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Some Properties of Circles on the Plane[1]

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The articles [30], [34], [1], [5], [35], [7], [6], [23], [29], [17], [4], [33], [2], [27], [24], [26], [31], [9], [25], [37], [12], [18], [11], [10], [28], [3], [14], [36], [15], [32], [13], [16], [20], [19], [21], [8], and [22] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we follow the rules: n is a natural number, i is an integer, a, b, r are real numbers, and x is a point of $\mathcal{E}^n_{\rm T}$.

One can check the following observations:

* $]0,1[$ is non empty,
* $[-1,1]$ is non empty, and
* $]\frac{1}{2},\frac{3}{2}[$ is non empty.

One can verify the following observations:

* the function sin is continuous,
* the function cos is continuous,
* the function arcsin is continuous, and
* the function arccos is continuous.

Next we state two propositions:

(1) $\sin(a \cdot r + b) = (($the function sin$) \cdot {\rm AffineMap}(a,b))(r)$.

(2) $\cos(a \cdot r + b) = (($the function cos$) \cdot {\rm AffineMap}(a,b))(r)$.

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Let a be a non zero real number and let b be a real number. Note that AffineMap(a, b) is onto and one-to-one.

Let a, b be real numbers. The functor IntIntervals(a, b) is defined as follows:

(Def. 1) IntIntervals$(a, b) = \{]a + n, b + n[: n$ ranges over elements of $\mathbb{Z}\}$.

One can prove the following proposition

(3) For every set x holds $x \in$ IntIntervals(a, b) iff there exists an element n of $\mathbb{Z}$ such that $x =]a + n, b + n[$.

Let a, b be real numbers. Observe that IntIntervals(a, b) is non empty.

Next we state the proposition

(4) If $b - a \leq 1$, then IntIntervals(a, b) is mutually-disjoint.

Let a, b be real numbers. Then IntIntervals(a, b) is a family of subsets of $\mathbb{R}^{\mathbf{1}}$.

Let a, b be real numbers. Then IntIntervals(a, b) is an open family of subsets of $\mathbb{R}^{\mathbf{1}}$.

2. Correspondence between $\mathbb{R}$ and $\mathbb{R}^{\mathbf{1}}$

Let r be a real number. The functor $R^1 r$ yielding a point of $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def. 2) $R^1 r = r$.

Let A be a subset of $\mathbb{R}$. The functor $R^1 A$ yielding a subset of $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def. 3) $R^1 A = A$.

Let A be a non empty subset of $\mathbb{R}$. Observe that $R^1 A$ is non empty.

Let A be an open subset of $\mathbb{R}$. Note that $R^1 A$ is open.

Let A be a closed subset of $\mathbb{R}$. Observe that $R^1 A$ is closed.

Let A be an open subset of $\mathbb{R}$. Observe that $\mathbb{R}^{\mathbf{1}} \restriction R^1 A$ is open.

Let A be a closed subset of $\mathbb{R}$. One can verify that $\mathbb{R}^{\mathbf{1}} \restriction R^1 A$ is closed.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $R^1 f$ yielding a map from $\mathbb{R}^{\mathbf{1}} \restriction R^1 \operatorname{dom} f$ into $\mathbb{R}^{\mathbf{1}} \restriction R^1 \operatorname{rng} f$ is defined as follows:

(Def. 4) $R^1 f = f$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. One can check that $R^1 f$ is onto.

Let f be an one-to-one partial function from $\mathbb{R}$ to $\mathbb{R}$. Observe that $R^1 f$ is one-to-one.

One can prove the following four propositions:

(5) $\mathbb{R}^{\mathbf{1}} \restriction R^1(\Omega_{\mathbb{R}}) = \mathbb{R}^{\mathbf{1}}$.

(6) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} f = \mathbb{R}$ holds $\mathbb{R}^{\mathbf{1}} \restriction R^1 \operatorname{dom} f = \mathbb{R}^{\mathbf{1}}$.

(7) Every function f from $\mathbb{R}$ into $\mathbb{R}$ is a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}} \restriction R^1 \operatorname{rng} f$.

(8) Every function from $\mathbb{R}$ into $\mathbb{R}$ is a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$.

