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Lines on Planes in n-Dimensional Euclidean Spaces

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Summary. In the paper we introduce basic properties of lines in the plane on this space. Lines and planes are expressed by the vector equation and are the image of $\mathbb{R}$ and $\mathbb{R}^2$. By this, we can say that the properties of the classic Euclid geometry are satisfied also in $\mathcal{R}^n$ as we know them intuitively. Next, we define the metric between the point and the line of this space.

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The notation and terminology used here are introduced in the following papers: [1], [5], [12], [4], [9], [14], [13], [8], [15], [6], [2], [3], [7], [11], and [10].

We follow the rules: a, a_1, a_2, a_3, b, b_1, b_2, b_3, r, s, t, u are real numbers, n is a natural number, and x_0, x, x_1, x_2, x_3, y_0, y, y_1, y_2, y_3 are elements of $\mathcal{R}^n$.

One can prove the following propositions:

(1) $\frac{s}{t} \cdot (u \cdot x) = \frac{s \cdot u}{t} \cdot x$ and $\frac{1}{t} \cdot (u \cdot x) = \frac{u}{t} \cdot x$.

(2) $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$.

(3) $x - \langle \underbrace{0, \ldots, 0}_{n} \rangle = x$.

(4) $\langle \underbrace{0, \ldots, 0}_{n} \rangle - x = -x$.

(5) $x_1 - (x_2 + x_3) = x_1 - x_2 - x_3$.

(6) $x_1 - x_2 = x_1 + -x_2$.

(7) $x - x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$ and $x + -x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(8) $-a \cdot x = (-a) \cdot x$ and $-a \cdot x = a \cdot -x$.

(9) $x_1 - (x_2 - x_3) = (x_1 - x_2) + x_3$.

(10) $x_1 + (x_2 - x_3) = (x_1 + x_2) - x_3$.

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(11) $x_1 = x_2 + x_3$ iff $x_2 = x_1 - x_3$.

(12) $x = x_1 + x_2 + x_3$ iff $x - x_1 = x_2 + x_3$.

(13) $-(x_1 + x_2 + x_3) = -x_1 + -x_2 + -x_3$.

(14) $x_1 = x_2$ iff $x_1 - x_2 = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(15) If $x_1 - x_0 = t \cdot x$ and $x_1 \neq x_0$, then $t \neq 0$.

(16) $(a-b) \cdot x = a \cdot x + (-b) \cdot x$ and $(a-b) \cdot x = a \cdot x + -b \cdot x$ and $(a-b) \cdot x = a \cdot x - b \cdot x$.

(17) $a \cdot (x-y) = a \cdot x + -a \cdot y$ and $a \cdot (x-y) = a \cdot x + (-a) \cdot y$ and $a \cdot (x-y) = a \cdot x - a \cdot y$.

(18) $(s - t - u) \cdot x = s \cdot x - t \cdot x - u \cdot x$.

(19) $x - (a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3) = x + ((-a_1) \cdot x_1 + (-a_2) \cdot x_2 + (-a_3) \cdot x_3)$.

(20) $x - (s + t + u) \cdot y = x + (-s) \cdot y + (-t) \cdot y + (-u) \cdot y$.

(21) $(x_1 + x_2) + (y_1 + y_2) = x_1 + y_1 + (x_2 + y_2)$.

(22) $(x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = x_1 + y_1 + (x_2 + y_2) + (x_3 + y_3)$.

(23) $(x_1 + x_2) - (y_1 + y_2) = (x_1 - y_1) + (x_2 - y_2)$.

(24) $(x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = (x_1 - y_1) + (x_2 - y_2) + (x_3 - y_3)$.

(25) $a \cdot (x_1 + x_2 + x_3) = a \cdot x_1 + a \cdot x_2 + a \cdot x_3$.

(26) $a \cdot (b_1 \cdot x_1 + b_2 \cdot x_2) = a \cdot b_1 \cdot x_1 + a \cdot b_2 \cdot x_2$.

