FORMALIZED MATHEMATICS
Volume 13, Number 2, Pages 315–323
University of Białystok, 2005

Properties of Connected Subsets of the Real Line

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MML identifier: RCOMP_3, version: 7.5.01 4.39.921

The papers [31], [36], [3], [37], [27], [18], [9], [38], [10], [22], [14], [4], [34], [5], [39], [1], [33], [30], [2], [23], [21], [6], [20], [35], [29], [24], [28], [40], [17], [13], [12], [26], [15], [8], [11], [16], [19], [25], [32], and [7] provide the notation and terminology for this paper.

1. Preliminaries

Let X be a non empty set. Observe that Ω_X is non empty.

Let us observe that every subspace of the metric space of real numbers is real-membered.

Let S be a real-membered 1-sorted structure. One can check that the carrier of S is real-membered.

One can check that there exists a real-membered set which is non empty, finite, lower bounded, and upper bounded.

We now state three propositions:

(1) For every non empty lower bounded real-membered set X and for every closed subset Y of $\mathbb{R}$ such that $X \subseteq Y$ holds $\inf X \in Y$.

(2) For every non empty upper bounded real-membered set X and for every closed subset Y of $\mathbb{R}$ such that $X \subseteq Y$ holds $\sup X \in Y$.

(3) For all subsets X, Y of $\mathbb{R}$ holds $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$.

[1]The paper was written during the author's post-doctoral fellowship granted by Shinshu University, Japan.

ISSN 1426–2630

2. Intervals

In the sequel a, b, r, s are real numbers.

Let us consider r, s. One can check the following observations:

* $[r, s[$ is bounded,
* $]r, s]$ is bounded, and
* $]r, s[$ is bounded.

Let us consider r, s. One can verify the following observations:

* $[r, s]$ is connected,
* $[r, s[$ is connected,
* $]r, s]$ is connected, and
* $]r, s[$ is connected.

Let us observe that there exists a subset of $\mathbb{R}$ which is open, bounded, connected, and non empty.

One can prove the following propositions:

(4) If $r < s$, then $\inf[r, s[= r$.

(5) If $r < s$, then $\sup[r, s[= s$.

(6) If $r < s$, then $\inf]r, s] = r$.

(7) If $r < s$, then $\sup]r, s] = s$.

(8) If $a \leq b$ or $r \leq s$ and if $[a, b] = [r, s]$, then $a = r$ and $b = s$.

(9) If $a < b$ or $r < s$ and if $]a, b[=]r, s[$, then $a = r$ and $b = s$.

(10) If $a < b$ or $r < s$ and if $]a, b] =]r, s]$, then $a = r$ and $b = s$.

(11) If $a < b$ or $r < s$ and if $[a, b[= [r, s[$, then $a = r$ and $b = s$.

(12) If $a < b$ and $[a, b[\subseteq [r, s]$, then $r \leq a$ and $b \leq s$.

(13) If $a < b$ and $[a, b[\subseteq [r, s[$, then $r \leq a$ and $b \leq s$.

(14) If $a < b$ and $]a, b] \subseteq [r, s]$, then $r \leq a$ and $b \leq s$.

(15) If $a < b$ and $]a, b] \subseteq]r, s]$, then $r \leq a$ and $b \leq s$.

3. Halflines

One can prove the following propositions:

(16) $[a, b]^{\rm c} =]-\infty, a[\cup]b, +\infty[$.

(17) $]a, b[^{\rm c} =]-\infty, a] \cup [b, +\infty[$.

(18) $[a, b[^{\rm c} =]-\infty, a[\cup [b, +\infty[$.

(19) $]a, b]^{\rm c} =]-\infty, a] \cup]b, +\infty[$.

(20) If $a \leq b$, then $[a, b] \cap (]-\infty, a] \cup [b, +\infty[) = \{a, b\}$.

Let us consider a. One can verify the following observations:

* $]-\infty, a]$ is non lower bounded, upper bounded, and connected,
* $]-\infty, a[$ is non lower bounded, upper bounded, and connected,
* $[a, +\infty[$ is lower bounded, non upper bounded, and connected, and
* $]a, +\infty[$ is lower bounded, non upper bounded, and connected.

The following propositions are true:

(21) $\sup]-\infty, a] = a$.

(22) $\sup]-\infty, a[= a$.

(23) $\inf[a, +\infty[= a$.

(24) $\inf]a, +\infty[= a$.

4. Connectedness

Let us observe that $\Omega_{\mathbb{R}}$ is connected, non lower bounded, and non upper bounded.

One can prove the following propositions:

(25) For every bounded connected subset X of $\mathbb{R}$ such that $\inf X \in X$ and $\sup X \in X$ holds $X = [\inf X, \sup X]$.

