FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Spaces of Pencils, Grassmann Spaces, and Generalized Veronese Spaces[1]

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Summary. In this paper we construct several examples of partial linear spaces. First, we define two algebraic structures, namely the spaces of k-pencils and Grassmann spaces for vector spaces over an arbitrary field. Then we introduce the notion of generalized Veronese spaces following the definition presented in the paper [8] by Naumowicz and Prażmowski. For all spaces defined, we state the conditions under which they are not degenerated to a single line.

MML Identifier: PENCIL_4.

The terminology and notation used here are introduced in the following articles: [11], [16], [4], [2], [9], [3], [1], [5], [10], [7], [15], [6], [14], [13], [12], and [17].

1. Spaces of k-Pencils

One can prove the following propositions:

(1) For all natural numbers k, n such that $1 \leq k$ and $k < n$ and $3 \leq n$ holds $k+1 < n$ or $2 \leq k$.

(2) For every finite set X and for every natural number n such that $n \leq \operatorname{card} X$ there exists a subset Y of X such that $\operatorname{card} Y = n$.

(3) For every field F and for every vector space V over F holds every subspace of V is a subspace of Ω_V.

(4) For every field F and for every vector space V over F holds every subspace of Ω_V is a subspace of V.

[1]This work has been partially supported by the KBN grant 4 T11C 039 24OB and the FP6 IST grant TYPES No. 510096.

ISSN 1426–2630

(5) For every field F and for every vector space V over F and for every subspace W of V holds Ω_W is a subspace of V.

(6) Let F be a field and V, W be vector spaces over F. If Ω_W is a subspace of V, then W is a subspace of V.

Let F be a field, let V be a vector space over F, and let W_1, W_2 be subspaces of V. The functor segment(W_1, W_2) yielding a subset of Subspaces V is defined by:

(Def. 1)(i) For every strict subspace W of V holds $W \in \text{segment}(W_1, W_2)$ iff W_1 is a subspace of W and W is a subspace of W_2 if W_1 is a subspace of W_2,

(ii) $\text{segment}(W_1, W_2) = \emptyset$, otherwise.

We now state the proposition

(7) Let F be a field, V be a vector space over F, and W_1, W_2, W_3, W_4 be subspaces of V. Suppose W_1 is a subspace of W_2 and W_3 is a subspace of W_4 and $\Omega_{(W_1)} = \Omega_{(W_3)}$ and $\Omega_{(W_2)} = \Omega_{(W_4)}$. Then $\text{segment}(W_1, W_2) = \text{segment}(W_3, W_4)$.

Let F be a field, let V be a vector space over F, and let W_1, W_2 be subspaces of V. The functor pencil(W_1, W_2) yielding a subset of Subspaces V is defined by:

(Def. 2) $\text{pencil}(W_1, W_2) = \text{segment}(W_1, W_2) \setminus \{\Omega_{(W_1)}, \Omega_{(W_2)}\}$.

Next we state the proposition

(8) Let F be a field, V be a vector space over F, and W_1, W_2, W_3, W_4 be subspaces of V. Suppose W_1 is a subspace of W_2 and W_3 is a subspace of W_4 and $\Omega_{(W_1)} = \Omega_{(W_3)}$ and $\Omega_{(W_2)} = \Omega_{(W_4)}$. Then $\text{pencil}(W_1, W_2) = \text{pencil}(W_3, W_4)$.

Let F be a field, let V be a finite dimensional vector space over F, let W_1, W_2 be subspaces of V, and let k be a natural number. The functor pencil(W_1, W_2, k) yielding a subset of $\text{Sub}_k(V)$ is defined by:

(Def. 3) $\text{pencil}(W_1, W_2, k) = \text{pencil}(W_1, W_2) \cap \text{Sub}_k(V)$.

We now state two propositions:

(9) Let F be a field, V be a finite dimensional vector space over F, k be a natural number, and W_1, W_2, W be subspaces of V. If $W \in \text{pencil}(W_1, W_2, k)$, then W_1 is a subspace of W and W is a subspace of W_2.

(10) Let F be a field, V be a finite dimensional vector space over F, k be a natural number, and W_1, W_2, W_3, W_4 be subspaces of V. Suppose W_1 is a subspace of W_2 and W_3 is a subspace of W_4 and $\Omega_{(W_1)} = \Omega_{(W_3)}$ and $\Omega_{(W_2)} = \Omega_{(W_4)}$. Then $\text{pencil}(W_1, W_2, k) = \text{pencil}(W_3, W_4, k)$.