(27) $a \cdot (b_1 \cdot x_1 + b_2 \cdot x_2 + b_3 \cdot x_3) = a \cdot b_1 \cdot x_1 + a \cdot b_2 \cdot x_2 + a \cdot b_3 \cdot x_3$.

(28) $a_1 \cdot x_1 + a_2 \cdot x_2 + (b_1 \cdot x_1 + b_2 \cdot x_2) = (a_1 + b_1) \cdot x_1 + (a_2 + b_2) \cdot x_2$.

(29) $a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3 + (b_1 \cdot x_1 + b_2 \cdot x_2 + b_3 \cdot x_3) = ((a_1 + b_1) \cdot x_1 + (a_2 + b_2) \cdot x_2) + (a_3 + b_3) \cdot x_3$.

(30) $(a_1 \cdot x_1 + a_2 \cdot x_2) - (b_1 \cdot x_1 + b_2 \cdot x_2) = (a_1 - b_1) \cdot x_1 + (a_2 - b_2) \cdot x_2$.

(31) $(a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3) - (b_1 \cdot x_1 + b_2 \cdot x_2 + b_3 \cdot x_3) = (a_1 - b_1) \cdot x_1 + (a_2 - b_2) \cdot x_2 + (a_3 - b_3) \cdot x_3$.

(32) If $a_1 + a_2 + a_3 = 1$, then $a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3 = x_1 + a_2 \cdot (x_2 - x_1) + a_3 \cdot (x_3 - x_1)$.

(33) If $x = x_1 + a_2 \cdot (x_2 - x_1) + a_3 \cdot (x_3 - x_1)$, then there exists a real number a_1 such that $x = a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3$ and $a_1 + a_2 + a_3 = 1$.

(34) For every natural number n such that $n \geq 1$ holds $1 * n \neq \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(35) For every subset A of $\mathcal{R}^n$ and for all x_1, x_2 such that A is a line and $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$ holds $A = \mathrm{Line}(x_1, x_2)$.

(36) For all elements x_1, x_2 of $\mathcal{R}^n$ such that $y_1 \in \mathrm{Line}(x_1, x_2)$ and $y_2 \in \mathrm{Line}(x_1, x_2)$ there exists a such that $y_2 - y_1 = a \cdot (x_2 - x_1)$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. The predicate $x_1 \parallel x_2$ is defined as follows:

(Def. 1) $x_1 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $x_2 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and there exists r such that $x_1 = r \cdot x_2$.

One can prove the following proposition

(37) For all elements x_1, x_2 of $\mathcal{R}^n$ such that $x_1 \parallel x_2$ there exists a such that $a \neq 0$ and $x_1 = a \cdot x_2$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. Let us note that the predicate $x_1 \parallel x_2$ is symmetric.

The following proposition is true

(38) If $x_1 \parallel x_2$ and $x_2 \parallel x_3$, then $x_1 \parallel x_3$.

Let n be a natural number and let x_1, x_2 be elements of $\mathcal{R}^n$. We say that x_1 and x_2 are linearly independent if and only if:

(Def. 2) For all real numbers a_1, a_2 such that $a_1 \cdot x_1 + a_2 \cdot x_2 = \langle\underbrace{0,\ldots,0}_{n}\rangle$ holds $a_1 = 0$ and $a_2 = 0$.

Let us note that the predicate x_1 and x_2 are linearly independent is symmetric.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. We introduce x_1 and x_2 are linearly dependent as an antonym of x_1 and x_2 are linearly independent.

Next we state a number of propositions:

(39) If x_1 and x_2 are linearly independent, then $x_1 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $x_2 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(40) For all x_1, x_2 such that x_1 and x_2 are linearly independent holds if $a_1 \cdot x_1 + a_2 \cdot x_2 = b_1 \cdot x_1 + b_2 \cdot x_2$, then $a_1 = b_1$ and $a_2 = b_2$.

