FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Construction of Gröbner Bases: Avoiding S-Polynomials – Buchberger's First Criterium[1]

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Summary. We continue the formalization of Groebner bases following the book "Groebner Bases – A Computational Approach to Commutative Algebra" by Becker and Weispfenning. Here we prove Buchberger's first criterium on avoiding S-polynomials: S-polynomials for polynomials with disjoint head terms need not be considered when constructing Groebner bases. In the course of formalizing this theorem we also introduced the splitting of a polynomial in an upper and a lower polynomial containing the greater resp. smaller terms of the original polynomial with respect to a given term order.

MML Identifier: `GROEB_3`.

The terminology and notation used in this paper have been introduced in the following articles: [24], [28], [29], [31], [1], [3], [12], [2], [8], [30], [9], [10], [17], [25], [16], [26], [11], [7], [5], [15], [13], [19], [27], [6], [4], [14], [23], [20], [22], [21], and [18].

1. Preliminaries

One can prove the following propositions:

(1) For every set X and for all bags b_1, b_2 of X holds $\frac{b_1+b_2}{b_2} = b_1$.

(2) Let n be an ordinal number, T be an admissible term order of n, and b_1, b_2, b_3 be bags of n. If $b_1 \leq_T b_2$, then $b_1 + b_3 \leq_T b_2 + b_3$.

(3) Let n be an ordinal number, T be a term order of n, and b_1, b_2, b_3 be bags of n. If $b_1 \leq_T b_2$ and $b_2 <_T b_3$, then $b_1 <_T b_3$.

[1]This work has been partially supported by grant BW 5100-5-0147-4.

ISSN 1426–2630

(4) Let n be an ordinal number, T be an admissible term order of n, and b_1, b_2, b_3 be bags of n. If $b_1 <_T b_2$, then $b_1 + b_3 <_T b_2 + b_3$.

(5) Let n be an ordinal number, T be an admissible term order of n, and b_1, b_2, b_3, b_4 be bags of n. If $b_1 <_T b_2$ and $b_3 \leq_T b_4$, then $b_1 + b_3 <_T b_2 + b_4$.

(6) Let n be an ordinal number, T be an admissible term order of n, and b_1, b_2, b_3, b_4 be bags of n. If $b_1 \leq_T b_2$ and $b_3 <_T b_4$, then $b_1 + b_3 <_T b_2 + b_4$.

2. More on Polynomials

One can prove the following propositions:

(7) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, and m_1, m_2 be non-zero monomials of n, L. Then $\operatorname{term} m_1 * m_2 = \operatorname{term} m_1 + \operatorname{term} m_2$.

(8) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, m be a non-zero monomial of n, L, and b be a bag of n. Then $b \in \operatorname{Support} p$ if and only if $\operatorname{term} m + b \in \operatorname{Support}(m * p)$.

(9) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, and m be a non-zero monomial of n, L. Then $\operatorname{Support}(m * p) = \{\operatorname{term} m + b; b$ ranges over elements of $\operatorname{Bags} n : b \in \operatorname{Support} p\}$.

(10) Let n be an ordinal number, L be an add-associative right complementable left zeroed right zeroed unital distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, and m be a non-zero monomial of n, L. Then $\operatorname{card} \operatorname{Support} p = \operatorname{card} \operatorname{Support}(m * p)$.

(11) Let n be an ordinal number, T be a connected term order of n, and L be an add-associative right complementable right zeroed non trivial loop structure. Then $\operatorname{Red}(0_n L, T) = 0_n L$.

(12) Let n be an ordinal number, L be an Abelian add-associative right zeroed right complementable commutative unital distributive non trivial double loop structure, and p, q be polynomials of n, L. If $p - q = 0_n L$, then $p = q$.

(13) Let X be a set and L be an add-associative right zeroed right complementable non empty loop structure. Then $-0_X L = 0_X L$.

(14) Let X be a set, L be an add-associative right zeroed right complementable non empty loop structure, and f be a series of X, L. Then $0_X L - f = -f$.

(15) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial double loop structure, and p be a polynomial of n, L. Then $p - \mathrm{Red}(p,T) = \mathrm{HM}(p,T)$.

Let n be an ordinal number, let L be an add-associative right complementable right zeroed non empty loop structure, and let p be a polynomial of n, L. Observe that $\mathrm{Support}\, p$ is finite.

Let n be an ordinal number, let L be a right zeroed add-associative right complementable unital distributive non trivial double loop structure, and let p, q be polynomials of n, L. Then $\{p, q\}$ is a non empty finite subset of $\mathrm{Polynom\text{-}Ring}(n, L)$.