Let F be a field, let V be a finite dimensional vector space over F, and let k be a natural number. k pencils of V yields a family of subsets of $\text{Sub}_k(V)$ and is defined by the condition (Def. 4).

(Def. 4) Let X be a set. Then $X \in k$ pencils of V if and only if there exist subspaces W_1, W_2 of V such that W_1 is a subspace of W_2 and $\dim(W_1) + 1 = k$ and $\dim(W_2) = k + 1$ and $X = \mathrm{pencil}(W_1, W_2, k)$.

We now state several propositions:

(11) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $1 \leq k$ and $k < \dim(V)$, then k pencils of V is non empty.

(12) Let F be a field, V be a finite dimensional vector space over F, W_1, W_2, P, Q be subspaces of V, and k be a natural number. Suppose $1 \leq k$ and $k < \dim(V)$ and $\dim(W_1) + 1 = k$ and $\dim(W_2) = k + 1$ and $P \in \mathrm{pencil}(W_1, W_2, k)$ and $Q \in \mathrm{pencil}(W_1, W_2, k)$ and $P \neq Q$. Then $P \cap Q = \Omega_{(W_1)}$ and $P + Q = \Omega_{(W_2)}$.

(13) Let F be a field, V be a finite dimensional vector space over F, and v be a vector of V. If $v \neq 0_V$, then $\dim(\mathrm{Lin}(\{v\})) = 1$.

(14) Let F be a field, V be a finite dimensional vector space over F, W be a subspace of V, and v be a vector of V. If $v \notin W$, then $\dim(W + \mathrm{Lin}(\{v\})) = \dim(W) + 1$.

(15) Let F be a field, V be a finite dimensional vector space over F, W be a subspace of V, and v, u be vectors of V. Suppose $v \notin W$ and $u \notin W$ and $v \neq u$ and $\{v, u\}$ is linearly independent and $W \cap \mathrm{Lin}(\{v, u\}) = \mathbf{0}_V$. Then $\dim(W + \mathrm{Lin}(\{v, u\})) = \dim(W) + 2$.

(16) Let F be a field, V be a finite dimensional vector space over F, and W_1, W_2 be subspaces of V. Suppose W_1 is a subspace of W_2. Let k be a natural number. Suppose $1 \leq k$ and $k < \dim(V)$ and $\dim(W_1) + 1 = k$ and $\dim(W_2) = k + 1$. Let v be a vector of V. If $v \in W_2$ and $v \notin W_1$, then $W_1 + \mathrm{Lin}(\{v\}) \in \mathrm{pencil}(W_1, W_2, k)$.

(17) Let F be a field, V be a finite dimensional vector space over F, and W_1, W_2 be subspaces of V. Suppose W_1 is a subspace of W_2. Let k be a natural number. If $1 \leq k$ and $k < \dim(V)$ and $\dim(W_1) + 1 = k$ and $\dim(W_2) = k + 1$, then $\mathrm{pencil}(W_1, W_2, k)$ is non trivial.

Let F be a field, let V be a finite dimensional vector space over F, and let k be a natural number. The functor $\mathrm{PencilSpace}(V, k)$ yielding a strict topological structure is defined by:

(Def. 5) $\mathrm{PencilSpace}(V, k) = \langle \mathrm{Sub}_k(V), k \text{ pencils of } V \rangle$.

Next we state several propositions:

(18) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $k \leq \dim(V)$, then $\mathrm{PencilSpace}(V, k)$ is non empty.

(19) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $1 \leq k$ and $k < \dim(V)$, then $\mathrm{PencilSpace}(V, k)$ is non void.

(20) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $1 \leq k$ and $k < \dim(V)$ and $3 \leq \dim(V)$, then $\mathrm{PencilSpace}(V, k)$ is non degenerated.

(21) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $1 \leq k$ and $k < \dim(V)$, then $\mathrm{PencilSpace}(V, k)$ has non trivial blocks.

(22) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $1 \leq k$ and $k < \dim(V)$, then $\mathrm{PencilSpace}(V, k)$ is identifying close blocks.

(23) Let F be a field, V be a finite dimensional vector space over F, and k be a natural number. If $1 \leq k$ and $k < \dim(V)$ and $3 \leq \dim(V)$, then $\mathrm{PencilSpace}(V, k)$ is a PLS.

2. Grassmann Spaces

Let F be a field, let V be a finite dimensional vector space over F, and let m, n be natural numbers. The functor $\mathrm{SubspaceSet}(V, m, n)$ yields a family of subsets of $\mathrm{Sub}_m(V)$ and is defined by:

(Def. 6) For every set X holds $X \in \mathrm{SubspaceSet}(V, m, n)$ iff there exists a subspace W of V such that $\dim(W) = n$ and $X = \mathrm{Sub}_m(W)$.