Let f be a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$. Note that $R^1 f$ is continuous.

Let a be a non zero real number and let b be a real number. One can verify that R^1 AffineMap(a, b) is open.

3. Circles

Let S be a subspace of $\mathcal{E}^2_{\rm T}$. We say that S satisfies conditions of simple closed curve if and only if:

(Def. 5) The carrier of S is a simple closed curve.

Let us note that every subspace of $\mathcal{E}^2_{\rm T}$ which satisfies conditions of simple closed curve is also non empty, arcwise connected, and compact.

Let r be a positive real number and let x be a point of $\mathcal{E}^2_{\rm T}$. Observe that Sphere(x, r) satisfies conditions of simple closed curve.

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\rm T}$, and let r be a real number. The functor Tcircle(x, r) yielding a subspace of $\mathcal{E}^n_{\rm T}$ is defined by:

(Def. 6) $\mathrm{Tcircle}(x, r) = (\mathcal{E}^n_{\rm T}) \restriction \mathrm{Sphere}(x, r)$.

Let n be a non empty natural number, let x be a point of $\mathcal{E}^n_{\rm T}$, and let r be a non negative real number. Note that Tcircle(x, r) is strict and non empty.

One can prove the following proposition

(9) The carrier of $\mathrm{Tcircle}(x, r) = \mathrm{Sphere}(x, r)$.

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\rm T}$, and let r be an empty real number. Note that Tcircle(x, r) is trivial.

Next we state the proposition

(10) $\mathrm{Tcircle}(0_{\mathcal{E}^2_{\rm T}}, r)$ is a subspace of $\mathrm{Trectangle}(-r, r, -r, r)$.

Let x be a point of $\mathcal{E}^2_{\rm T}$ and let r be a positive real number. One can verify that Tcircle(x, r) satisfies conditions of simple closed curve.

Let us mention that there exists a subspace of $\mathcal{E}^2_{\rm T}$ which is strict and satisfies conditions of simple closed curve.

Next we state the proposition

(11) For all subspaces S, T of $\mathcal{E}^2_{\rm T}$ satisfying conditions of simple closed curve holds S and T are homeomorphic.

Let n be a natural number. The functor TopUnitCircle n yields a subspace of $\mathcal{E}^n_{\rm T}$ and is defined by:

(Def. 7) $\mathrm{TopUnitCircle}\, n = \mathrm{Tcircle}(0_{\mathcal{E}^n_{\rm T}}, 1)$.

Let n be a non empty natural number. Note that TopUnitCircle n is non empty.

We now state several propositions:

(12) For every non empty natural number n and for every point x of $\mathcal{E}^n_{\mathrm{T}}$ such that x is a point of TopUnitCircle n holds $|x| = 1$.

(13) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopUnitCircle 2 holds $-1 \leq x_{\mathbf{1}}$ and $x_{\mathbf{1}} \leq 1$ and $-1 \leq x_{\mathbf{2}}$ and $x_{\mathbf{2}} \leq 1$.

(14) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopUnitCircle 2 and $x_{\mathbf{1}} = 1$ holds $x_{\mathbf{2}} = 0$.

(15) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopUnitCircle 2 and $x_{\mathbf{1}} = -1$ holds $x_{\mathbf{2}} = 0$.

(16) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopUnitCircle 2 and $x_{\mathbf{2}} = 1$ holds $x_{\mathbf{1}} = 0$.

(17) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopUnitCircle 2 and $x_{\mathbf{2}} = -1$ holds $x_{\mathbf{1}} = 0$.

The following propositions are true:

(18) TopUnitCircle 2 is a subspace of Trectangle$(-1, 1, -1, 1)$.

(19) Let n be a non empty natural number, r be a positive real number, x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and f be a map from TopUnitCircle n into Tcircle(x, r). Suppose that for every point a of TopUnitCircle n and for every point b of $\mathcal{E}^n_{\mathrm{T}}$ such that $a = b$ holds $f(a) = r \cdot b + x$. Then f is a homeomorphism.

Let us observe that TopUnitCircle 2 satisfies conditions of simple closed curve.

