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Brouwer Fixed Point Theorem for Disks on the Plane

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Summary. The article formalizes the proof of Brouwer's Fixed Point Theorem for 2-dimensional disks. Assuming, on the contrary, that the theorem is false, we show that a circle is a retract of a disk. Next, using the retraction, we prove that any loop in the circle is homotopic to the constant loop what contradicts with infiniteness of the fundamental group of a circle, see [15].

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The terminology and notation used in this paper are introduced in the following papers: [26], [9], [29], [2], [22], [28], [30], [6], [8], [7], [5], [4], [12], [3], [25], [16], [23], [21], [20], [27], [11], [13], [14], [18], [17], [19], [10], [1], and [24].

In this paper n is a natural number, a, r are real numbers, and x is a point of $\mathcal{E}^n_{\rm T}$.

Let S, T be non empty topological spaces. The functor DiffElems(S,T) yielding a subset of $[\!: S, T :\!]$ is defined by:

(Def. 1) DiffElems$(S,T) = \{\langle s, t\rangle; s$ ranges over points of S, t ranges over points of T: $s \neq t\}$.

One can prove the following proposition

(1) Let S, T be non empty topological spaces and x be a set. Then $x \in$ DiffElems(S,T) if and only if there exists a point s of S and there exists a point t of T such that $x = \langle s, t\rangle$ and $s \neq t$.

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Let S be a non trivial non empty topological space and let T be a non empty topological space. One can check that $\mathrm{DiffElems}(S,T)$ is non empty.

Let S be a non empty topological space and let T be a non trivial non empty topological space. Note that $\mathrm{DiffElems}(S,T)$ is non empty.

We now state the proposition

(2) $\overline{\mathrm{Ball}}(x,0) = \{x\}$.

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a real number. The functor $\mathrm{Tdisk}(x,r)$ yields a subspace of $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 2) $\mathrm{Tdisk}(x,r) = (\mathcal{E}^n_{\mathrm{T}}) \restriction \overline{\mathrm{Ball}}(x,r)$.

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a non negative real number. Note that $\mathrm{Tdisk}(x,r)$ is non empty.

We now state the proposition

(3) The carrier of $\mathrm{Tdisk}(x,r) = \overline{\mathrm{Ball}}(x,r)$.

Let n be a natural number, let x be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a real number. Note that $\mathrm{Tdisk}(x,r)$ is convex.

We adopt the following convention: n denotes a natural number, r denotes a non negative real number, and s, t, x denote points of $\mathcal{E}^n_{\mathrm{T}}$.

One can prove the following two propositions:

(4) If $s \neq t$ and s is a point of $\mathrm{Tdisk}(x,r)$ and s is not a point of $\mathrm{Tcircle}(x,r)$, then there exists a point e of $\mathrm{Tcircle}(x,r)$ such that $\{e\} = \mathrm{halfline}(s,t) \cap \mathrm{Sphere}(x,r)$.

(5) Suppose $s \neq t$ and $s \in$ the carrier of $\mathrm{Tcircle}(x,r)$ and t is a point of $\mathrm{Tdisk}(x,r)$. Then there exists a point e of $\mathrm{Tcircle}(x,r)$ such that $e \neq s$ and $\{s,e\} = \mathrm{halfline}(s,t) \cap \mathrm{Sphere}(x,r)$.

Let n be a non empty natural number, let o be a point of $\mathcal{E}^n_{\mathrm{T}}$, let s, t be points of $\mathcal{E}^n_{\mathrm{T}}$, and let r be a non negative real number. Let us assume that s is a point of $\mathrm{Tdisk}(o,r)$, and t is a point of $\mathrm{Tdisk}(o,r)$ and $s \neq t$. The functor $\mathrm{HC}(s,t,o,r)$ yields a point of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def. 3) $\mathrm{HC}(s,t,o,r) \in \mathrm{halfline}(s,t) \cap \mathrm{Sphere}(o,r)$ and $\mathrm{HC}(s,t,o,r) \neq s$.

In the sequel n is a non empty natural number and s, t, o are points of $\mathcal{E}^n_{\mathrm{T}}$.

We now state three propositions:

(6) If s is a point of $\mathrm{Tdisk}(o,r)$ and t is a point of $\mathrm{Tdisk}(o,r)$ and $s \neq t$, then $\mathrm{HC}(s,t,o,r)$ is a point of $\mathrm{Tcircle}(o,r)$.