One can prove the following propositions:

(24) Let F be a field, V be a finite dimensional vector space over F, and m, n be natural numbers. If $n \leq \dim(V)$, then $\mathrm{SubspaceSet}(V, m, n)$ is non empty.

(25) Let F be a field and W_1, W_2 be finite dimensional vector spaces over F. If $\Omega_{(W_1)} = \Omega_{(W_2)}$, then for every natural number m holds $\mathrm{Sub}_m(W_1) = \mathrm{Sub}_m(W_2)$.

(26) Let F be a field, V be a finite dimensional vector space over F, W be a subspace of V, and m be a natural number. If $1 \leq m$ and $m \leq \dim(V)$ and $\mathrm{Sub}_m(V) \subseteq \mathrm{Sub}_m(W)$, then $\Omega_V = \Omega_W$.

Let F be a field, let V be a finite dimensional vector space over F, and let m, n be natural numbers. The functor $\mathrm{GrassmannSpace}(V, m, n)$ yields a strict topological structure and is defined as follows:

(Def. 7) $\mathrm{GrassmannSpace}(V, m, n) = \langle \mathrm{Sub}_m(V), \mathrm{SubspaceSet}(V, m, n) \rangle$.

We now state several propositions:

(27) Let F be a field, V be a finite dimensional vector space over F, and m, n be natural numbers. If $m \leq \dim(V)$, then $\mathrm{GrassmannSpace}(V, m, n)$ is non empty.

(28) Let F be a field, V be a finite dimensional vector space over F, and m, n be natural numbers. If $n \leq \dim(V)$, then GrassmannSpace(V, m, n) is non void.

(29) Let F be a field, V be a finite dimensional vector space over F, and m, n be natural numbers. If $1 \leq m$ and $m < n$ and $n < \dim(V)$, then GrassmannSpace(V, m, n) is non degenerated.

(30) Let F be a field, V be a finite dimensional vector space over F, and m, n be natural numbers. If $1 \leq m$ and $m < n$ and $n \leq \dim(V)$, then GrassmannSpace(V, m, n) has non trivial blocks.

(31) Let F be a field, V be a finite dimensional vector space over F, and m be a natural number. If $1 \leq m$ and $m + 1 \leq \dim(V)$, then GrassmannSpace$(V, m, m + 1)$ is identifying close blocks.

(32) Let F be a field, V be a finite dimensional vector space over F, and m be a natural number. If $1 \leq m$ and $m + 1 < \dim(V)$, then GrassmannSpace$(V, m, m + 1)$ is a PLS.

3. Veronese Spaces

Let X be a set. The functor PairSet X is defined as follows:

(Def. 8) For every set z holds $z \in$ PairSet X iff there exist sets x, y such that $x \in X$ and $y \in X$ and $z = \{x, y\}$.

Let X be a non empty set. One can verify that PairSet X is non empty.

Let t, X be sets. The functor PairSet(t, X) is defined as follows:

(Def. 9) For every set z holds $z \in$ PairSet(t, X) iff there exists a set y such that $y \in X$ and $z = \{t, y\}$.

Let t be a set and let X be a non empty set. One can verify that PairSet(t, X) is non empty.

Let t be a set and let X be a non trivial set. One can verify that PairSet(t, X) is non trivial.

Let X be a set and let L be a family of subsets of X. The functor PairSetFamily L yields a family of subsets of PairSet X and is defined as follows:

(Def. 10) For every set S holds $S \in$ PairSetFamily L iff there exists a set t and there exists a subset l of X such that $t \in X$ and $l \in L$ and $S =$ PairSet(t, l).

Let X be a non empty set and let L be a non empty family of subsets of X. Note that PairSetFamily L is non empty.

Let S be a topological structure. The functor VeroneseSpace S yielding a strict topological structure is defined by:

(Def. 11) VeroneseSpace $S = \langle$PairSet (the carrier of S),PairSetFamily (the topology of S)$\rangle$.

Let S be a non empty topological structure. One can verify that VeroneseSpace S is non empty.

Let S be a non empty non void topological structure. One can check that VeroneseSpace S is non void.

Let S be a non empty non void non degenerated topological structure. Note that VeroneseSpace S is non degenerated.

Let S be a non empty non void topological structure with non trivial blocks. One can check that VeroneseSpace S has non trivial blocks.

Let S be an identifying close blocks topological structure. Note that VeroneseSpace S is identifying close blocks.

Let S be a PLS. Then VeroneseSpace S is a strict PLS.

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Received November 8, 2004