One can prove the following proposition

(20) Let n be a non empty natural number, r, s be positive real numbers, and x, y be points of $\mathcal{E}^n_{\mathrm{T}}$. Then Tcircle$(x, r)$ and Tcircle(y, s) are homeomorphic.

Let x be a point of $\mathcal{E}^2_{\mathrm{T}}$ and let r be a non negative real number. Observe that Tcircle(x, r) is arcwise connected.

The point $c[10]$ of TopUnitCircle 2 is defined as follows:

(Def. 8) $c[10] = [1, 0]$.

The point $c[-10]$ of TopUnitCircle 2 is defined as follows:

(Def. 9) $c[-10] = [-1, 0]$.

Next we state the proposition

(21) $c[10] \neq c[-10]$.

Let p be a point of TopUnitCircle 2. The functor TopOpenUnitCircle p yielding a strict subspace of TopUnitCircle 2 is defined by:

(Def. 10) The carrier of TopOpenUnitCircle p = (the carrier of TopUnitCircle 2) $\setminus$ $\{p\}$.

Let p be a point of TopUnitCircle 2. Note that TopOpenUnitCircle p is non empty.

We now state several propositions:

(22) For every point p of TopUnitCircle 2 holds p is not a point of TopOpenUnitCircle p.

(23) For every point p of TopUnitCircle 2 holds TopOpenUnitCircle p = TopUnitCircle $2 \restriction (\Omega_{\mathrm{TopUnitCircle}\,2} \setminus \{p\})$.

(24) For all points p, q of TopUnitCircle 2 such that $p \neq q$ holds q is a point of TopOpenUnitCircle p.

(25) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that p is a point of TopOpenUnitCircle $c[10]$ and $p_{\mathbf{2}} = 0$ holds $p = c[-10]$.

(26) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that p is a point of TopOpenUnitCircle $c[-10]$ and $p_{\mathbf{2}} = 0$ holds $p = c[10]$.

Next we state three propositions:

(27) Let p be a point of TopUnitCircle 2 and x be a point of $\mathcal{E}^2_{\mathrm{T}}$. If x is a point of TopOpenUnitCircle p, then $-1 \leq x_{\mathbf{1}}$ and $x_{\mathbf{1}} \leq 1$ and $-1 \leq x_{\mathbf{2}}$ and $x_{\mathbf{2}} \leq 1$.

(28) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopOpenUnitCircle $c[10]$ holds $-1 \leq x_{\mathbf{1}}$ and $x_{\mathbf{1}} < 1$.

(29) For every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that x is a point of TopOpenUnitCircle $c[-10]$ holds $-1 < x_{\mathbf{1}}$ and $x_{\mathbf{1}} \leq 1$.

Let p be a point of TopUnitCircle 2. Note that TopOpenUnitCircle p is open.

We now state two propositions:

(30) For every point p of TopUnitCircle 2 holds TopOpenUnitCircle p and $I(01)$ are homeomorphic.

(31) For all points p, q of TopUnitCircle 2 holds TopOpenUnitCircle p and TopOpenUnitCircle q are homeomorphic.

4. Correspondence between the Real Line and Circles

The map CircleMap from $\mathbb{R}^{\mathbf{1}}$ into TopUnitCircle 2 is defined by:

(Def. 11) For every real number x holds $\mathrm{CircleMap}(x) = [\cos(2 \cdot \pi \cdot x), \sin(2 \cdot \pi \cdot x)]$.

Next we state several propositions:

(32) $\mathrm{CircleMap}(r) = \mathrm{CircleMap}(r + i)$.

(33) $\mathrm{CircleMap}(i) = c[10]$.

(34) $\mathrm{CircleMap}^{-1}(\{c[10]\}) = \mathbb{Z}$.

(35) If $\mathrm{frac}\, r = \frac{1}{2}$, then $\mathrm{CircleMap}(r) = [-1, 0]$.

(36) If $\mathrm{frac}\, r = \frac{1}{4}$, then $\mathrm{CircleMap}(r) = [0, 1]$.

(37) If $\mathrm{frac}\, r = \frac{3}{4}$, then $\mathrm{CircleMap}(r) = [0, -1]$.

