$ class, or of the $3^{\rm rd}$ class.

Let T be a topological space. One can verify the following observations:

- every subset of T which is of the $1^{\rm st}$ class is also non of the $2^{\rm nd}$ class and non of the $3^{\rm rd}$ class,
- every subset of T which is of the $2^{\rm nd}$ class is also non of the $1^{\rm st}$ class and non of the $3^{\rm rd}$ class, and
- every subset of T which is of the $3^{\rm rd}$ class is also non of the $1^{\rm st}$ class and non of the $2^{\rm nd}$ class.

One can prove the following proposition

(47) A is of the $1^{\rm st}$ class iff $\operatorname{Border} A$ is empty.

Let T be a topological space. Note that every subset of T which is supercondensed is also of the $1^{\rm st}$ class and every subset of T which is subcondensed is also of the $1^{\rm st}$ class.

Let T be a topological space. We say that T has subsets of the $1^{\rm st}$ class if and only if:

(Def. 9) There exists a subset of T which is of the $1^{\rm st}$ class.

We say that T has subsets of the $2^{\rm nd}$ class if and only if:

(Def. 10) There exists a subset of T which is of the $2^{\rm nd}$ class.

We say that T has subsets of the $3^{\rm rd}$ class if and only if:

(Def. 11) There exists a subset of T which is of the $3^{\rm rd}$ class.

Let T be an anti-discrete non empty topological space. Note that every subset of T which is proper and non empty is also of the $2^{\rm nd}$ class.

Let T be an anti-discrete non trivial non empty strict topological space. Observe that there exists a subset of T which is of the $2^{\rm nd}$ class.

One can verify that there exists a topological space which is non empty, strict, and non trivial and has subsets of the $1^{\rm st}$ class and subsets of the $2^{\rm nd}$ class and there exists a topological space which is non empty and strict and has subsets of the $3^{\rm rd}$ class.

Let us consider T. Observe that there exists a subset of T which is of the $1^{\rm st}$ class.

Let T be a topological space with subsets of the $2^{\rm nd}$ class. One can verify that there exists a subset of T which is of the $2^{\rm nd}$ class.

Let T be a topological space with subsets of the $3^{\rm rd}$ class. Observe that there exists a subset of T which is of the $3^{\rm rd}$ class.

The following propositions are true:

(48) A is of the 1st class iff A^{c} is of the 1st class.

(49) A is of the 2nd class iff A^{c} is of the 2nd class.

(50) A is of the 3rd class iff A^{c} is of the 3rd class.

Let us consider T and let A be an of the 1st class subset of T. Observe that A^{c} is of the 1st class.

Let T be a topological space with subsets of the 2nd class and let A be an of the 2nd class subset of T. Note that A^{c} is of the 2nd class.

Let T be a topological space with subsets of the 3rd class and let A be an of the 3rd class subset of T. Note that A^{c} is of the 3rd class.

Next we state four propositions:

(51) If A is of the 1st class, then $\mathrm{Int}\,\overline{A} = \mathrm{Int}\,\overline{\mathrm{Int}\,A}$ and $\overline{\mathrm{Int}\,A} = \overline{\mathrm{Int}\,\overline{A}}$.

(52) If $\mathrm{Int}\,\overline{A} = \mathrm{Int}\,\overline{\mathrm{Int}\,A}$ or $\overline{\mathrm{Int}\,A} = \overline{\mathrm{Int}\,\overline{A}}$, then A is of the 1st class.

(53) Suppose A is of the 1st class and B is of the 1st class. Then $\mathrm{Int}\,\overline{A} \cap \mathrm{Int}\,\overline{B} = \mathrm{Int}\,\overline{A \cap B}$ and $\overline{\mathrm{Int}\,A} \cup \overline{\mathrm{Int}\,B} = \overline{\mathrm{Int}(A \cup B)}$.

(54) Suppose A is of the 1st class and B is of the 1st class. Then $A \cup B$ is of the 1st class and $A \cap B$ is of the 1st class.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.

[3] Adam Grabowski. On the subcontinua of a real line. *Formalized Mathematics*, 11(**3**):313–322, 2003.

[4] Yoshinori Isomichi. New concepts in the theory of topological space – supercondensed set, subcondensed set, and condensed set. *Pacific Journal of Mathematics*, 38(3):657–668, 1971.

[5] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.

[6] Jarosław Kotowicz. The limit of a real function at infinity. *Formalized Mathematics*, 2(**1**):17–28, 1991.

[7] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[8] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[9] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[11] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[12] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received March 31, 2005

Index of MML Identifiers

Contents

Formaliz. Math. 13 (**2**)

Continued on inside back cover