3. Restriction and Splitting of Polynomials

Let X be a set, let L be a non empty zero structure, let s be a series of X, L, and let Y be a subset of $\mathrm{Bags}\, X$. The functor $s{\upharpoonright}Y$ yields a series of X, L and is defined as follows:

(Def. 1) $s{\upharpoonright}Y = s{+\cdot}(\mathrm{Support}\, s \setminus Y \longmapsto 0_L)$.

Let n be an ordinal number, let L be a non empty zero structure, let p be a polynomial of n, L, and let Y be a subset of $\mathrm{Bags}\, n$. Note that $p{\upharpoonright}Y$ is finite-Support.

Next we state three propositions:

(16) Let X be a set, L be a non empty zero structure, s be a series of X, L, and Y be a subset of $\mathrm{Bags}\, X$. Then $\mathrm{Support}(s{\upharpoonright}Y) = \mathrm{Support}\, s \cap Y$ and for every bag b of X such that $b \in \mathrm{Support}(s{\upharpoonright}Y)$ holds $(s{\upharpoonright}Y)(b) = s(b)$.

(17) Let X be a set, L be a non empty zero structure, s be a series of X, L, and Y be a subset of $\mathrm{Bags}\, X$. Then $\mathrm{Support}(s{\upharpoonright}Y) \subseteq \mathrm{Support}\, s$.

(18) For every set X and for every non empty zero structure L and for every series s of X, L holds $s{\upharpoonright}\mathrm{Support}\, s = s$ and $s{\upharpoonright}\emptyset_{\mathrm{Bags}\, X} = 0_X L$.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right zeroed right complementable non empty loop structure, let p be a polynomial of n, L, and let i be a natural number. Let us assume that $i \leq \mathrm{card}\, \mathrm{Support}\, p$. The functor $\mathrm{UpperSupport}(p, T, i)$ yielding a finite subset of $\mathrm{Bags}\, n$ is defined by the conditions (Def. 2).

(Def. 2)(i) $\mathrm{UpperSupport}(p, T, i) \subseteq \mathrm{Support}\, p$,

(ii) $\mathrm{card}\, \mathrm{UpperSupport}(p, T, i) = i$, and

(iii) for all bags b, b' of n such that $b \in \mathrm{UpperSupport}(p, T, i)$ and $b' \in \mathrm{Support}\, p$ and $b \leq_T b'$ holds $b' \in \mathrm{UpperSupport}(p, T, i)$.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right zeroed right complementable non empty loop structure, let p be a polynomial of n, L, and let i be a natural number. The functor $\mathrm{LowerSupport}(p, T, i)$ yielding a finite subset of $\mathrm{Bags}\, n$ is defined by:

(Def. 3) $\mathrm{LowerSupport}(p, T, i) = \mathrm{Support}\, p \setminus \mathrm{UpperSupport}(p, T, i)$.

We now state several propositions:

(19) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $i \leq \mathrm{card}\, \mathrm{Support}\, p$, then $\mathrm{UpperSupport}(p, T, i) \cup \mathrm{LowerSupport}(p, T, i) = \mathrm{Support}\, p$ and $\mathrm{UpperSupport}(p, T, i) \cap \mathrm{LowerSupport}(p, T, i) = \emptyset$.

(20) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. Suppose $i \leq \mathrm{card}\, \mathrm{Support}\, p$. Let b, b' be bags of n. If $b \in \mathrm{UpperSupport}(p, T, i)$ and $b' \in \mathrm{LowerSupport}(p, T, i)$, then $b' <_T b$.

(21) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, and p be a polynomial of n, L. Then $\mathrm{UpperSupport}(p, T, 0) = \emptyset$ and $\mathrm{LowerSupport}(p, T, 0) = \mathrm{Support}\, p$.

(22) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, and p be a polynomial of n, L. Then $\mathrm{UpperSupport}(p, T, \mathrm{card}\, \mathrm{Support}\, p) = \mathrm{Support}\, p$ and $\mathrm{LowerSupport}(p, T, \mathrm{card}\, \mathrm{Support}\, p) = \emptyset$.

(23) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non trivial loop structure, p be a non-zero polynomial of n, L, and i be a natural number. If $1 \leq i$ and $i \leq \mathrm{card}\, \mathrm{Support}\, p$, then $\mathrm{HT}(p, T) \in \mathrm{UpperSupport}(p, T, i)$.

