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Homeomorphism between Finite Topological Spaces, Two-Dimensional Lattice Spaces and a Fixed Point Theorem

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Summary. In this paper we first introduced the notion of homeomorphism between finite topological spaces. We also gave a fixed point theorem in finite topological space. Next, we showed two 2-dimensional concrete models of lattice spaces. One was 2-dimensional linear finite topological space. Another was 2-dimensional small finite topological space.

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The articles [10], [6], [12], [1], [13], [4], [5], [2], [7], [9], [8], [3], and [11] provide the notation and terminology for this paper.

The following propositions are true:

(1) Let X be a set, Y be a non empty set, f be a function from X into Y, and A be a subset of X. If f is one-to-one, then $(f^{-1})^\circ f^\circ A = A$.

(2) For every natural number n holds $n > 0$ iff $\mathrm{Seg}\, n \neq \emptyset$.

Let F_1, F_2 be finite topology spaces and let h be a map from F_1 into F_2. We say that h is a homeomorphism if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) h is one-to-one and onto, and

(ii) for every element x of F_1 holds h°(the neighbour-map of F_1)(x) = (the neighbour-map of F_2)($h(x)$).

One can prove the following propositions:

(3) Let F_1, F_2 be non empty finite topology spaces and h be a map from F_1 into F_2. Suppose h is a homeomorphism. Then there exists a map g from F_2 into F_1 such that $g = h^{-1}$ and g is a homeomorphism.

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(4) Let F_1, F_2 be non empty finite topology spaces, h be a map from F_1 into F_2, n be a natural number, x be an element of F_1, and y be an element of F_2. Suppose h is a homeomorphism and $y = h(x)$. Let z be an element of F_1. Then $z \in U(x, n)$ if and only if $h(z) \in U(y, n)$.

(5) Let F_1, F_2 be non empty finite topology spaces, h be a map from F_1 into F_2, n be a natural number, x be an element of F_1, and y be an element of F_2. Suppose h is a homeomorphism and $y = h(x)$. Let v be an element of F_2. Then $h^{-1}(v) \in U(x, n)$ if and only if $v \in U(y, n)$.

(6) Let n be a non zero natural number and f be a map from $\mathrm{FTSL1}(n)$ into $\mathrm{FTSL1}(n)$. If f is continuous 0, then there exists an element p of $\mathrm{FTSL1}(n)$ such that $f(p) \in U(p, 0)$.

(7) Let T be a non empty finite topology space, p be an element of T, and k be a natural number. If T is filled, then $U(p, k) \subseteq U(p, k + 1)$.

(8) Let T be a non empty finite topology space, p be an element of T, and k be a natural number. If T is filled, then $U(p, 0) \subseteq U(p, k)$.

(9) Let n be a non zero natural number, j_1, j, k be natural numbers, and p be an element of $\mathrm{FTSL1}(n)$. If $p = j_1$, then $j \in U(p, k)$ iff $j \in \mathrm{Seg}\, n$ and $|j_1 - j| \leq k + 1$.

(10) Let k_1, k_2 be natural numbers, n be a non zero natural number, and f be a map from $\mathrm{FTSL1}(n)$ into $\mathrm{FTSL1}(n)$. Suppose f is continuous k_1 and $k_2 = \lceil \frac{k_1}{2} \rceil$. Then there exists an element p of $\mathrm{FTSL1}(n)$ such that $f(p) \in U(p, k_2)$.

Let n, m be natural numbers. The functor $\mathrm{Nbdl2}(n, m)$ yields a function from $[\!:\mathrm{Seg}\, n, \mathrm{Seg}\, m:\!]$ into $2^{[\!:\mathrm{Seg}\, n, \mathrm{Seg}\, m:\!]}$ and is defined by:

(Def. 2) For every set x such that $x \in [\!:\mathrm{Seg}\, n, \mathrm{Seg}\, m:\!]$ and for all natural numbers i, j such that $x = \langle i, j \rangle$ holds $(\mathrm{Nbdl2}(n, m))(x) = [\!:(\mathrm{Nbdl1}(n))(i), (\mathrm{Nbdl1}(m))(j):\!]$.

Let n, m be natural numbers. The functor $\mathrm{FTSL2}(n, m)$ yielding a strict finite topology space is defined as follows:

(Def. 3) $\mathrm{FTSL2}(n, m) = \langle [\!:\mathrm{Seg}\, n, \mathrm{Seg}\, m:\!], \mathrm{Nbdl2}(n, m) \rangle$.

Let n, m be non zero natural numbers. One can verify that $\mathrm{FTSL2}(n, m)$ is non empty.

We now state three propositions:

(11) For all non zero natural numbers n, m holds $\mathrm{FTSL2}(n, m)$ is filled.

(12) For all non zero natural numbers n, m holds $\mathrm{FTSL2}(n, m)$ is symmetric.

(13) For every non zero natural number n holds there exists a map from $\mathrm{FTSL2}(n, 1)$ into $\mathrm{FTSL1}(n)$ which is a homeomorphism.

Let n, m be natural numbers. The functor $\mathrm{Nbds2}(n, m)$ yielding a function from $[\!:\mathrm{Seg}\, n, \mathrm{Seg}\, m:\!]$ into $2^{[\!:\mathrm{Seg}\, n, \mathrm{Seg}\, m:\!]}$ is defined by:

(Def. 4) For every set x such that $x \in [: \operatorname{Seg} n, \operatorname{Seg} m :]$ and for all natural numbers i, j such that $x = \langle i, j \rangle$ holds $(\mathrm{Nbds2}(n, m))(x) = [: \{i\}, (\mathrm{Nbdl1}(m))(j) :] \cup [: (\mathrm{Nbdl1}(n))(i), \{j\} :]$.

Let n, m be natural numbers. The functor $\mathrm{FTSS2}(n, m)$ yielding a strict finite topology space is defined as follows:

(Def. 5) $\mathrm{FTSS2}(n, m) = \langle [: \operatorname{Seg} n, \operatorname{Seg} m :], \mathrm{Nbds2}(n, m) \rangle$.

Let n, m be non zero natural numbers. Note that $\mathrm{FTSS2}(n, m)$ is non empty.

One can prove the following propositions:

(14) For all non zero natural numbers n, m holds $\mathrm{FTSS2}(n, m)$ is filled.

(15) For all non zero natural numbers n, m holds $\mathrm{FTSS2}(n, m)$ is symmetric.

(16) For every non zero natural number n holds there exists a map from $\mathrm{FTSS2}(n, 1)$ into $\mathrm{FTSL1}(n)$ which is a homeomorphism.

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