(41) Let given x_1, x_2, y_1, y_1. Suppose y_1 and y_2 are linearly independent. Suppose $y_1 = a_1 \cdot x_1 + a_2 \cdot x_2$ and $y_2 = b_1 \cdot x_1 + b_2 \cdot x_2$. Then there exist real numbers c_1, c_2, d_1, d_2 such that $x_1 = c_1 \cdot y_1 + c_2 \cdot y_2$ and $x_2 = d_1 \cdot y_1 + d_2 \cdot y_2$.

(42) If x_1 and x_2 are linearly independent, then $x_1 \neq x_2$.

(43) If $x_2 - x_1$ and $x_3 - x_1$ are linearly independent, then $x_2 \neq x_3$.

(44) If x_1 and x_2 are linearly independent, then $x_1 + t \cdot x_2$ and x_2 are linearly independent.

(45) Suppose $x_1 - x_0$ and $x_3 - x_2$ are linearly independent and $y_0 \in \mathrm{Line}(x_0, x_1)$ and $y_1 \in \mathrm{Line}(x_0, x_1)$ and $y_0 \neq y_1$ and $y_2 \in \mathrm{Line}(x_2, x_3)$ and $y_3 \in \mathrm{Line}(x_2, x_3)$ and $y_2 \neq y_3$. Then $y_1 - y_0$ and $y_3 - y_2$ are linearly independent.

(46) If $x_1 \parallel x_2$, then x_1 and x_2 are linearly dependent and $x_1 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $x_2 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(47) If x_1 and x_2 are linearly dependent, then $x_1 = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ or $x_2 = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ or $x_1 \parallel x_2$.

(48) For all elements x_1, x_2, y_1 of $\mathcal{R}^n$ there exists an element y_2 of $\mathcal{R}^n$ such that $y_2 \in \mathrm{Line}(x_1, x_2)$ and $x_1 - x_2$, $y_1 - y_2$ are orthogonal.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. The predicate $x_1 \perp x_2$ is defined by:

(Def. 3) $x_1 \neq \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ and $x_2 \neq \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ and x_1, x_2 are orthogonal.

Let us note that the predicate $x_1 \perp x_2$ is symmetric.

The following propositions are true:

(49) If $x \perp y_0$ and $y_0 \parallel y_1$, then $x \perp y_1$.

(50) If $x \perp y$, then x and y are linearly independent.

(51) If $x_1 \parallel x_2$, then $x_1 \not\perp x_2$.

(52) If $x_1 \perp x_2$, then $x_1 \nparallel x_2$.

Let us consider n. The functor $\mathrm{Lines}(\mathcal{R}^n)$ yields a family of subsets of $\mathcal{R}^n$ and is defined by:

(Def. 4) $\mathrm{Lines}(\mathcal{R}^n) = \{\mathrm{Line}(x_1, x_2)\}$.

Let us consider n. Note that $\mathrm{Lines}(\mathcal{R}^n)$ is non empty.

The following proposition is true

(53) $\mathrm{Line}(x_1, x_2) \in \mathrm{Lines}(\mathcal{R}^n)$.

In the sequel L, L_0, L_1, L_2 are elements of $\mathrm{Lines}(\mathcal{R}^n)$.

The following propositions are true:

(54) If $x_1 \in L$ and $x_2 \in L$, then $\mathrm{Line}(x_1, x_2) \subseteq L$.

(55) L_1 meets L_2 iff there exists x such that $x \in L_1$ and $x \in L_2$.

(56) If L_0 misses L_1 and $x \in L_0$, then $x \notin L_1$.

(57) There exist x_1, x_2 such that $L = \mathrm{Line}(x_1, x_2)$.

(58) There exists x such that $x \in L$.

(59) If $x_0 \in L$ and L is a line, then there exists x_1 such that $x_1 \neq x_0$ and $x_1 \in L$.

(60) If $x \notin L$ and L is a line, then there exist x_1, x_2 such that $L = \mathrm{Line}(x_1, x_2)$ and $x - x_1 \perp x_2 - x_1$.

