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Subsequences of Almost, Weakly and Poorly One-to-one Finite Sequences[1]

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The articles [21], [24], [1], [3], [2], [23], [4], [11], [9], [22], [16], [20], [19], [6], [7], [12], [8], [13], [17], [14], [15], [5], [18], and [10] provide the terminology and notation for this paper.

In this paper n is a natural number.

The following three propositions are true:

(1) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $\mathrm{len} \downharpoonright p, f \geq 1$.

(2) For every non empty finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for every point p of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{len} \downharpoonleft f, p \geq 1$.

(3) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for all points p, q of $\mathcal{E}^2_{\rm T}$ holds $\downharpoonright\downharpoonleft p, f, q \neq \emptyset$.

Let x be a set. One can check that $\langle x \rangle$ is one-to-one.

Let f be a finite sequence. We say that f is almost one-to-one if and only if:

(Def. 1) For all natural numbers i, j such that $i \in \mathrm{dom}\, f$ and $j \in \mathrm{dom}\, f$ and $i \neq 1$ or $j \neq \mathrm{len}\, f$ and $i \neq \mathrm{len}\, f$ or $j \neq 1$ and $f(i) = f(j)$ holds $i = j$.

Let f be a finite sequence. We say that f is weakly one-to-one if and only if:

(Def. 2) For every natural number i such that $1 \leq i$ and $i < \mathrm{len}\, f$ holds $f(i) \neq f(i+1)$.

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Let f be a finite sequence. We say that f is poorly one-to-one if and only if:

(Def. 3)(i) For every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $f(i) \neq f(i+1)$ if $\operatorname{len} f \neq 2$,

(ii) TRUE, otherwise.

The following three propositions are true:

(4) Let D be a set and f be a finite sequence of elements of D. Then f is almost one-to-one if and only if for all natural numbers i, j such that $i \in \operatorname{dom} f$ and $j \in \operatorname{dom} f$ and $i \neq 1$ or $j \neq \operatorname{len} f$ and $i \neq \operatorname{len} f$ or $j \neq 1$ and $f_i = f_j$ holds $i = j$.

(5) Let D be a set and f be a finite sequence of elements of D. Then f is weakly one-to-one if and only if for every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $f_i \neq f_{i+1}$.

(6) Let D be a set and f be a finite sequence of elements of D. Then f is poorly one-to-one if and only if if $\operatorname{len} f \neq 2$, then for every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $f_i \neq f_{i+1}$.

Let us note that every finite sequence which is one-to-one is also almost one-to-one.

One can check that every finite sequence which is almost one-to-one is also poorly one-to-one.

The following proposition is true

(7) For every finite sequence f such that $\operatorname{len} f \neq 2$ holds f is weakly one-to-one iff f is poorly one-to-one.

Let us note that $\emptyset$ is weakly one-to-one.

Let x be a set. One can verify that $\langle x \rangle$ is weakly one-to-one.

Let x, y be sets. Observe that $\langle x, y \rangle$ is poorly one-to-one.

Let us mention that there exists a finite sequence which is weakly one-to-one and non empty.

Let D be a non empty set. Observe that there exists a finite sequence of elements of D which is weakly one-to-one, circular, and non empty.

We now state three propositions:

(8) For every finite sequence f such that f is almost one-to-one holds $\operatorname{Rev}(f)$ is almost one-to-one.

(9) For every finite sequence f such that f is weakly one-to-one holds $\operatorname{Rev}(f)$ is weakly one-to-one.

(10) For every finite sequence f such that f is poorly one-to-one holds $\operatorname{Rev}(f)$ is poorly one-to-one.

Let us observe that there exists a finite sequence which is one-to-one and non empty.

Let f be an almost one-to-one finite sequence. Observe that $\operatorname{Rev}(f)$ is almost one-to-one.

Let f be a weakly one-to-one finite sequence. Observe that $\mathrm{Rev}(f)$ is weakly one-to-one.

Let f be a poorly one-to-one finite sequence. Observe that $\mathrm{Rev}(f)$ is poorly one-to-one.

One can prove the following three propositions:

(11) Let D be a non empty set and f be a finite sequence of elements of D. Suppose f is almost one-to-one. Let p be an element of D. Then $f \circlearrowleft p$ is almost one-to-one.

(12) Let D be a non empty set and f be a finite sequence of elements of D. Suppose f is weakly one-to-one and circular. Let p be an element of D. Then $f \circlearrowleft p$ is weakly one-to-one.

(13) Let D be a non empty set and f be a finite sequence of elements of D. Suppose f is poorly one-to-one and circular. Let p be an element of D. Then $f \circlearrowleft p$ is poorly one-to-one.

