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On the Characterization of Collineations of the Segre Product of Strongly Connected Partial Linear Spaces[1]

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Summary. In this paper we characterize the automorphisms (collineations) of the Segre product of partial linear spaces. In particular, we show that if all components of the product are strongly connected, then every collineation is determined by a set of isomorphisms between its components. The formalization follows the ideas presented in the *Journal of Geometry* paper [16] by Naumowicz and Prażmowski.

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The articles [20], [10], [2], [23], [22], [6], [8], [9], [19], [24], [7], [1], [11], [5], [3], [17], [21], [12], [13], [18], [4], [15], and [14] provide the terminology and notation for this paper.

1. Preliminaries

The following propositions are true:

(1) Let S be a non empty non void topological structure, f be a collineation of S, and p, q be points of S. If p, q are collinear, then $f(p)$, $f(q)$ are collinear.

(2) Let I be a non empty set, i be an element of I, and A be a non-Trivial-yielding 1-sorted yielding many sorted set indexed by I. Then $A(i)$ is non trivial.

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(3) Let S be a non void identifying close blocks topological structure such that S is strongly connected. Then S has no isolated points.

(4) Let S be a non empty non void identifying close blocks topological structure. If S is strongly connected, then S is connected.

(5) Let S be a non void non degenerated topological structure and L be a block of S. Then there exists a point x of S such that $x \notin L$.

(6) Let I be a non empty set and A be a nonempty TopStruct-yielding many sorted set indexed by I. Then every point of SegreProduct A is an element of the support of A.

(7) Let I be a non empty set, A be a 1-sorted yielding many sorted set indexed by I, and x be an element of I. Then (the support of A)$(x) = \Omega_{A(x)}$.

(8) Let I be a non empty set, i be an element of I, and A be a non-Trivial-yielding TopStruct-yielding many sorted set indexed by I. Then there exists a Segre-like non trivial-yielding many sorted subset L indexed by the support of A such that $\mathrm{index}(L) = i$ and $\prod L$ is a Segre coset of A.

(9) Let I be a non empty set, i be an element of I, A be a non-Trivial-yielding TopStruct-yielding many sorted set indexed by I, and p be a point of SegreProduct A. Then there exists a Segre-like non trivial-yielding many sorted subset L indexed by the support of A such that $\mathrm{index}(L) = i$ and $\prod L$ is a Segre coset of A and $p \in \prod L$.

(10) Let I be a non empty set, A be a non-Trivial-yielding TopStruct-yielding many sorted set indexed by I, and b be a Segre-like non trivial-yielding many sorted subset indexed by the support of A. If $\prod b$ is a Segre coset of A, then $b(\mathrm{index}(b)) = \Omega_{A(\mathrm{index}(b))}$.

(11) Let I be a non empty set, A be a non-Trivial-yielding TopStruct-yielding many sorted set indexed by I, and L_1, L_2 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A. Suppose $\prod L_1$ is a Segre coset of A and $\prod L_2$ is a Segre coset of A and $\mathrm{index}(L_1) = \mathrm{index}(L_2)$ and $\prod L_1$ meets $\prod L_2$. Then $L_1 = L_2$.

(12) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, L be a Segre-like non trivial-yielding many sorted subset indexed by the support of A, and B be a block of $A(\mathrm{index}(L))$. Then $\prod(L +\cdot (\mathrm{index}(L), B))$ is a block of SegreProduct A.

(13) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, i be an element of I, p be a point of $A(i)$, and L be a Segre-like non trivial-yielding many sorted subset indexed by the support of A. Suppose $i \neq \mathrm{index}(L)$. Then $L +\cdot (i, \{p\})$ is a Segre-like non trivial-yielding many sorted subset indexed by the support of A.

(14) Let I be a non empty set, A be a PLS-yielding many sorted set indexed

by I, i be an element of I, S be a subset of $A(i)$, and L be a Segre-like non trivial-yielding many sorted subset indexed by the support of A. Then $\prod(L +\cdot (i, S))$ is a subset of SegreProduct A.

(15) Let I be a non empty set, P be a many sorted set indexed by I, and i be an element of I. Then $\{P\}(i)$ is non empty and trivial.

(16) Let I be a non empty set, i be an element of I, A be a PLS-yielding many sorted set indexed by I, B be a block of $A(i)$, and P be an element of the support of A. Then $\prod(\{P\} +\cdot (i, B))$ is a block of SegreProduct A.

(17) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, and p, q be points of SegreProduct A. Suppose $p \neq q$. Then p, q are collinear if and only if for all many sorted sets p_1, q_1 indexed by I such that $p_1 = p$ and $q_1 = q$ there exists an element i of I such that for all points a, b of $A(i)$ such that $a = p_1(i)$ and $b = q_1(i)$ holds $a \neq b$ and a, b are collinear and for every element j of I such that $j \neq i$ holds $p_1(j) = q_1(j)$.