(61) If $x \notin L$ and L is a line, then there exist x_1, x_2 such that $L = \mathrm{Line}(x_1, x_2)$ and $x - x_1$ and $x_2 - x_1$ are linearly independent.

Let n be a natural number, let x be an element of $\mathcal{R}^n$, and let L be an element of $\mathrm{Lines}(\mathcal{R}^n)$. The functor $\rho(x, L)$ yields a real number and is defined by:

(Def. 5) There exists a subset S of $\mathbb{R}$ such that $S = \{|x - x_0|; x_0$ ranges over elements of $\mathcal{R}^n$: $x_0 \in L\}$ and $\rho(x, L) = \inf S$.

Next we state three propositions:

(62) There exists x_0 such that $x_0 \in L$ and $|x - x_0| = \rho(x, L)$.

(63) $\rho(x, L) \geq 0$.

(64) $x \in L$ iff $\rho(x, L) = 0$.

Let us consider n and let us consider L_1, L_2. The predicate $L_1 \parallel L_2$ is defined as follows:

(Def. 6) There exist elements x_1, x_2, y_1, y_2 of $\mathcal{R}^n$ such that $L_1 = \mathrm{Line}(x_1, x_2)$ and $L_2 = \mathrm{Line}(y_1, y_2)$ and $x_2 - x_1 \parallel y_2 - y_1$.

Let us note that the predicate $L_1 \parallel L_2$ is symmetric.

The following proposition is true

(65) If $L_0 \parallel L_1$ and $L_1 \parallel L_2$, then $L_0 \parallel L_2$.

Let us consider n and let us consider L_1, L_2. The predicate $L_1 \perp L_2$ is defined by:

(Def. 7) There exist elements x_1, x_2, y_1, y_2 of $\mathcal{R}^n$ such that $L_1 = \mathrm{Line}(x_1, x_2)$ and $L_2 = \mathrm{Line}(y_1, y_2)$ and $x_2 - x_1 \perp y_2 - y_1$.

Let us note that the predicate $L_1 \perp L_2$ is symmetric.

We now state a number of propositions:

(66) If $L_0 \perp L_1$ and $L_1 \parallel L_2$, then $L_0 \perp L_2$.

(67) If $x \notin L$ and L is a line, then there exists L_0 such that $x \in L_0$ and $L_0 \perp L$ and L_0 meets L.

(68) If L_1 misses L_2, then there exists x such that $x \in L_1$ and $x \notin L_2$.

(69) If $x_1 \in L$ and $x_2 \in L$ and $x_1 \neq x_2$, then $\mathrm{Line}(x_1, x_2) = L$ and L is a line.

(70) If L_1 is a line and L_2 is a line and $L_1 = L_2$, then $L_1 \parallel L_2$.

(71) If $L_1 \parallel L_2$, then L_1 is a line and L_2 is a line.

(72) If $L_1 \perp L_2$, then L_1 is a line and L_2 is a line.

(73) If $x \in L$ and $a \neq 1$ and $a \cdot x \in L$, then $\langle \underbrace{0, \ldots, 0}_{n} \rangle \in L$.

(74) If $x_1 \in L$ and $x_2 \in L$, then there exists x_3 such that $x_3 \in L$ and $x_3 - x_1 = a \cdot (x_2 - x_1)$.

(75) If $x_1 \in L$ and $x_2 \in L$ and $x_3 \in L$ and $x_1 \neq x_2$, then there exists a such that $x_3 - x_1 = a \cdot (x_2 - x_1)$.

(76) If $L_1 \parallel L_2$ and $L_1 \neq L_2$, then L_1 misses L_2.

(77) If $L_1 \parallel L_2$, then $L_1 = L_2$ or L_1 misses L_2.

(78) If $L_1 \parallel L_2$ and L_1 meets L_2, then $L_1 = L_2$.

(79) If L is a line, then there exists L_0 such that $x \in L_0$ and $L_0 \parallel L$.