(7) Let S, T, O be elements of $\mathcal{R}^n$. Suppose $S = s$ and $T = t$ and $O = o$. Suppose s is a point of $\mathrm{Tdisk}(o,r)$ and t is a point of $\mathrm{Tdisk}(o,r)$ and $s \neq t$ and $a = \frac{-|(t-s,s-o)|+\sqrt{|(t-s,s-o)|^{\mathbf{2}}-\sum^2(T-S)\cdot(\sum^2(S-O)-r^{\mathbf{2}})}}{\sum^2(T-S)}$. Then $\mathrm{HC}(s,t,o,r) = (1-a)\cdot s + a \cdot t$.

(8) Let r_1, r_2, s_1, s_2 be real numbers and s, t, o be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that s is a point of $\mathrm{Tdisk}(o,r)$ and t is a point of $\mathrm{Tdisk}(o,r)$ and

$s \neq t$ and $r_1 = t_{\mathbf{1}} - s_{\mathbf{1}}$ and $r_2 = t_{\mathbf{2}} - s_{\mathbf{2}}$ and $s_1 = s_{\mathbf{1}} - o_{\mathbf{1}}$ and $s_2 = s_{\mathbf{2}} - o_{\mathbf{2}}$ and $a = \frac{-(s_1 \cdot r_1 + s_2 \cdot r_2) + \sqrt{(s_1 \cdot r_1 + s_2 \cdot r_2)^{\mathbf{2}} - (r_1{}^{\mathbf{2}} + r_2{}^{\mathbf{2}}) \cdot ((s_1{}^{\mathbf{2}} + s_2{}^{\mathbf{2}}) - r^{\mathbf{2}})}}{r_1{}^{\mathbf{2}} + r_2{}^{\mathbf{2}}}$. Then $\mathrm{HC}(s, t, o, r) = [s_{\mathbf{1}} + a \cdot r_1, s_{\mathbf{2}} + a \cdot r_2]$.

Let n be a non empty natural number, let o be a point of $\mathcal{E}^n_{\mathrm{T}}$, let r be a non negative real number, let x be a point of $\mathrm{Tdisk}(o, r)$, and let f be a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. Let us assume that x is not a fixpoint of f. The functor $\mathrm{HC}(x, f)$ yielding a point of $\mathrm{Tcircle}(o, r)$ is defined as follows:

(Def. 4) There exist points y, z of $\mathcal{E}^n_{\mathrm{T}}$ such that $y = x$ and $z = f(x)$ and $\mathrm{HC}(x, f) = \mathrm{HC}(z, y, o, r)$.

The following two propositions are true:

(9) Let x be a point of $\mathrm{Tdisk}(o, r)$ and f be a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. If x is not a fixpoint of f and x is a point of $\mathrm{Tcircle}(o, r)$, then $\mathrm{HC}(x, f) = x$.

(10) Let r be a positive real number, o be a point of $\mathcal{E}^2_{\mathrm{T}}$, and Y be a non empty subspace of $\mathrm{Tdisk}(o, r)$. If $Y = \mathrm{Tcircle}(o, r)$, then Y is not a retract of $\mathrm{Tdisk}(o, r)$.

Let n be a non empty natural number, let r be a non negative real number, let o be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let f be a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. The functor BR-map f yielding a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tcircle}(o, r)$ is defined as follows:

(Def. 5) For every point x of $\mathrm{Tdisk}(o, r)$ holds $(\text{BR-map}\, f)(x) = \mathrm{HC}(x, f)$.

The following propositions are true:

(11) Let o be a point of $\mathcal{E}^n_{\mathrm{T}}$, x be a point of $\mathrm{Tdisk}(o, r)$, and f be a map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. If x is not a fixpoint of f and x is a point of $\mathrm{Tcircle}(o, r)$, then $(\text{BR-map}\, f)(x) = x$.

(12) For every continuous map f from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$ such that f has no fixpoint holds $\text{BR-map}\, f \restriction \mathrm{Sphere}(o, r) = \mathrm{id}_{\mathrm{Tcircle}(o, r)}$.

(13) Let r be a positive real number, o be a point of $\mathcal{E}^2_{\mathrm{T}}$, and f be a continuous map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. If f has no fixpoint, then BR-map f is continuous.

(14) For every non negative real number r and for every point o of $\mathcal{E}^2_{\mathrm{T}}$ holds every continuous map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$ has a fixpoint.

(15) Let r be a non negative real number, o be a point of $\mathcal{E}^2_{\mathrm{T}}$, and f be a continuous map from $\mathrm{Tdisk}(o, r)$ into $\mathrm{Tdisk}(o, r)$. Then there exists a point x of $\mathrm{Tdisk}(o, r)$ such that $f(x) = x$.

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