(24) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. Suppose $i \leq \mathrm{card}\, \mathrm{Support}\, p$. Then $\mathrm{LowerSupport}(p, T, i) \subseteq \mathrm{Support}\, p$ and $\mathrm{card}\, \mathrm{LowerSupport}(p, T, i) = \mathrm{card}\, \mathrm{Support}\, p - i$ and for all bags b, b' of n such that $b \in \mathrm{LowerSupport}(p, T, i)$ and $b' \in \mathrm{Support}\, p$ and $b' \leq_T b$ holds $b' \in \mathrm{LowerSupport}(p, T, i)$.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right zeroed right complementable non empty loop structure, let p be a polynomial of n, L, and let i be a natural number. The functor $\mathrm{Up}(p, T, i)$ yields a polynomial of n, L and is defined by:

(Def. 4) $\mathrm{Up}(p, T, i) = p \restriction \mathrm{UpperSupport}(p, T, i)$.

The functor $\mathrm{Low}(p, T, i)$ yielding a polynomial of n, L is defined by:

(Def. 5) $\mathrm{Low}(p, T, i) = p \restriction \mathrm{LowerSupport}(p, T, i)$.

One can prove the following propositions:

(25) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $i \leq \operatorname{card} \operatorname{Support} p$, then $\operatorname{Support} \operatorname{Up}(p, T, i) = \operatorname{UpperSupport}(p, T, i)$ and $\operatorname{Support} \operatorname{Low}(p, T, i) = \operatorname{LowerSupport}(p, T, i)$.

(26) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $i \leq \operatorname{card} \operatorname{Support} p$, then $\operatorname{Support} \operatorname{Up}(p, T, i) \subseteq \operatorname{Support} p$ and $\operatorname{Support} \operatorname{Low}(p, T, i) \subseteq \operatorname{Support} p$.

(27) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, p be a polynomial of n, L, and i be a natural number. If $1 \leq i$ and $i \leq \operatorname{card} \operatorname{Support} p$, then $\operatorname{Support} \operatorname{Low}(p, T, i) \subseteq \operatorname{Support} \operatorname{Red}(p, T)$.

(28) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. Suppose $i \leq \operatorname{card} \operatorname{Support} p$. Let b be a bag of n. If $b \in \operatorname{Support} p$, then $b \in \operatorname{Support} \operatorname{Up}(p, T, i)$ or $b \in \operatorname{Support} \operatorname{Low}(p, T, i)$ but $b \notin \operatorname{Support} \operatorname{Up}(p, T, i) \cap \operatorname{Support} \operatorname{Low}(p, T, i)$.

(29) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. Suppose $i \leq \operatorname{card} \operatorname{Support} p$. Let b, b' be bags of n. If $b \in \operatorname{Support} \operatorname{Low}(p, T, i)$ and $b' \in \operatorname{Support} \operatorname{Up}(p, T, i)$, then $b <_T b'$.

(30) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $1 \leq i$ and $i \leq \operatorname{card} \operatorname{Support} p$, then $\operatorname{HT}(p, T) \in \operatorname{Support} \operatorname{Up}(p, T, i)$.

(31) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. Suppose $i \leq \operatorname{card} \operatorname{Support} p$. Let b be a bag of n. If $b \in \operatorname{Support} \operatorname{Low}(p, T, i)$, then $(\operatorname{Low}(p, T, i))(b) = p(b)$ and $(\operatorname{Up}(p, T, i))(b) = 0_L$.

(32) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. Suppose $i \leq \operatorname{card} \operatorname{Support} p$. Let b be a bag of n. If $b \in \operatorname{Support} \operatorname{Up}(p, T, i)$, then $(\operatorname{Up}(p, T, i))(b) = p(b)$ and $(\operatorname{Low}(p, T, i))(b) = 0_L$.

(33) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop

structure, p be a polynomial of n, L, and i be a natural number. If $i \leq \operatorname{card} \operatorname{Support} p$, then $\operatorname{Up}(p, T, i) + \operatorname{Low}(p, T, i) = p$.

(34) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, and p be a polynomial of n, L. Then $\operatorname{Up}(p, T, 0) = 0_n L$ and $\operatorname{Low}(p, T, 0) = p$.

(35) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable Abelian non empty double loop structure, and p be a polynomial of n, L. Then $\operatorname{Up}(p, T, \operatorname{card} \operatorname{Support} p) = p$ and $\operatorname{Low}(p, T, \operatorname{card} \operatorname{Support} p) = 0_n L$.

