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The Properties of Supercondensed Sets, Subcondensed Sets and Condensed Sets

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Summary. We formalized the article "New concepts in the theory of topological space – supercondensed set, subcondensed set, and condensed set" by Yoshinori Isomichi [4]. First we defined supercondensed, subcondensed, and condensed sets and then gradually, defining other attributes such as regular open set or regular closed set, we formalized all the theorems and remarks that one can find in Isomichi's article.

In the last section, the classification of subsets of a topological space is given, depending on the inclusion relation between the interior of the closure and the closure of the interior of a given subset.

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The notation and terminology used in this paper are introduced in the following papers: [10], [11], [1], [6], [8], [9], [7], [12], [2], [3], and [5].

1. Preliminaries

In this paper T denotes a topological space and A, B denote subsets of T.

Let D be a non trivial set. Note that $\mathrm{ADTS}(D)$ is non trivial.

One can check that there exists a topological space which is anti-discrete, non trivial, non empty, and strict.

One can prove the following propositions:

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(1) $\operatorname{Int}\overline{\operatorname{Int} A} \cap \operatorname{Int}\overline{\operatorname{Int} B} = \operatorname{Int}\overline{\operatorname{Int}(A \cap B)}$.

(2) $\overline{\operatorname{Int}\overline{A \cup B}} = \overline{\operatorname{Int}\overline{A}} \cup \overline{\operatorname{Int}\overline{B}}$.

2. Connections between Supercondensed, Condensed, and Subcondensed Sets

Let T be a topological structure and let A be a subset of T. We say that A is supercondensed if and only if:

(Def. 1) $\operatorname{Int}\overline{A} = \operatorname{Int} A$.

We say that A is subcondensed if and only if:

(Def. 2) $\overline{\operatorname{Int} A} = \overline{A}$.

Next we state two propositions:

(3) If A is closed, then A is supercondensed.

(4) If A is open, then A is subcondensed.

Let T be a topological space and let A be a subset of T. Let us observe that A is condensed if and only if:

(Def. 3) $\overline{\operatorname{Int} A} = \overline{A}$ and $\operatorname{Int}\overline{A} = \operatorname{Int} A$.

We now state the proposition

(5) A is condensed iff A is subcondensed and supercondensed.

Let T be a topological space. One can verify that every subset of T which is condensed is also subcondensed and supercondensed and every subset of T which is subcondensed and supercondensed is also condensed.

Let T be a topological space. Observe that there exists a subset of T which is condensed, subcondensed, and supercondensed.

One can prove the following propositions:

(6) If A is supercondensed, then A^{c} is subcondensed.

(7) If A is subcondensed, then A^{c} is supercondensed.

(8) A is supercondensed iff $\operatorname{Int}\overline{A} \subseteq A$.

(9) A is subcondensed iff $A \subseteq \overline{\operatorname{Int} A}$.

Let T be a topological space. Note that every subset of T which is subcondensed is also semi-open and every subset of T which is semi-open is also subcondensed.

We now state the proposition

(10) A is condensed iff $\operatorname{Int}\overline{A} \subseteq A$ and $A \subseteq \overline{\operatorname{Int} A}$.

3. Regular Open and Regular Closed Sets

Let T be a topological structure and let A be a subset of T. We introduce A is regular open as a synonym of A is open condensed.

Let T be a topological structure and let A be a subset of T. We introduce A is regular closed as a synonym of A is closed condensed.

The following proposition is true

(11) For every topological space T holds Ω_T is regular open and Ω_T is regular closed.

Let T be a topological space. Note that Ω_T is regular open and regular closed.

We now state the proposition

(12) For every topological space X holds $\emptyset_X$ is regular open and $\emptyset_X$ is regular closed.

Let T be a topological space. One can verify that $\emptyset_T$ is regular open and regular closed.

The following propositions are true:

(14)[2] $\operatorname{Int} \overline{\emptyset_T} = \emptyset_T$.