(80) For all x, L such that $x \notin L$ and L is a line there exists L_0 such that $x \in L_0$ and $L_0 \parallel L$ and $L_0 \neq L$.

(81) For all x_0, x_1, y_0, y_1, L_1, L_2 such that $x_0 \in L_1$ and $x_1 \in L_1$ and $x_0 \neq x_1$ and $y_0 \in L_2$ and $y_1 \in L_2$ and $y_0 \neq y_1$ and $L_1 \perp L_2$ holds $x_1 - x_0 \perp y_1 - y_0$.

(82) For all L_1, L_2 such that $L_1 \perp L_2$ holds $L_1 \neq L_2$.

(83) For all x_1, x_2, L such that L is a line and $L = \mathrm{Line}(x_1, x_2)$ holds $x_1 \neq x_2$.

(84) If $x_0 \in L_1$ and $x_1 \in L_1$ and $x_0 \neq x_1$ and $y_0 \in L_2$ and $y_1 \in L_2$ and $y_0 \neq y_1$ and $L_1 \parallel L_2$, then $x_1 - x_0 \parallel y_1 - y_0$.

(85) Suppose $x_2 - x_1$ and $x_3 - x_1$ are linearly independent and $y_2 \in \mathrm{Line}(x_1, x_2)$ and $y_3 \in \mathrm{Line}(x_1, x_3)$ and $L_1 = \mathrm{Line}(x_2, x_3)$ and $L_2 = \mathrm{Line}(y_2, y_3)$. Then $L_1 \parallel L_2$ if and only if there exists a such that $a \neq 0$ and $y_2 - x_1 = a \cdot (x_2 - x_1)$ and $y_3 - x_1 = a \cdot (x_3 - x_1)$.

(86) For all L_1, L_2 such that L_1 is a line and L_2 is a line and $L_1 \neq L_2$ there exists x such that $x \in L_1$ and $x \notin L_2$.

(87) For all x, L_1, L_2 such that $L_1 \perp L_2$ and $x \in L_2$ there exists L_0 such that $x \in L_0$ and $L_0 \perp L_2$ and $L_0 \parallel L_1$.

(88) For all x, L_1, L_2 such that $x \in L_1$ and $x \in L_2$ and $L_1 \perp L_2$ there exists x_0 such that $x \neq x_0$ and $x_0 \in L_1$ and $x_0 \notin L_2$.

Let n be a natural number and let x_1, x_2, x_3 be elements of $\mathcal{R}^n$. The functor $\mathrm{Plane}(x_1, x_2, x_3)$ yielding a subset of $\mathcal{R}^n$ is defined as follows:

(Def. 8) $\mathrm{Plane}(x_1, x_2, x_3) = \{a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3 : a_1 + a_2 + a_3 = 1\}$.

Let n be a natural number and let x_1, x_2, x_3 be elements of $\mathcal{R}^n$. One can check that $\mathrm{Plane}(x_1, x_2, x_3)$ is non empty.

Let us consider n and let A be a subset of $\mathcal{R}^n$. We say that A is plane if and only if:

(Def. 9) There exist x_1, x_2, x_3 such that $x_2 - x_1$ and $x_3 - x_1$ are linearly independent and $A = \mathrm{Plane}(x_1, x_2, x_3)$.

One can prove the following propositions:

(89) $x_1 \in \mathrm{Plane}(x_1, x_2, x_3)$ and $x_2 \in \mathrm{Plane}(x_1, x_2, x_3)$ and $x_3 \in \mathrm{Plane}(x_1, x_2, x_3)$.

(90) If $x_1 \in \mathrm{Plane}(y_1, y_2, y_3)$ and $x_2 \in \mathrm{Plane}(y_1, y_2, y_3)$ and $x_3 \in \mathrm{Plane}(y_1, y_2, y_3)$, then $\mathrm{Plane}(x_1, x_2, x_3) \subseteq \mathrm{Plane}(y_1, y_2, y_3)$.