(36) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable Abelian non trivial double loop structure, and p be a non-zero polynomial of n, L. Then $\operatorname{Up}(p, T, 1) = \operatorname{HM}(p, T)$ and $\operatorname{Low}(p, T, 1) = \operatorname{Red}(p, T)$.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right zeroed right complementable non trivial loop structure, and let p be a non-zero polynomial of n, L. Observe that $\operatorname{Up}(p, T, 0)$ is monomial-like.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right zeroed right complementable Abelian non trivial double loop structure, and let p be a non-zero polynomial of n, L. Note that $\operatorname{Up}(p, T, 1)$ is non-zero and monomial-like and $\operatorname{Low}(p, T, \operatorname{card} \operatorname{Support} p)$ is monomial-like.

The following propositions are true:

(37) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non trivial loop structure, p be a polynomial of n, L, and j be a natural number. If $j = \operatorname{card} \operatorname{Support} p - 1$, then $\operatorname{Low}(p, T, j)$ is a non-zero monomial of n, L.

(38) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $i < \operatorname{card} \operatorname{Support} p$, then $\operatorname{HT}(\operatorname{Low}(p, T, i + 1), T) \leq_T \operatorname{HT}(\operatorname{Low}(p, T, i), T)$.

(39) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $0 < i$ and $i < \operatorname{card} \operatorname{Support} p$, then $\operatorname{HT}(\operatorname{Low}(p, T, i), T) <_T \operatorname{HT}(p, T)$.

(40) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, m be a non-zero monomial of n, L, and i be a natural number. Suppose $i \leq \operatorname{card} \operatorname{Support} p$. Let b be a bag of n. Then $\operatorname{term} m + b \in \operatorname{Support} \operatorname{Low}(m * p, T, i)$ if and only if $b \in \operatorname{Support} \operatorname{Low}(p, T, i)$.

(41) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $i < \mathrm{card\,Support}\,p$, then $\mathrm{Support\,Low}(p, T, i+1) \subseteq \mathrm{Support\,Low}(p, T, i)$.

(42) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right zeroed right complementable non empty loop structure, p be a polynomial of n, L, and i be a natural number. If $i < \mathrm{card\,Support}\,p$, then $\mathrm{Support\,Low}(p, T, i) \setminus \mathrm{Support\,Low}(p, T, i+1) = \{\mathrm{HT}(\mathrm{Low}(p, T, i), T)\}$.

(43) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right zeroed right complementable non trivial loop structure, p be a polynomial of n, L, and i be a natural number. If $i < \mathrm{card\,Support}\,p$, then $\mathrm{Low}(p, T, i+1) = \mathrm{Red}(\mathrm{Low}(p, T, i), T)$.

(44) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, m be a non-zero monomial of n, L, and i be a natural number. If $i \leq \mathrm{card\,Support}\,p$, then $\mathrm{Low}(m * p, T, i) = m * \mathrm{Low}(p, T, i)$.

4. More on Polynomial Reduction

Next we state several propositions:

(45) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and f, g, p be polynomials of n, L. If f reduces to g, p, T, then $-f$ reduces to $-g$, p, T.

(46) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and f, f_1, g, p be polynomials of n, L. Suppose f reduces to f_1, $\{p\}$, T and for every bag b_1 of n such that $b_1 \in \mathrm{Support}\,g$ holds $\mathrm{HT}(p, T) \nmid b_1$. Then $f + g$ reduces to $f_1 + g$, $\{p\}$, T.

(47) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and f, g be non-zero polynomials of n, L. Then $f * g$ reduces to $\mathrm{Red}(f, T) * g$, $\{g\}$, T.

(48) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative

associative left unital right unital distributive Abelian field-like non trivial double loop structure, f, g be non-zero polynomials of n, L, and p be a polynomial of n, L. If $p(\mathrm{HT}(f * g, T)) = 0_L$, then $f * g + p$ reduces to $\mathrm{Red}(f, T) * g + p$, $\{g\}$, T.

(49) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, P be a subset of $\mathrm{Polynom\text{-}Ring}(n, L)$, and f, g be polynomials of n, L. If $\mathrm{PolyRedRel}(P, T)$ reduces f to g, then $\mathrm{PolyRedRel}(P, T)$ reduces $-f$ to $-g$.

(50) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and f, f_1, g, p be polynomials of n, L. Suppose $\mathrm{PolyRedRel}(\{p\}, T)$ reduces f to f_1 and for every bag b_1 of n such that $b_1 \in \mathrm{Support}\, g$ holds $\mathrm{HT}(p, T) \nmid b_1$. Then $\mathrm{PolyRedRel}(\{p\}, T)$ reduces $f+g$ to $f_1 + g$.

(51) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and f, g be non-zero polynomials of n, L. Then $\mathrm{PolyRedRel}(\{g\}, T)$ reduces $f * g$ to $0_n L$.