Let D be a non empty set. One can check that there exists a finite sequence of elements of D which is one-to-one, circular, and non empty.

Let D be a non empty set, let f be an almost one-to-one finite sequence of elements of D, and let p be an element of D. Note that $f \circlearrowleft p$ is almost one-to-one.

Let D be a non empty set, let f be a circular weakly one-to-one finite sequence of elements of D, and let p be an element of D. Note that $f \circlearrowleft p$ is weakly one-to-one.

Let D be a non empty set, let f be a circular poorly one-to-one finite sequence of elements of D, and let p be an element of D. One can verify that $f \circlearrowleft p$ is poorly one-to-one.

The following proposition is true

(14) Let D be a non empty set and f be a finite sequence of elements of D. Then f is almost one-to-one if and only if $f_{\lfloor 1}$ is one-to-one and $f \restriction (\mathrm{len}\, f -' 1)$ is one-to-one.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. Observe that $\mathrm{Cage}(C, n)$ is almost one-to-one.

Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and let n be a natural number. One can check that $\mathrm{Cage}(C, n)$ is weakly one-to-one.

The following propositions are true:

(15) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and f is weakly one-to-one, then $\downharpoonleft\!\downharpoonleft p, f, p = \langle p \rangle$.

(16) For every finite sequence f such that f is one-to-one holds f is weakly one-to-one.

One can check that every finite sequence which is one-to-one is also weakly one-to-one.

The following propositions are true:

(17) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is weakly one-to-one. Let p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonleft\downharpoonright p, f, q = \mathrm{Rev}(\downharpoonleft\downharpoonright q, f, p)$.

(18) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$, p be a point of $\mathcal{E}^2_{\rm T}$, and i_1 be a natural number. Suppose f is poorly one-to-one, unfolded, and s.n.c. and $1 < i_1$ and $i_1 \leq \mathrm{len}\, f$ and $p = f(i_1)$. Then $\mathrm{Index}(p, f) + 1 = i_1$.

(19) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is weakly one-to-one. Let p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$, then $(\downharpoonleft\downharpoonright p, f, q)_1 = p$.

(20) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is weakly one-to-one. Let p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$, then $(\downharpoonleft\downharpoonright p, f, q)_{\mathrm{len}\downharpoonleft\downharpoonright p, f, q} = q$.

(21) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $\widetilde{\mathcal{L}}(\downharpoonleft p, f) \subseteq \widetilde{\mathcal{L}}(f)$.

(22) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and f is weakly one-to-one, then $\widetilde{\mathcal{L}}(\downharpoonleft\downharpoonright p, f, q) \subseteq \widetilde{\mathcal{L}}(f)$.

(23) For all finite sequences f, g holds $\mathrm{dom}\, f \subseteq \mathrm{dom}(f \frown g)$.

(24) For every non empty finite sequence f and for every finite sequence g holds $\mathrm{dom}\, g \subseteq \mathrm{dom}(f \frown g)$.

(25) For all finite sequences f, g such that $f \frown g$ is constant holds f is constant.

(26) For all finite sequences f, g such that $f \frown g$ is constant and $f(\mathrm{len}\, f) = g(1)$ and $f \neq \emptyset$ holds g is constant.

(27) For every special finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for all natural numbers i, j holds $\mathrm{mid}(f, i, j)$ is special.

(28) For every unfolded finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ and for all natural numbers i, j holds $\mathrm{mid}(f, i, j)$ is unfolded.

(29) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is special. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonleft p, f$ is special.

(30) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is special. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonright f, p$ is special.

(31) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is special and weakly one-to-one. Let p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonleft\downharpoonright p, f, q$ is special.

(32) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is unfolded. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonleft p, f$ is unfolded.

(33) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is unfolded. Let

p be a point of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonleft f, p$ is unfolded.

(34) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is unfolded and weakly one-to-one. Let p, q be points of $\mathcal{E}^2_{\rm T}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$, then $\downharpoonright\downharpoonleft p, f, q$ is unfolded.

(35) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is almost one-to-one, special, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$ and $g = (\mathrm{mid}(f, 1, \mathrm{Index}(p, f))) \frown \langle p \rangle$. Then g is a special sequence joining f_1, p.

(36) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is poorly one-to-one, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p = f(\mathrm{Index}(p, f)+1)$ and $p \neq f(\mathrm{len}\, f)$. Then $\mathrm{Index}(p, \mathrm{Rev}(f)) + \mathrm{Index}(p, f) + 1 = \mathrm{len}\, f$.