(18) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, b be a Segre-like non trivial-yielding many sorted subset indexed by the support of A, and x be a point of $A(\mathrm{index}(b))$. Then there exists a many sorted set p indexed by I such that $p \in \prod b$ and $\{(p +\cdot (\mathrm{index}(b), x)\ \textbf{qua}\ \mathrm{set})\ \} = \prod(b +\cdot (\mathrm{index}(b), \{x\}))$.

Let I be a finite non empty set and let b_1, b_2 be many sorted sets indexed by I. The functor ${b_1}'(b_2)$ yields a natural number and is defined by:

(Def. 1) ${b_1}'(b_2) = \overline{\overline{\{i; i \text{ ranges over elements of } I\colon b_1(i) \neq b_2(i)\}}}$.

One can prove the following proposition

(19) Let I be a finite non empty set, b_1, b_2 be many sorted sets indexed by I, and i be an element of I. If $b_1(i) \neq b_2(i)$, then ${b_1}'(b_2) = {b_1}'(b_2 +\cdot (i, b_1(i))) + 1$.

2. The Adherence of Segre Cosets

Let I be a non empty set, let A be a PLS-yielding many sorted set indexed by I, and let B_1, B_2 be Segre cosets of A. The predicate $B_1 || B_2$ is defined as follows:

(Def. 2) For every point x of SegreProduct A such that $x \in B_1$ there exists a point y of SegreProduct A such that $y \in B_2$ and x, y are collinear.

Next we state several propositions:

(20) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, and B_1, B_2 be Segre cosets of A. Suppose $B_1 || B_2$. Let f be a collineation of SegreProduct A and C_1, C_2 be Segre cosets of A. If $C_1 = f^{\circ} B_1$ and $C_2 = f^{\circ} B_2$, then $C_1 || C_2$.

(21) Let I be a non empty set, A be a PLS-yielding many sorted set indexed by I, and B_1, B_2 be Segre cosets of A. Suppose B_1 misses B_2. Then $B_1 || B_2$ if and only if for all Segre-like non trivial-yielding many sorted subsets b_1, b_2 indexed by the support of A such that $B_1 = \prod b_1$ and $B_2 = \prod b_2$ holds $\mathrm{index}(b_1) = \mathrm{index}(b_2)$ and there exists an element r of I such that $r \neq \mathrm{index}(b_1)$ and for every element i of I such that $i \neq r$ holds $b_1(i) = b_2(i)$ and for all points c_1, c_2 of $A(r)$ such that $b_1(r) = \{c_1\}$ and $b_2(r) = \{c_2\}$ holds c_1, c_2 are collinear.

(22) Let I be a finite non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is connected. Let i be an element of I, p be a point of $A(i)$, and b_1, b_2 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A. Suppose $\prod b_1$ is a Segre coset of A and $\prod b_2$ is a Segre coset of A and $b_1 = b_2 +\cdot (i, \{p\})$ and $p \notin b_2(i)$. Then there exists a finite sequence D of elements of $2^{\text{the carrier of SegreProduct}\,A}$ such that

(i) $D(1) = \prod b_1$,

(ii) $D(\mathrm{len}\,D) = \prod b_2$,

(iii) for every natural number i such that $i \in \mathrm{dom}\,D$ holds $D(i)$ is a Segre coset of A, and

(iv) for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\,D$ and for all Segre cosets D_1, D_2 of A such that $D_1 = D(i)$ and $D_2 = D(i+1)$ holds D_1 misses D_2 and $D_1 || D_2$.

(23) Let I be a finite non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is connected. Let B_1, B_2 be Segre cosets of A. Suppose B_1 misses B_2. Let b_1, b_2 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A. Suppose $B_1 = \prod b_1$ and $B_2 = \prod b_2$. Then $\mathrm{index}(b_1) = \mathrm{index}(b_2)$ if and only if there exists a finite sequence D of elements of $2^{\text{the carrier of SegreProduct}\,A}$ such that $D(1) = B_1$ and $D(\mathrm{len}\,D) = B_2$ and for every natural number i such that $i \in \mathrm{dom}\,D$ holds $D(i)$ is a Segre coset of A and for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\,D$ and for all Segre cosets D_1, D_2 of A such that $D_1 = D(i)$ and $D_2 = D(i+1)$ holds D_1 misses D_2 and $D_1 || D_2$.