5. The Criterium

We now state several propositions:

(52) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive field-like non trivial double loop structure, and p_1, p_2 be polynomials of n, L. Suppose $\mathrm{HT}(p_1, T)$, $\mathrm{HT}(p_2, T)$ are disjoint. Let b_1, b_2 be bags of n. If $b_1 \in \mathrm{Support}\,\mathrm{Red}(p_1, T)$ and $b_2 \in \mathrm{Support}\,\mathrm{Red}(p_2, T)$, then $\mathrm{HT}(p_1, T) + b_2 \neq \mathrm{HT}(p_2, T) + b_1$.

(53) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and p_1, p_2 be polynomials of n, L. If $\mathrm{HT}(p_1, T)$, $\mathrm{HT}(p_2, T)$ are disjoint, then $\mathrm{S\text{-}Poly}(p_1, p_2, T) = \mathrm{HM}(p_2, T) * \mathrm{Red}(p_1, T) - \mathrm{HM}(p_1, T) * \mathrm{Red}(p_2, T)$.

(54) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double

loop structure, and p_1, p_2 be polynomials of n, L. If $\mathrm{HT}(p_1,T)$, $\mathrm{HT}(p_2,T)$ are disjoint, then $\text{S-Poly}(p_1,p_2,T) = \mathrm{Red}(p_1,T) * p_2 - \mathrm{Red}(p_2,T) * p_1$.

(55) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and p_1, p_2 be non-zero polynomials of n, L. Suppose $\mathrm{HT}(p_1,T)$, $\mathrm{HT}(p_2,T)$ are disjoint and $\mathrm{Red}(p_1,T)$ is non-zero and $\mathrm{Red}(p_2,T)$ is non-zero. Then $\mathrm{PolyRedRel}(\{p_1\},T)$ reduces $\mathrm{HM}(p_2,T) * \mathrm{Red}(p_1,T) - \mathrm{HM}(p_1,T) * \mathrm{Red}(p_2,T)$ to $p_2 * \mathrm{Red}(p_1,T)$.

(56) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non trivial double loop structure, and p_1, p_2 be polynomials of n, L. If $\mathrm{HT}(p_1,T)$, $\mathrm{HT}(p_2,T)$ are disjoint, then $\mathrm{PolyRedRel}(\{p_1,p_2\},T)$ reduces $\text{S-Poly}(p_1,p_2,T)$ to 0_nL.

(57) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non degenerated non empty double loop structure, and G be a subset of $\text{Polynom-Ring}(n,L)$. Suppose G is a Groebner basis wrt T. Let g_1, g_2 be polynomials of n, L. Suppose $g_1 \in G$ and $g_2 \in G$ and $\mathrm{HT}(g_1,T)$, $\mathrm{HT}(g_2,T)$ are not disjoint. Then $\mathrm{PolyRedRel}(G,T)$ reduces $\text{S-Poly}(g_1,g_2,T)$ to 0_nL.

(58) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non degenerated non trivial double loop structure, and G be a subset of $\text{Polynom-Ring}(n,L)$. Suppose $0_nL \notin G$. Suppose that for all polynomials g_1, g_2 of n, L such that $g_1 \in G$ and $g_2 \in G$ and $\mathrm{HT}(g_1,T)$, $\mathrm{HT}(g_2,T)$ are not disjoint holds $\mathrm{PolyRedRel}(G,T)$ reduces $\text{S-Poly}(g_1,g_2,T)$ to 0_nL. Let g_1, g_2, h be polynomials of n, L. Suppose $g_1 \in G$ and $g_2 \in G$ and $\mathrm{HT}(g_1,T)$, $\mathrm{HT}(g_2,T)$ are not disjoint and h is a normal form of $\text{S-Poly}(g_1,g_2,T)$ w.r.t. $\mathrm{PolyRedRel}(G,T)$. Then $h = 0_nL$.

(59) Let n be a natural number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative left unital right unital distributive Abelian field-like non degenerated non empty double loop structure, and G be a subset of $\text{Polynom-Ring}(n,L)$. Suppose $0_nL \notin G$. Suppose that for all polynomials g_1, g_2, h of n, L such that $g_1 \in G$ and $g_2 \in G$ and $\mathrm{HT}(g_1,T)$, $\mathrm{HT}(g_2,T)$ are not disjoint and h is a normal form of $\text{S-Poly}(g_1,g_2,T)$ w.r.t. $\mathrm{PolyRedRel}(G,T)$ holds $h = 0_nL$. Then G is a Groebner basis wrt T.

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Received December 10, 2004