(15) If A is regular open, then A^{c} is regular closed.

Let T be a topological space. Observe that there exists a subset of T which is regular open and regular closed.

Let T be a topological space and let A be a regular open subset of T. Observe that A^{c} is regular closed.

One can prove the following proposition

(16) If A is regular closed, then A^{c} is regular open.

Let T be a topological space and let A be a regular closed subset of T. One can check that A^{c} is regular open.

Let T be a topological space. Note that every subset of T which is regular open is also open and every subset of T which is regular closed is also closed.

Next we state the proposition

(17) $\operatorname{Int} \overline{A}$ is regular open and $\overline{\operatorname{Int} A}$ is regular closed.

Let T be a topological space and let A be a subset of T. Observe that $\operatorname{Int} \overline{A}$ is regular open and $\overline{\operatorname{Int} A}$ is regular closed.

Next we state two propositions:

(18) A is regular open iff A is supercondensed and open.

(19) A is regular closed iff A is subcondensed and closed.

Let T be a topological space. One can check the following observations:

* every subset of T which is regular open is also condensed and open,

[2]The proposition (13) has been removed.

* every subset of T which is condensed and open is also regular open,
* every subset of T which is regular closed is also condensed and closed, and
* every subset of T which is condensed and closed is also regular closed.

One can prove the following two propositions:

(20) A is condensed iff there exists B such that B is regular open and $B \subseteq A$ and $A \subseteq \overline{B}$.

(21) A is condensed iff there exists B such that B is regular closed and $\operatorname{Int} B \subseteq A$ and $A \subseteq B$.

4. Boundaries and Borders

Let T be a topological structure and let A be a subset of T. We introduce $\operatorname{Bound} A$ as a synonym of $\operatorname{Fr} A$.

Let T be a topological structure and let A be a subset of T. Then $\operatorname{Fr} A$ can be characterized by the condition:

(Def. 4) $\operatorname{Fr} A = \overline{A} \setminus \operatorname{Int} A$.

One can prove the following proposition

(22) $\operatorname{Fr} A$ is closed.

Let T be a topological space and let A be a subset of T. Observe that $\operatorname{Fr} A$ is closed.

One can prove the following proposition

(23) A is condensed iff $\operatorname{Fr} A = \overline{\operatorname{Int} A} \setminus \operatorname{Int} \overline{A}$ and $\operatorname{Fr} A = \overline{\operatorname{Int} A} \cap \overline{\operatorname{Int}(A^{c})}$.

Let T be a topological structure and let A be a subset of T. The functor $\operatorname{Border} A$ yields a subset of T and is defined by:

(Def. 5) $\operatorname{Border} A = \operatorname{Int} \operatorname{Fr} A$.

One can prove the following proposition

(24) $\operatorname{Border} A$ is regular open and $\operatorname{Border} A = \operatorname{Int} \overline{A} \setminus \overline{\operatorname{Int} A}$ and $\operatorname{Border} A = \operatorname{Int} \overline{A} \cap \operatorname{Int} \overline{A^{c}}$.

Let T be a topological space and let A be a subset of T. One can verify that $\operatorname{Border} A$ is regular open.

One can prove the following two propositions:

(25) A is supercondensed iff $\operatorname{Int} A$ is regular open and $\operatorname{Border} A$ is empty.

(26) A is subcondensed iff $\overline{A}$ is regular closed and $\operatorname{Border} A$ is empty.

Let T be a topological space and let A be a subset of T. One can verify that $\operatorname{Border} \operatorname{Border} A$ is empty.

The following proposition is true

(27) A is condensed iff $\operatorname{Int} A$ is regular open and $\overline{A}$ is regular closed and $\operatorname{Border} A$ is empty.

5. Auxiliary Theorems about Intervals

Next we state a number of propositions:

(28) For every subset A of $\mathbb{R}^1$ and for every real number a such that $A =]-\infty, a]$ holds $\operatorname{Int} A =]-\infty, a[$.