(91) Let A be a subset of $\mathcal{R}^n$ and given x, x_1, x_2, x_3. Suppose $x \in \mathrm{Plane}(x_1, x_2, x_3)$ and there exist real numbers c_1, c_2, c_3 such that $c_1 + c_2 + c_3 = 0$ and $x = c_1 \cdot x_1 + c_2 \cdot x_2 + c_3 \cdot x_3$. Then $\langle \underbrace{0, \ldots, 0}_{n} \rangle \in \mathrm{Plane}(x_1, x_2, x_3)$.

(92) If $y_1 \in \mathrm{Plane}(x_1, x_2, x_3)$ and $y_2 \in \mathrm{Plane}(x_1, x_2, x_3)$, then $\mathrm{Line}(y_1, y_2) \subseteq \mathrm{Plane}(x_1, x_2, x_3)$.

(93) For every subset A of $\mathcal{R}^n$ and for every x such that A is plane and $x \in A$ and there exists a such that $a \neq 1$ and $a \cdot x \in A$ holds $\langle\underbrace{0,\ldots,0}_{n}\rangle \in A$.

(94) If $x_1 - x_1$ and $x_3 - x_1$ are linearly independent and $x \in \mathrm{Plane}(x_1, x_2, x_3)$ and $x = a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3$, then $a_1 + a_2 + a_3 = 1$ or $\langle\underbrace{0,\ldots,0}_{n}\rangle \in \mathrm{Plane}(x_1, x_2, x_3)$.

(95) $x \in \mathrm{Plane}(x_1, x_2, x_3)$ iff there exist a_1, a_2, a_3 such that $a_1 + a_2 + a_3 = 1$ and $x = a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3$.

(96) Suppose that

(i) $x_2 - x_1$ and $x_3 - x_1$ are linearly independent,

(ii) $x \in \mathrm{Plane}(x_1, x_2, x_3)$,

(iii) $a_1 + a_2 + a_3 = 1$,

(iv) $x = a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3$,

(v) $b_1 + b_2 + b_3 = 1$, and

(vi) $x = b_1 \cdot x_1 + b_2 \cdot x_2 + b_3 \cdot x_3$.

Then $a_1 = b_1$ and $a_2 = b_2$ and $a_3 = b_3$.

Let us consider n. The functor $\mathrm{Planes}(\mathcal{R}^n)$ yielding a family of subsets of $\mathcal{R}^n$ is defined by:

(Def. 10) $\mathrm{Planes}(\mathcal{R}^n) = \{\mathrm{Plane}(x_1, x_2, x_3)\}$.

Let us consider n. Note that $\mathrm{Planes}(\mathcal{R}^n)$ is non empty.

The following proposition is true

(97) $\mathrm{Plane}(x_1, x_2, x_3) \in \mathrm{Planes}(\mathcal{R}^n)$.

In the sequel P, P_0, P_1, P_2 are elements of $\mathrm{Planes}(\mathcal{R}^n)$.

Next we state several propositions:

(98) If $x_1 \in P$ and $x_2 \in P$ and $x_3 \in P$, then $\mathrm{Plane}(x_1, x_2, x_3) \subseteq P$.

(99) If $x_1 \in P$ and $x_2 \in P$ and $x_3 \in P$ and $x_2 - x_1$ and $x_3 - x_1$ are linearly independent, then $P = \mathrm{Plane}(x_1, x_2, x_3)$.

(100) If P_1 is plane and $P_1 \subseteq P_2$, then $P_1 = P_2$.

(101) $\mathrm{Line}(x_1, x_2) \subseteq \mathrm{Plane}(x_1, x_2, x_3)$ and $\mathrm{Line}(x_2, x_3) \subseteq \mathrm{Plane}(x_1, x_2, x_3)$ and $\mathrm{Line}(x_3, x_1) \subseteq \mathrm{Plane}(x_1, x_2, x_3)$.

(102) If $x_1 \in P$ and $x_2 \in P$, then $\mathrm{Line}(x_1, x_2) \subseteq P$.