(26) For every bounded subset X of $\mathbb{R}$ such that $\inf X \notin X$ holds $X \subseteq]\inf X, \sup X]$.

(27) For every bounded connected subset X of $\mathbb{R}$ such that $\inf X \notin X$ and $\sup X \in X$ holds $X =]\inf X, \sup X]$.

(28) For every bounded subset X of $\mathbb{R}$ such that $\sup X \notin X$ holds $X \subseteq [\inf X, \sup X[$.

(29) For every bounded connected subset X of $\mathbb{R}$ such that $\inf X \in X$ and $\sup X \notin X$ holds $X = [\inf X, \sup X[$.

(30) For every bounded subset X of $\mathbb{R}$ such that $\inf X \notin X$ and $\sup X \notin X$ holds $X \subseteq]\inf X, \sup X[$.

(31) For every non empty bounded connected subset X of $\mathbb{R}$ such that $\inf X \notin X$ and $\sup X \notin X$ holds $X =]\inf X, \sup X[$.

(32) For every subset X of $\mathbb{R}$ such that X is upper bounded holds $X \subseteq]-\infty, \sup X]$.

(33) For every connected subset X of $\mathbb{R}$ such that X is not lower bounded and X is upper bounded and $\sup X \in X$ holds $X =]-\infty, \sup X]$.

(34) For every subset X of $\mathbb{R}$ such that X is upper bounded and $\sup X \notin X$ holds $X \subseteq]-\infty, \sup X[$.

(35) For every non empty connected subset X of $\mathbb{R}$ such that X is not lower bounded and X is upper bounded and $\sup X \notin X$ holds $X =]-\infty, \sup X[$.

(36) For every subset X of $\mathbb{R}$ such that X is lower bounded holds $X \subseteq [\inf X, +\infty[$.

(37) For every connected subset X of $\mathbb{R}$ such that X is lower bounded and X is not upper bounded and $\inf X \in X$ holds $X = [\inf X, +\infty[$.

(38) For every subset X of $\mathbb{R}$ such that X is lower bounded and $\inf X \notin X$ holds $X \subseteq]\inf X, +\infty[$.

(39) For every non empty connected subset X of $\mathbb{R}$ such that X is lower bounded and X is not upper bounded and $\inf X \notin X$ holds $X =]\inf X, +\infty[$.

(40) For every connected subset X of $\mathbb{R}$ such that X is not upper bounded and X is not lower bounded holds $X = \mathbb{R}$.

(41) Let X be a connected subset of $\mathbb{R}$. Then X is empty or $X = \mathbb{R}$ or there exists a such that $X =]-\infty, a]$ or there exists a such that $X =]-\infty, a[$ or there exists a such that $X = [a, +\infty[$ or there exists a such that $X =]a, +\infty[$ or there exist a, b such that $a \leq b$ and $X = [a, b]$ or there exist a, b such that $a < b$ and $X = [a, b[$ or there exist a, b such that $a < b$ and $X =]a, b]$ or there exist a, b such that $a < b$ and $X =]a, b[$.

(42) For every non empty connected subset X of $\mathbb{R}$ such that $r \notin X$ holds $r \leq \inf X$ or $\sup X \leq r$.

(43) Let X, Y be non empty bounded connected subsets of $\mathbb{R}$. Suppose $\inf X \leq \inf Y$ and $\sup Y \leq \sup X$ and if $\inf X = \inf Y$ and $\inf Y \in Y$, then $\inf X \in X$ and if $\sup X = \sup Y$ and $\sup Y \in Y$, then $\sup X \in X$. Then $Y \subseteq X$.

Let us observe that there exists a subset of $\mathbb{R}$ which is open, closed, connected, non empty, and non bounded.

5. $\mathbb{R}^1$

Next we state several propositions:

(44) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $a \leq b$ and $X = [a, b]$ holds $\operatorname{Fr} X = \{a, b\}$.

(45) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $a < b$ and $X =]a, b[$ holds $\operatorname{Fr} X = \{a, b\}$.

(46) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $a < b$ and $X = [a, b[$ holds $\operatorname{Fr} X = \{a, b\}$.

(47) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $a < b$ and $X =]a, b]$ holds $\operatorname{Fr} X = \{a, b\}$.

(48) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $X = [a, b]$ holds $\operatorname{Int} X =]a, b[$.

(49) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $X =]a, b[$ holds $\operatorname{Int} X =]a, b[$.

(50) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $X = [a, b[$ holds $\operatorname{Int} X =]a, b[$.

(51) For every subset X of $\mathbb{R}^{\mathbf{1}}$ such that $X =]a, b]$ holds $\operatorname{Int} X =]a, b[$.