(38) For all integers i, j holds $\mathrm{CircleMap}(r) = [((\text{the function cos}) \cdot \mathrm{AffineMap}(2 \cdot \pi, 2 \cdot \pi \cdot i))(r), ((\text{the function sin}) \cdot \mathrm{AffineMap}(2 \cdot \pi, 2 \cdot \pi \cdot j))(r)]$.

Let us note that CircleMap is continuous.

The following proposition is true

(39) For every subset B of $\mathbb{R}^{\mathbf{1}}$ and for every map f from $\mathbb{R}^{\mathbf{1}} \restriction B$ into TopUnitCircle 2 such that $[0,1[\subseteq B$ and $f = \mathrm{CircleMap} \restriction B$ holds f is onto.

Let us observe that CircleMap is onto.

Let r be a real number. One can verify that CircleMap $\restriction [r, r+1[$ is one-to-one.

Let r be a real number. One can verify that CircleMap $\restriction]r, r+1[$ is one-to-one.

One can prove the following two propositions:

(40) If $b - a \leq 1$, then for every set d such that $d \in \mathrm{IntIntervals}(a, b)$ holds CircleMap $\restriction d$ is one-to-one.

(41) For every set d such that $d \in \mathrm{IntIntervals}(a, b)$ holds $\mathrm{CircleMap}^{\circ} d = \mathrm{CircleMap}^{\circ} \bigcup \mathrm{IntIntervals}(a, b)$.

Let r be a point of $\mathbb{R}^{\mathbf{1}}$. The functor CircleMap r yielding a map from $\mathbb{R}^{\mathbf{1}} \restriction R^1]r, r+1[$ into TopOpenUnitCircle CircleMap(r) is defined by:

(Def. 12) $\mathrm{CircleMap}\, r = \mathrm{CircleMap} \restriction]r, r+1[$.

One can prove the following proposition

(42) $\mathrm{CircleMap}\, R^1(a+i) = \mathrm{CircleMap}\, R^1 a \cdot (\mathrm{AffineMap}(1, -i) \restriction]a+i, a+i+1[)$.

Let r be a point of $\mathbb{R}^{\mathbf{1}}$. One can check that CircleMap r is one-to-one, onto, and continuous.

The map Circle2IntervalR from TopOpenUnitCircle $c[10]$ into $\mathbb{R}^{\mathbf{1}} \restriction R^1]0, 1[$ is defined by the condition (Def. 13).

(Def. 13) Let p be a point of TopOpenUnitCircle $c[10]$. Then there exist real numbers x, y such that $p = [x, y]$ and if $y \geq 0$, then $\mathrm{Circle2IntervalR}(p) = \frac{\arccos x}{2 \cdot \pi}$ and if $y \leq 0$, then $\mathrm{Circle2IntervalR}(p) = 1 - \frac{\arccos x}{2 \cdot \pi}$.

The map Circle2IntervalL from TopOpenUnitCircle $c[-10]$ into $\mathbb{R}^{\mathbf{1}} \restriction R^1]\frac{1}{2}, \frac{3}{2}[$ is defined by the condition (Def. 14).

(Def. 14) Let p be a point of TopOpenUnitCircle $c[-10]$. Then there exist real numbers x, y such that $p = [x, y]$ and if $y \geq 0$, then $\mathrm{Circle2IntervalL}(p) = 1 + \frac{\arccos x}{2 \cdot \pi}$ and if $y \leq 0$, then $\mathrm{Circle2IntervalL}(p) = 1 - \frac{\arccos x}{2 \cdot \pi}$.

We now state two propositions:

(43) $(\mathrm{CircleMap}\, R^1 0)^{-1} = \mathrm{Circle2IntervalR}$.

(44) $(\mathrm{CircleMap}\, R^1(\frac{1}{2}))^{-1} = \mathrm{Circle2IntervalL}$.

Let us observe that Circle2IntervalR is one-to-one, onto, and continuous and Circle2IntervalL is one-to-one, onto, and continuous.

Let i be an integer. Observe that CircleMap $R^1 i$ is open and CircleMap $R^1(\frac{1}{2} + i)$ is open.