Let n be a natural number and let L_1, L_2 be elements of $\mathrm{Lines}(\mathcal{R}^n)$. We say that L_1 and L_2 are coplanar if and only if:

(Def. 11) There exist elements x_1, x_2, x_3 of $\mathcal{R}^n$ such that $L_1 \subseteq \mathrm{Plane}(x_1, x_2, x_3)$ and $L_2 \subseteq \mathrm{Plane}(x_1, x_2, x_3)$.

We now state a number of propositions:

(103) L_1 and L_2 are coplanar iff there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$.

(104) If $L_1 \parallel L_2$, then L_1 and L_2 are coplanar.

(105) Suppose L_1 is a line and L_2 is a line and L_1 and L_2 are coplanar and L_1 misses L_2. Then there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$ and P is plane.

(106) There exists P such that $x \in P$ and $L \subseteq P$.

(107) If $x \notin L$ and L is a line, then there exists P such that $x \in P$ and $L \subseteq P$ and P is plane.

(108) If $x \in P$ and $L \subseteq P$ and $x \notin L$ and L is a line, then P is plane.

(109) If $x \notin L$ and L is a line and $x \in P_0$ and $L \subseteq P_0$ and $x \in P_1$ and $L \subseteq P_1$, then $P_0 = P_1$.

(110) If L_1 is a line and L_2 is a line and L_1 and L_2 are coplanar and $L_1 \neq L_2$, then there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$ and P is plane.

(111) For all L_1, L_2 such that L_1 is a line and L_2 is a line and $L_1 \neq L_2$ and L_1 meets L_2 there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$ and P is plane.

(112) If L_1 is a line and L_2 is a line and $L_1 \neq L_2$ and L_1 meets L_2 and $L_1 \subseteq P_1$ and $L_2 \subseteq P_1$ and $L_1 \subseteq P_2$ and $L_2 \subseteq P_2$, then $P_1 = P_2$.

(113) If $L_1 \parallel L_2$ and $L_1 \neq L_2$, then there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$ and P is plane.

(114) If $L_1 \perp L_2$ and L_1 meets L_2, then there exists P such that P is plane and $L_1 \subseteq P$ and $L_2 \subseteq P$.

(115) If $L_0 \subseteq P$ and $L_1 \subseteq P$ and $L_2 \subseteq P$ and $x \in L_0$ and $x \in L_1$ and $x \in L_2$ and $L_0 \perp L_2$ and $L_1 \perp L_2$, then $L_0 = L_1$.

(116) If L_1 and L_2 are coplanar and $L_1 \perp L_2$, then L_1 meets L_2.

(117) If $L_1 \subseteq P$ and $L_2 \subseteq P$ and $L_1 \perp L_2$ and $x \in P$ and $L_0 \parallel L_2$ and $x \in L_0$, then $L_0 \subseteq P$ and $L_0 \perp L_1$.

(118) If $L \subseteq P$ and $L_1 \subseteq P$ and $L_2 \subseteq P$ and $L \perp L_1$ and $L \perp L_2$, then $L_1 \parallel L_2$.

(119) Suppose $L_0 \subseteq P$ and $L_1 \subseteq P$ and $L_2 \subseteq P$ and $L_0 \parallel L_1$ and $L_1 \parallel L_2$ and $L_0 \neq L_1$ and $L_1 \neq L_2$ and $L_2 \neq L_0$ and L meets L_0 and L meets L_1. Then L meets L_2.

(120) If L_1 and L_2 are coplanar and L_1 is a line and L_2 is a line and L_1 misses L_2, then $L_1 \parallel L_2$.

(121) If $x_1 \in P$ and $x_2 \in P$ and $y_1 \in P$ and $y_2 \in P$ and $x_2 - x_1$ and $y_2 - y_1$ are linearly independent, then $\mathrm{Line}(x_1, x_2)$ meets $\mathrm{Line}(y_1, y_2)$.

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