Let X be a convex subset of $\mathbb{R}^{\mathbf{1}}$. Observe that $\mathbb{R}^{\mathbf{1}}\restriction X$ is convex.

Let A be a connected subset of $\mathbb{R}$. One can check that $R^1 A$ is convex.

We now state the proposition

(52) Let X be a subset of $\mathbb{R}^{\mathbf{1}}$ and Y be a subset of $\mathbb{R}$. If $X = Y$, then X is connected iff Y is connected.

6. Topology of Closed Intervals

Let us consider r. Note that $[r, r]_{\rm T}$ is trivial.

The following four propositions are true:

(53) If $r \leq s$, then every subset of $[r, s]_{\rm T}$ is a bounded subset of $\mathbb{R}$.

(54) If $r \leq s$, then for every subset X of $[r, s]_{\rm T}$ such that $X = [a, b[$ and $r < a$ and $b \leq s$ holds $\operatorname{Int} X =]a, b[$.

(55) If $r \leq s$, then for every subset X of $[r, s]_{\rm T}$ such that $X =]a, b]$ and $r \leq a$ and $b < s$ holds $\operatorname{Int} X =]a, b[$.

(56) Let X be a subset of $[r, s]_{\rm T}$ and Y be a subset of $\mathbb{R}$. If $X = Y$, then X is connected iff Y is connected.

Let T be a topological space. Observe that there exists a subset of T which is open, closed, and connected.

Let T be a non empty connected topological space. Observe that there exists a subset of T which is non empty, open, closed, and connected.

We now state the proposition

(57) Suppose $r \leq s$. Let X be an open connected subset of $[r, s]_{\rm T}$. Then

(i) X is empty, or

(ii) $X = [r, s]$, or

(iii) there exists a real number a such that $r < a$ and $a \leq s$ and $X = [r, a[$, or

(iv) there exists a real number a such that $r \leq a$ and $a < s$ and $X =]a, s]$, or

(v) there exist real numbers a, b such that $r \leq a$ and $a < b$ and $b \leq s$ and $X =]a, b[$.

7. Minimal Cover of Intervals

Next we state three propositions:

(58) Let T be a 1-sorted structure and F be a family of subsets of T. Then F is a cover of T if and only if F is a cover of Ω_T.

(59) Let T be a 1-sorted structure, F be a finite family of subsets of T, and F_1 be a family of subsets of T. Suppose F is a cover of T and $F_1 = F \setminus \{X;X$

ranges over subsets of T: $X \in F \wedge \bigvee_{Y:\text{subset of } T} (Y \in F \wedge X \subseteq Y \wedge X \neq Y)\}$. Then F_1 is a cover of T.

(60) Let S be a trivial non empty 1-sorted structure, s be a point of S, and F be a family of subsets of S. If F is a cover of S, then $\{s\} \in F$.

Let T be a topological structure and let F be a family of subsets of T. We say that F is connected if and only if:

(Def. 1) For every subset X of T such that $X \in F$ holds X is connected.

Let T be a topological space. Note that there exists a family of subsets of T which is non empty, open, closed, and connected.

In the sequel n, m are natural numbers and F is a family of subsets of $[r, s]_{\rm T}$.

The following two propositions are true:

(61) Let L be a topological space and G, G_1 be families of subsets of L. Suppose G is a cover of L and finite. Let A_1 be a set such that $G_1 = G \setminus \{X; X$ ranges over subsets of L: $X \in G \wedge \bigvee_{Y:\text{subset of } L} (Y \in G \wedge X \subseteq Y \wedge X \neq Y)\}$ and $A_1 = \{C; C$ ranges over families of subsets of L: C is a cover of $L \wedge C \subseteq G_1\}$. Then A_1 has the lower Zorn property w.r.t. $\subseteq_{(A_1)}$.

(62) Let L be a topological space and G, A_1 be sets. Suppose $A_1 = \{C; C$ ranges over families of subsets of L: C is a cover of $L \wedge C \subseteq G\}$. Let M be a set. Suppose M is minimal in $\subseteq_{(A_1)}$ and $M \in \text{field}(\subseteq_{(A_1)})$. Let A_4 be a subset of L. Suppose $A_4 \in M$. Then it is not true that there exist subsets A_2, A_3 of L such that $A_2 \in M$ and $A_3 \in M$ and $A_4 \subseteq A_2 \cup A_3$ and $A_4 \neq A_2$ and $A_4 \neq A_3$.

Let r, s be real numbers and let F be a family of subsets of $[r, s]_{\rm T}$. Let us assume that F is a cover of $[r, s]_{\rm T}$ F is open F is connected and $r \leq s$. A finite sequence of elements of $2^{\mathbb{R}}$ is said to be an interval cover of F if it satisfies the conditions (Def. 2).