(29) For every subset A of $\mathbb{R}^1$ and for every real number a such that $A = [a, +\infty[$ holds $\operatorname{Int} A =]a, +\infty[$.

(30) For every subset A of $\mathbb{R}^1$ and for all real numbers a, b such that $A =]-\infty, a] \cup]a, b[_{\mathbb{IQ}} \cup [b, +\infty[$ holds $\overline{A} =$ the carrier of $\mathbb{R}^1$.

(31) For every subset A of $\mathbb{R}^1$ and for all real numbers a, b such that $A =]a, b[_{\mathbb{Q}}$ holds $\operatorname{Int} A = \emptyset$.

(32) For every subset A of $\mathbb{R}^1$ and for all real numbers a, b such that $A =]a, b[_{\mathbb{IQ}}$ holds $\operatorname{Int} A = \emptyset$.

(33) For all real numbers a, b holds $]-\infty, a] \setminus]-\infty, b[= [b, a]$.

(34) For all real numbers a, b such that $a < b$ holds $[b, +\infty[$ misses $]-\infty, a[$.

(35) For all real numbers a, b such that $a \geq b$ holds $]a, b[_{\mathbb{IQ}} = \emptyset$.

(36) For all real numbers a, b holds $]a, b[_{\mathbb{IQ}} \subseteq [a, +\infty[$.

(37) For every subset A of $\mathbb{R}^1$ and for all real numbers a, b, c such that $A =]-\infty, a[\cup]b, c[_{\mathbb{Q}}$ and $a < b$ and $b < c$ holds $\operatorname{Int} A =]-\infty, a[$.

(38) For all real numbers a, b holds $[a, b]$ misses $]b, +\infty[$.

(39) For every real number b holds $[b, +\infty[\setminus]b, +\infty[= \{b\}$.

(40) For all real numbers a, b such that $a < b$ holds $[a, b] = [a, +\infty[\setminus]b, +\infty[$.

(41) For all real numbers a, b such that $a < b$ holds $\mathbb{R} =]-\infty, a[\cup [a, b] \cup]b, +\infty[$.

(42) For all real numbers a, b holds $]a, b[=]a, +\infty[\setminus [b, +\infty[$.

(43) For all real numbers a, b, c such that $b < c$ and $c < a$ holds $]-\infty, a[\setminus [b, c] =]-\infty, b[\cup]c, a[$.

(44) For every subset A of $\mathbb{R}^1$ and for all real numbers a, b, c such that $A =]-\infty, a] \cup [b, c]$ and $a < b$ and $b < c$ holds $\operatorname{Int} A =]-\infty, a[\cup]b, c[$.

6. Classification of Subsets

Let A, B be sets. We introduce A and B are $\subseteq$-incomparable as an antonym of A and B are $\subseteq$-comparable.

We now state the proposition

(45) For all sets A, B holds A and B are $\subseteq$-incomparable or $A \subseteq B$ or $B \subset A$.

Let us consider T, A. We say that A is of the 1st class if and only if:

(Def. 6) $\operatorname{Int} \overline{A} \subseteq \overline{\operatorname{Int} A}$.

We say that A is of the $2^{\rm nd}$ class if and only if:

(Def. 7) $\overline{\operatorname{Int} A} \subset \operatorname{Int} \overline{A}$.

We say that A is of the $3^{\rm rd}$ class if and only if:

(Def. 8) $\overline{\operatorname{Int} A}$ and $\operatorname{Int} \overline{A}$ are $\subseteq$-incomparable.

The following proposition is true

(46) A is of the $1^{\rm st}$ class, or of the $2^{\rm nd}$ class, or of the $3^{\rm rd}$ class.