(37) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. If f is weakly one-to-one and $\mathrm{len}\, f \geq 2$, then $\downharpoonright f_1, f = f$.

(38) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is poorly one-to-one, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$. Then $\downharpoonright p, \mathrm{Rev}(f) = \mathrm{Rev}(\downharpoonleft f, p)$.

(39) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is almost one-to-one, special, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $\downharpoonleft f, p$ is a special sequence joining f_1, p.

(40) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is almost one-to-one, special, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$ and $p \neq f(1)$. Then $\downharpoonright p, f$ is a special sequence joining p, $f_{\mathrm{len}\, f}$.

(41) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is almost one-to-one, special, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(1)$. Then $\downharpoonleft f, p$ is a special sequence.

(42) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose f is almost one-to-one, special, unfolded, and s.n.c. and $p \in \widetilde{\mathcal{L}}(f)$ and $p \neq f(\mathrm{len}\, f)$ and $p \neq f(1)$. Then $\downharpoonright p, f$ is a special sequence.

(43) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. Suppose that f is almost one-to-one, special, unfolded, and s.n.c. and $\mathrm{len}\, f \neq 2$ and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$ and $p \neq f(1)$ and $q \neq f(1)$. Then $\downharpoonright\downharpoonleft p, f, q$ is a special sequence joining p, q.

(44) Let f be a non empty finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. Suppose that f is almost one-to-one, special, unfolded, and s.n.c. and $\mathrm{len}\, f \neq 2$ and $p \in \widetilde{\mathcal{L}}(f)$ and $q \in \widetilde{\mathcal{L}}(f)$ and $p \neq q$ and $p \neq f(1)$ and $q \neq f(1)$. Then $\downharpoonright\downharpoonleft p, f, q$ is a special sequence.

(45) Let C be a compact non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and p, q be points of $\mathcal{E}^2_{\rm T}$. Suppose $p \in \mathrm{BDD}\, \widetilde{\mathcal{L}}(\mathrm{Cage}(C, n))$. Then there exists a

S-sequence B in $\mathbb{R}^2$ such that

(i) $B = \downharpoonleft\downharpoonright$ South-Bound$(p, \widetilde{\mathcal{L}}(\text{Cage}(C,n)))$, $(\text{Cage}(C,n) \circlearrowleft (\text{Cage}(C,n))_{\text{Index}(\text{South-Bound}(p,\widetilde{\mathcal{L}}(\text{Cage}(C,n))),\text{Cage}(C,n))}) \upharpoonright (\text{len}$ $(\text{Cage}(C,n) \circlearrowleft (\text{Cage}(C,n))_{\text{Index}(\text{South-Bound}(p,\widetilde{\mathcal{L}}(\text{Cage}(C,n))),\text{Cage}(C,n))}) -' 1)$, North-Bound$(p, \widetilde{\mathcal{L}}(\text{Cage}(C,n)))$, and

(ii) there exists a S-sequence P in $\mathbb{R}^2$ such that P is a sequence which elements belong to the Go-board of $B \frown \langle$North-Bound$(p, \widetilde{\mathcal{L}}(\text{Cage}(C,n)))$, South-Bound$(p, \widetilde{\mathcal{L}}(\text{Cage}(C,n)))\rangle$ and $\widetilde{\mathcal{L}}(\langle$North-Bound$(p, \widetilde{\mathcal{L}}(\text{Cage}(C,n)))$, South-Bound$(p, \widetilde{\mathcal{L}}(\text{Cage}(C,n)))\rangle) = \widetilde{\mathcal{L}}(P)$ and $P_1 =$ North-Bound$(p, \widetilde{\mathcal{L}}(\text{Cage}(C,n)))$ and $P_{\text{len}\,P} =$ South-Bound$(p, \widetilde{\mathcal{L}}(\text{Cage}(C,n)))$ and $\text{len}\,P \geq 2$ and there exists a S-sequence B_1 in $\mathbb{R}^2$ such that B_1 is a sequence which elements belong to the Go-board of $B \frown \langle$North-Bound$(p, \widetilde{\mathcal{L}}(\text{Cage}(C,n)))$, South-Bound$(p, \widetilde{\mathcal{L}}(\text{Cage}(C,n)))\rangle$ and $\widetilde{\mathcal{L}}(B) = \widetilde{\mathcal{L}}(B_1)$ and $B_1 = (B_1)_1$ and $B_{\text{len}\,B} = (B_1)_{\text{len}\,B_1}$ and $\text{len}\,B \leq \text{len}\,B_1$ and there exists a non constant standard special circular sequence g such that $g = B_1 \frown P$.

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