(Def. 2)(i) $\text{rng}\,\text{it} \subseteq F$,

(ii) $\bigcup \text{rng}\,\text{it} = [r, s]$,

(iii) for every natural number n such that $1 \leq n$ holds if $n \leq \text{len}\,\text{it}$, then it_n is non empty and if $n + 1 \leq \text{len}\,\text{it}$, then $\inf(\text{it}_n) \leq \inf(\text{it}_{n+1})$ and $\sup(\text{it}_n) \leq \sup(\text{it}_{n+1})$ and $\inf(\text{it}_{n+1}) < \sup(\text{it}_n)$ and if $n + 2 \leq \text{len}\,\text{it}$, then $\sup(\text{it}_n) \leq \inf(\text{it}_{n+2})$,

(iv) if $[r, s] \in F$, then $\text{it} = \langle [r, s] \rangle$, and

(v) if $[r, s] \notin F$, then there exists a real number p such that $r < p$ and $p \leq s$ and $\text{it}(1) = [r, p[$ and there exists a real number p such that $r \leq p$ and $p < s$ and $\text{it}(\text{len}\,\text{it}) =]p, s]$ and for every natural number n such that $1 < n$ and $n < \text{len}\,\text{it}$ there exist real numbers p, q such that $r \leq p$ and $p < q$ and $q \leq s$ and $\text{it}(n) =]p, q[$.

We now state the proposition

(63) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$ and $[r, s] \in F$, then $\langle [r, s] \rangle$ is an interval cover of F.

In the sequel C denotes an interval cover of F.

One can prove the following propositions:

(64) Let F be a family of subsets of $[r, r]_{\rm T}$ and C be an interval cover of F. If F is a cover of $[r, r]_{\rm T}$, open, and connected, then $C = \langle [r, r] \rangle$.

(65) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$, then $1 \leq {\rm len}\, C$.

(66) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$ and ${\rm len}\, C = 1$, then $C = \langle [r, s] \rangle$.

(67) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$ and $n \in {\rm dom}\, C$ and $m \in {\rm dom}\, C$ and $n < m$, then $\inf(C_n) \leq \inf(C_m)$.

(68) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$ and $n \in {\rm dom}\, C$ and $m \in {\rm dom}\, C$ and $n < m$, then $\sup(C_n) \leq \sup(C_m)$.

(69) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$ and $1 \leq n$ and $n + 1 \leq {\rm len}\, C$, then $]\inf(C_{n+1}), \sup(C_n)[$ is non empty.

(70) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$, then $\inf(C_1) = r$.

(71) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$, then $r \in C_1$.

(72) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$, then $\sup(C_{{\rm len}\, C}) = s$.

(73) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$, then $s \in C_{{\rm len}\, C}$.

Let r, s be real numbers, let F be a family of subsets of $[r, s]_{\rm T}$, and let C be an interval cover of F. Let us assume that F is a cover of $[r, s]_{\rm T}$ F is open F is connected and $r \leq s$. A finite sequence of elements of $\mathbb{R}$ is said to be a chain of rivets in interval cover C if it satisfies the conditions (Def. 3).

(Def. 3)(i) ${\rm len}\, {\rm it} = {\rm len}\, C + 1$,

(ii) ${\rm it}(1) = r$,

(iii) ${\rm it}({\rm len}\, {\rm it}) = s$, and

(iv) for every natural number n such that $1 \leq n$ and $n + 1 < {\rm len}\, {\rm it}$ holds ${\rm it}(n + 1) \in]\inf(C_{n+1}), \sup(C_n)[$.

In the sequel G denotes a chain of rivets in interval cover C.

One can prove the following propositions:

(74) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$, then $2 \leq {\rm len}\, G$.

(75) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$ and ${\rm len}\, C = 1$, then $G = \langle r, s \rangle$.

(76) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$ and $1 \leq n$ and $n + 1 < {\rm len}\, G$, then $G(n + 1) < \sup(C_n)$.

(77) If F is a cover of $[r, s]_{\rm T}$, open, and connected and $r \leq s$ and $1 < n$ and $n \leq {\rm len}\, C$, then $\inf(C_n) < G(n)$.

(78) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$ and $1 \leq n$ and $n < \operatorname{len} C$, then $G(n) \leq \inf(C_{n+1})$.

(79) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r < s$, then G is increasing.

(80) If F is a cover of $[r, s]_{\mathrm{T}}$, open, and connected and $r \leq s$ and $1 \leq n$ and $n < \operatorname{len} G$, then $[G(n), G(n+1)] \subseteq C(n)$.

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Received February 22, 2005