Let T be a topological space. One can verify the following observations:

* every subset of T which is of the $1^{\rm st}$ class is also non of the $2^{\rm nd}$ class and non of the $3^{\rm rd}$ class,
* every subset of T which is of the $2^{\rm nd}$ class is also non of the $1^{\rm st}$ class and non of the $3^{\rm rd}$ class, and
* every subset of T which is of the $3^{\rm rd}$ class is also non of the $1^{\rm st}$ class and non of the $2^{\rm nd}$ class.

One can prove the following proposition

(47) A is of the $1^{\rm st}$ class iff $\operatorname{Border} A$ is empty.

Let T be a topological space. Note that every subset of T which is supercondensed is also of the $1^{\rm st}$ class and every subset of T which is subcondensed is also of the $1^{\rm st}$ class.

Let T be a topological space. We say that T has subsets of the $1^{\rm st}$ class if and only if:

(Def. 9) There exists a subset of T which is of the $1^{\rm st}$ class.

We say that T has subsets of the $2^{\rm nd}$ class if and only if:

(Def. 10) There exists a subset of T which is of the $2^{\rm nd}$ class.

We say that T has subsets of the $3^{\rm rd}$ class if and only if:

(Def. 11) There exists a subset of T which is of the $3^{\rm rd}$ class.

Let T be an anti-discrete non empty topological space. Note that every subset of T which is proper and non empty is also of the $2^{\rm nd}$ class.

Let T be an anti-discrete non trivial non empty strict topological space. Observe that there exists a subset of T which is of the $2^{\rm nd}$ class.

One can verify that there exists a topological space which is non empty, strict, and non trivial and has subsets of the $1^{\rm st}$ class and subsets of the $2^{\rm nd}$ class and there exists a topological space which is non empty and strict and has subsets of the $3^{\rm rd}$ class.

Let us consider T. Observe that there exists a subset of T which is of the $1^{\rm st}$ class.

Let T be a topological space with subsets of the $2^{\rm nd}$ class. One can verify that there exists a subset of T which is of the $2^{\rm nd}$ class.

Let T be a topological space with subsets of the $3^{\rm rd}$ class. Observe that there exists a subset of T which is of the $3^{\rm rd}$ class.

The following propositions are true:

(48) A is of the 1^{st} class iff A^{c} is of the 1^{st} class.

(49) A is of the 2^{nd} class iff A^{c} is of the 2^{nd} class.

(50) A is of the 3^{rd} class iff A^{c} is of the 3^{rd} class.

Let us consider T and let A be an of the 1^{st} class subset of T. Observe that A^{c} is of the 1^{st} class.

Let T be a topological space with subsets of the 2^{nd} class and let A be an of the 2^{nd} class subset of T. Note that A^{c} is of the 2^{nd} class.

Let T be a topological space with subsets of the 3^{rd} class and let A be an of the 3^{rd} class subset of T. Note that A^{c} is of the 3^{rd} class.

Next we state four propositions:

(51) If A is of the 1^{st} class, then $\mathrm{Int}\,\overline{A} = \mathrm{Int}\,\overline{\mathrm{Int}\,A}$ and $\overline{\mathrm{Int}\,A} = \overline{\mathrm{Int}\,\overline{A}}$.

(52) If $\mathrm{Int}\,\overline{A} = \mathrm{Int}\,\overline{\mathrm{Int}\,A}$ or $\overline{\mathrm{Int}\,A} = \overline{\mathrm{Int}\,\overline{A}}$, then A is of the 1^{st} class.

(53) Suppose A is of the 1^{st} class and B is of the 1^{st} class. Then $\mathrm{Int}\,\overline{A} \cap \mathrm{Int}\,\overline{B} = \mathrm{Int}\,\overline{A \cap B}$ and $\overline{\mathrm{Int}\,A} \cup \overline{\mathrm{Int}\,B} = \overline{\mathrm{Int}(A \cup B)}$.

(54) Suppose A is of the 1^{st} class and B is of the 1^{st} class. Then $A \cup B$ is of the 1^{st} class and $A \cap B$ is of the 1^{st} class.

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