Let us observe that Circle2IntervalR is open and Circle2IntervalL is open.

Next we state several propositions:

(45) CircleMap $R^1 0$ is a homeomorphism.

(46) CircleMap $R^1(\frac{1}{2})$ is a homeomorphism.

(47) Circle2IntervalR is a homeomorphism.

(48) Circle2IntervalL is a homeomorphism.

(49) There exists a family F of subsets of TopUnitCircle 2 such that

(i) $F = \{\text{CircleMap}^\circ]0,1[, \text{CircleMap}^\circ]\frac{1}{2},\frac{3}{2}[\}$,

(ii) F is a cover of TopUnitCircle 2 and open, and

(iii) for every subset U of TopUnitCircle 2 holds if $U = \text{CircleMap}^\circ]0,1[$, then $\bigcup \text{IntIntervals}(0,1) = \text{CircleMap}^{-1}(U)$ and for every subset d of $\mathbb{R}^\mathbf{1}$ such that $d \in \text{IntIntervals}(0,1)$ and for every map f from $\mathbb{R}^\mathbf{1} \restriction d$ into TopUnitCircle $2 \restriction U$ such that $f = \text{CircleMap} \restriction d$ holds f is a homeomorphism and if $U = \text{CircleMap}^\circ]\frac{1}{2},\frac{3}{2}[$, then $\bigcup \text{IntIntervals}(\frac{1}{2},\frac{3}{2}) = \text{CircleMap}^{-1}(U)$ and for every subset d of $\mathbb{R}^\mathbf{1}$ such that $d \in \text{IntIntervals}(\frac{1}{2},\frac{3}{2})$ and for every map f from $\mathbb{R}^\mathbf{1} \restriction d$ into TopUnitCircle $2 \restriction U$ such that $f = \text{CircleMap} \restriction d$ holds f is a homeomorphism.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[4] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[8] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.

[9] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.

[10] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[11] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.

[12] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_\mathrm{T}$. Simple closed curves. *Formalized Mathematics*, 2(**5**):663–664, 1991.

[13] Mariusz Giero. Hierarchies and classifications of sets. *Formalized Mathematics*, 9(**4**):865–869, 2001.

[14] Adam Grabowski. Introduction to the homotopy theory. *Formalized Mathematics*, 6(**4**):449–454, 1997.

[15] Adam Grabowski. Properties of the product of compact topological spaces. *Formalized Mathematics*, 8(**1**):55–59, 1999.

[16] Adam Grabowski. On the decompositions of intervals and simple closed curves. *Formalized Mathematics*, 10(**3**):145–151, 2002.

[17] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[18] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[19] Artur Korniłowicz. The fundamental group of convex subspaces of $\mathcal{E}^n_{\mathrm{T}}$. *Formalized Mathematics*, 12(**3**):295–299, 2004.
[20] Artur Korniłowicz and Yasunari Shidama. Intersections of intervals and balls in $\mathcal{E}^n_{\mathrm{T}}$. *Formalized Mathematics*, 12(**3**):301–306, 2004.
[21] Artur Korniłowicz and Yasunari Shidama. Inverse trigonometric functions arcsin and arccos. *Formalized Mathematics*, 13(**1**):73–79, 2005.
[22] Artur Korniłowicz and Yasunari Shidama. Some properties of rectangles on the plane. *Formalized Mathematics*, 13(**1**):109–115, 2005.
[23] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[24] Yatsuka Nakamura. Half open intervals in real numbers. *Formalized Mathematics*, 10(**1**):21–22, 2002.
[25] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.
[26] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[27] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[28] Agnieszka Sakowicz, Jarosław Gryko, and Adam Grabowski. Sequences in $\mathcal{E}^N_{\mathrm{T}}$. *Formalized Mathematics*, 5(**1**):93–96, 1996.
[29] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[30] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[31] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[32] Andrzej Trybulec and Yatsuka Nakamura. On the decomposition of a simple closed curve into two arcs. *Formalized Mathematics*, 10(**3**):163–167, 2002.
[33] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[34] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[35] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[36] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.
[37] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Formalized Mathematics*, 5(**1**):75–77, 1996.

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