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The Banach Space l^p

Yasumasa Suzuki
Take, Yokosuka-shi
Japan

Summary. We introduce the arithmetic addition and multiplication in the set of l^p real sequences and also introduce the norm. This set has the structure of the Banach space.

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The notation and terminology used in this paper have been introduced in the following articles: [16], [5], [19], [20], [3], [4], [1], [15], [7], [18], [2], [17], [10], [9], [8], [12], [11], [6], [14], and [13].

1. The Real Norm Space of l^p Real Sequences

Let x be a sequence of real numbers and let p be a real number. The functor x^p yielding a sequence of real numbers is defined as follows:

(Def. 1) For every natural number n holds $x^p(n) = |x(n)|^p$.

Let p be a real number. Let us assume that $p \geq 1$. The functor l^p yielding a non empty subset of the carrier of the linear space of real sequences is defined as follows:

(Def. 2) For every set x holds $x \in l^p$ iff $x \in$ the set of real sequences and $(\mathrm{id}_{\mathrm{seq}}(x))^p$ is summable.

In the sequel a, b, c are real numbers.

We now state several propositions:

(1) If $a \geq 0$ and $a < b$ and $c > 0$, then $a^c < b^c$.

(2) Let p be a real number. Suppose $1 \leq p$. Let a, b be sequences of real numbers and n be a natural number. Then $(\sum_{\alpha=0}^{\kappa}((a+b)^p)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}} \leq (\sum_{\alpha=0}^{\kappa}(a^p)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}} + (\sum_{\alpha=0}^{\kappa}(b^p)(\alpha))_{\kappa\in\mathbb{N}}(n)^{\frac{1}{p}}$.

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(3) Let a, b be sequences of real numbers and p be a real number. Suppose $1 \leq p$ and a^p is summable and b^p is summable. Then $(a+b)^p$ is summable and $(\sum((a+b)^p))^{\frac{1}{p}} \leq (\sum(a^p))^{\frac{1}{p}} + (\sum(b^p))^{\frac{1}{p}}$.

(4) For every real number p such that $1 \leq p$ holds l^p is linearly closed.

(5) Let p be a real number. Suppose $1 \leq p$. Then $\langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences})\rangle$ is a subspace of the linear space of real sequences.

(6) Let p be a real number. Suppose $1 \leq p$. Then $\langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences})\rangle$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

(7) Let p be a real number. Suppose $1 \leq p$. Then $\langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences})\rangle$ is a real linear space.

Let p be a real number. The functor l^p-norm yielding a function from l^p into $\mathbb{R}$ is defined by:

(Def. 3) For every set x such that $x \in l^p$ holds $l^p\text{-norm}(x) = (\sum((\mathrm{id}_{\mathrm{seq}}(x))^p))^{\frac{1}{p}}$.

The following two propositions are true:

(8) Let p be a real number. Suppose $1 \leq p$. Then $\langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$ is a real linear space.

(9) Let p be a real number. Suppose $p \geq 1$. Then $\langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$ is a subspace of the linear space of real sequences.

2. The Banach Space of l^p Real Sequences

Next we state several propositions:

(10) Let p be a real number. Suppose $1 \leq p$. Let l_1 be a non empty normed structure. Suppose $l_1 = \langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$. Then the carrier of $l_1 = l^p$ and for every set x holds x is a vector of l_1 iff x is a sequence of real numbers and $(\mathrm{id}_{\mathrm{seq}}(x))^p$ is summable and $0_{(l_1)} = \mathrm{Zeroseq}$ and for every vector x of l_1 holds $x = \mathrm{id}_{\mathrm{seq}}(x)$ and for all vectors x, y of l_1 holds $x + y = \mathrm{id}_{\mathrm{seq}}(x) + \mathrm{id}_{\mathrm{seq}}(y)$ and for every real number r and for every

vector x of l_1 holds $r \cdot x = r\ \mathrm{id_{seq}}(x)$ and for every vector x of l_1 holds $-x = -\mathrm{id_{seq}}(x)$ and $\mathrm{id_{seq}}(-x) = -\mathrm{id_{seq}}(x)$ and for all vectors x, y of l_1 holds $x-y = \mathrm{id_{seq}}(x)-\mathrm{id_{seq}}(y)$ and for every vector x of l_1 holds $(\mathrm{id_{seq}}(x))^p$ is summable and for every vector x of l_1 holds $\|x\| = (\sum((\mathrm{id_{seq}}(x))^p))^{\frac{1}{p}}$.

(11) Let p be a real number. Suppose $p \geq 1$. Let r_1 be a sequence of real numbers. Suppose that for every natural number n holds $r_1(n) = 0$. Then ${r_1}^p$ is summable and $(\sum({r_1}^p))^{\frac{1}{p}} = 0$.

(12) Let p be a real number. Suppose $1 \leq p$. Let r_1 be a sequence of real numbers. Suppose ${r_1}^p$ is summable and $(\sum({r_1}^p))^{\frac{1}{p}} = 0$. Let n be a natural number. Then $r_1(n) = 0$.

(13) Let p be a real number. Suppose $1 \leq p$. Let l_1 be a non empty normed structure. Suppose $l_1 = \langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$. Let x, y be points of l_1 and a be a real number. Then $\|x\| = 0$ iff $x = 0_{(l_1)}$ and $0 \leq \|x\|$ and $\|x+y\| \leq \|x\| + \|y\|$ and $\|a \cdot x\| = |a| \cdot \|x\|$.

(14) Let p be a real number. Suppose $p \geq 1$. Let l_1 be a non empty normed structure. Suppose $l_1 = \langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$. Then l_1 is real normed space-like.

(15) Let p be a real number. Suppose $p \geq 1$. Let l_1 be a non empty normed structure. Suppose $l_1 = \langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$. Then l_1 is a real normed space.

(16) Let p be a real number. Suppose $1 \leq p$. Let l_1 be a real normed space. Suppose $l_1 = \langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$. Let v_1 be a sequence of l_1. If v_1 is Cauchy sequence by norm, then v_1 is convergent.

Let p be a real number. Let us assume that $1 \leq p$. The functor l^p-space yielding a real Banach space is defined by the condition (Def. 4).

(Def. 4) l^p-space $= \langle l^p, \mathrm{Zero\_}(l^p, \text{the linear space of real sequences}), \mathrm{Add\_}(l^p, \text{the linear space of real sequences}), \mathrm{Mult\_}(l^p, \text{the linear space of real sequences}), l^p\text{-norm}\rangle$.

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