(24) Let I be a finite non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A, B_1, B_2 be Segre cosets of A, and b_1, b_2, b_3, b_4 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A. If $B_1 = \prod b_1$ and $B_2 = \prod b_2$ and $f^\circ B_1 = \prod b_3$ and $f^\circ B_2 = \prod b_4$, then if $\mathrm{index}(b_1) = \mathrm{index}(b_2)$, then $\mathrm{index}(b_3) = \mathrm{index}(b_4)$.

(25) Let I be a finite non empty set and A be a PLS-yielding many sorted set

indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A. Then there exists a permutation s of I such that for all elements i, j of I holds $s(i) = j$ if and only if for every Segre coset B_1 of A and for all Segre-like non trivial-yielding many sorted subsets b_1, b_2 indexed by the support of A such that $B_1 = \prod b_1$ and $f^\circ B_1 = \prod b_2$ holds if $\mathrm{index}(b_1) = i$, then $\mathrm{index}(b_2) = j$.

Let I be a finite non empty set and let A be a PLS-yielding many sorted set indexed by I. Let us assume that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A. The functor $\mathrm{IndPerm}(f)$ yields a permutation of I and is defined by the condition (Def. 3).

(Def. 3) Let i, j be elements of I. Then $(\mathrm{IndPerm}(f))(i) = j$ if and only if for every Segre coset B_1 of A and for all Segre-like non trivial-yielding many sorted subsets b_1, b_2 indexed by the support of A such that $B_1 = \prod b_1$ and $f^\circ B_1 = \prod b_2$ holds if $\mathrm{index}(b_1) = i$, then $\mathrm{index}(b_2) = j$.

3. Canonical Embeddings and Automorphisms of Segre Product

Let I be a finite non empty set and let A be a PLS-yielding many sorted set indexed by I. Let us assume that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A and let b_1 be a Segre-like non trivial-yielding many sorted subset indexed by the support of A. Let us assume that $\prod b_1$ is a Segre coset of A. The functor $\mathrm{CanEmb}(f, b_1)$ yields a map from $A(\mathrm{index}(b_1))$ into $A((\mathrm{IndPerm}(f))(\mathrm{index}(b_1)))$ and is defined by the conditions (Def. 4).

(Def. 4)(i) $\mathrm{CanEmb}(f, b_1)$ is isomorphic, and

(ii) for every many sorted set a indexed by I such that a is a point of SegreProduct A and $a \in \prod b_1$ and for every many sorted set b indexed by I such that $b = f(a)$ holds $b((\mathrm{IndPerm}(f))(\mathrm{index}(b_1))) = (\mathrm{CanEmb}(f, b_1))(a(\mathrm{index}(b_1)))$.

Next we state two propositions:

(26) Let I be a finite non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A and B_1, B_2 be Segre cosets of A. Suppose B_1 misses B_2 and $B_1 || B_2$. Let b_1, b_2 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A. If $\prod b_1 = B_1$ and $\prod b_2 = B_2$, then $\mathrm{CanEmb}(f, b_1) = \mathrm{CanEmb}(f, b_2)$.

(27) Let I be a finite non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A and b_1, b_2 be Segre-like non trivial-yielding many sorted subsets indexed by the support of A.

Suppose $\prod b_1$ is a Segre coset of A and $\prod b_2$ is a Segre coset of A and $\mathrm{index}(b_1) = \mathrm{index}(b_2)$. Then $\mathrm{CanEmb}(f, b_1) = \mathrm{CanEmb}(f, b_2)$.

Let I be a finite non empty set and let A be a PLS-yielding many sorted set indexed by I. Let us assume that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A and let i be an element of I. The functor $\mathrm{CanEmb}(f, i)$ yields a map from $A(i)$ into $A((\mathrm{IndPerm}(f))(i))$ and is defined by the condition (Def. 5).

(Def. 5) Let b be a Segre-like non trivial-yielding many sorted subset indexed by the support of A. If $\prod b$ is a Segre coset of A and $\mathrm{index}(b) = i$, then $\mathrm{CanEmb}(f, i) = \mathrm{CanEmb}(f, b)$.

Next we state the proposition

(28) Let I be a finite non empty set and A be a PLS-yielding many sorted set indexed by I. Suppose that for every element i of I holds $A(i)$ is strongly connected. Let f be a collineation of SegreProduct A. Then there exists a permutation s of I and there exists a function yielding many sorted set B indexed by I such that for every element i of I holds

(i) $B(i)$ is a map from $A(i)$ into $A(s(i))$,

(ii) for every map m from $A(i)$ into $A(s(i))$ such that $m = B(i)$ holds m is isomorphic, and

(iii) for every point p of SegreProduct A and for every many sorted set a indexed by I such that $a = p$ and for every many sorted set b indexed by I such that $b = f(p)$ and for every map m from $A(i)$ into $A(s(i))$ such that $m = B(i)$ holds $b(s(i)) = m(a(i))$.

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