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Formulas and Identities of Inverse Hyperbolic Functions

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Summary. This article describes definitions of inverse hyperbolic functions and their main properties, as well as some addition formulas with hyperbolic functions.

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The papers [1], [8], [4], [2], [9], [3], [6], [5], and [7] provide the terminology and notation for this paper.

1. Preliminaries

In this paper x, y, t denote real numbers.

Next we state a number of propositions:

(1) If $x > 0$, then $\frac{1}{x} = x^{-1}$.

(2) If $x > 1$, then $(\frac{\sqrt{x^{\mathbf{2}}-1}}{x})^{\mathbf{2}} < 1$.

(3) $(\frac{x}{\sqrt{x^{\mathbf{2}}+1}})^{\mathbf{2}} < 1$.

(4) $\sqrt{x^{\mathbf{2}}+1} > 0$.

(5) $\sqrt{x^{\mathbf{2}}+1} + x > 0$.

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(6) If $y \geq 0$ and $x \geq 1$, then $\frac{x+1}{y} \geq 0$.
(7) If $y \geq 0$ and $x \geq 1$, then $\frac{x-1}{y} \geq 0$.
(8) If $x \geq 1$, then $\sqrt{\frac{x+1}{2}} \geq 1$.
(9) If $y \geq 0$ and $x \geq 1$, then $\frac{x^{\mathbf{2}}-1}{y} \geq 0$.
(10) If $x \geq 1$, then $\sqrt{\frac{x+1}{2}} + \sqrt{\frac{x-1}{2}} > 0$.
(11) If $x^{\mathbf{2}} < 1$, then $x + 1 > 0$ and $1 - x > 0$.
(12) If $x \neq 1$, then $(1 - x)^{\mathbf{2}} > 0$.
(13) If $x^{\mathbf{2}} < 1$, then $\frac{x^{\mathbf{2}}+1}{1-x^{\mathbf{2}}} \geq 0$.
(14) If $x^{\mathbf{2}} < 1$, then $(\frac{2 \cdot x}{1+x^{\mathbf{2}}})^{\mathbf{2}} < 1$.
(15) If $0 < x$ and $x < 1$, then $\frac{1+x}{1-x} > 0$.
(16) If $0 < x$ and $x < 1$, then $x^{\mathbf{2}} < 1$.
(17) If $0 < x$ and $x < 1$, then $\frac{1}{\sqrt{1-x^{\mathbf{2}}}} > 1$.
(18) If $0 < x$ and $x < 1$, then $\frac{2 \cdot x}{1-x^{\mathbf{2}}} > 0$.
(19) If $0 < x$ and $x < 1$, then $0 < (1 - x^{\mathbf{2}})^{\mathbf{2}}$.
(20) If $0 < x$ and $x < 1$, then $\frac{1+x^{\mathbf{2}}}{1-x^{\mathbf{2}}} > 1$.
(21) If $1 < x^{\mathbf{2}}$, then $(\frac{1}{x})^{\mathbf{2}} < 1$.
(22) If $0 < x$ and $x \leq 1$, then $1 - x^{\mathbf{2}} \geq 0$.
(23) If $1 \leq x$, then $0 < x + \sqrt{x^{\mathbf{2}} - 1}$.
(24) If $1 \leq x$ and $1 \leq y$, then $0 \leq x \cdot \sqrt{y^{\mathbf{2}} - 1} + y \cdot \sqrt{x^{\mathbf{2}} - 1}$.
(25) If $1 \leq x$ and $1 \leq y$ and $|y| \leq |x|$, then $0 < y - \sqrt{y^{\mathbf{2}} - 1}$.
(26) If $1 \leq x$ and $1 \leq y$ and $|y| \leq |x|$, then $0 \leq y \cdot \sqrt{x^{\mathbf{2}} - 1} - x \cdot \sqrt{y^{\mathbf{2}} - 1}$.
(27) If $x^{\mathbf{2}} < 1$ and $y^{\mathbf{2}} < 1$, then $x \cdot y \neq -1$.
(28) If $x^{\mathbf{2}} < 1$ and $y^{\mathbf{2}} < 1$, then $x \cdot y \neq 1$.
(29) If $x \neq 0$, then $\exp x \neq 1$.
(30) If $0 \neq x$, then $(\exp x)^{\mathbf{2}} - 1 \neq 0$.
(31) If $0 < t$, then $\frac{t^{\mathbf{2}}-1}{t^{\mathbf{2}}+1} < 1$.
(32) If $-1 < t$ and $t < 1$, then $0 < \frac{t+1}{1-t}$.

2. Formulas and Identities of Inverse Hyperbolic Functions

Let x be a real number. The functor $\sinh' x$ yields a real number and is defined by:

(Def. 1) $\sinh' x = \log_e(x + \sqrt{x^{\mathbf{2}} + 1})$.

Let x be a real number. The functor $\cosh'_1 x$ yielding a real number is defined by:

(Def. 2) $\cosh_1' x = \log_e(x + \sqrt{x^2 - 1})$.

Let x be a real number. The functor $\cosh_2' x$ yields a real number and is defined by:

(Def. 3) $\cosh_2' x = -\log_e(x + \sqrt{x^2 - 1})$.

Let x be a real number. The functor $\tanh' x$ yields a real number and is defined by:

(Def. 4) $\tanh' x = \frac{1}{2} \cdot \log_e(\frac{1+x}{1-x})$.

Let x be a real number. The functor $\coth' x$ yielding a real number is defined as follows:

(Def. 5) $\coth' x = \frac{1}{2} \cdot \log_e(\frac{x+1}{x-1})$.

Let x be a real number. The functor $\mathrm{sech}_1' x$ yields a real number and is defined by:

(Def. 6) $\mathrm{sech}_1' x = \log_e(\frac{1+\sqrt{1-x^2}}{x})$.

Let x be a real number. The functor $\mathrm{sech}_2' x$ yielding a real number is defined as follows:

(Def. 7) $\mathrm{sech}_2' x = -\log_e(\frac{1+\sqrt{1-x^2}}{x})$.

Let x be a real number. The functor $\mathrm{csch}' x$ yielding a real number is defined by:

(Def. 8)(i) $\mathrm{csch}' x = \log_e(\frac{1+\sqrt{1+x^2}}{x})$ if $0 < x$,

(ii) $\mathrm{csch}' x = \log_e(\frac{1-\sqrt{1+x^2}}{x})$ if $x < 0$,

(iii) $x < 0$, otherwise.

The following propositions are true:

(33) If $0 \leq x$, then $\sinh' x = \cosh_1' \sqrt{x^2 + 1}$.

(34) If $x < 0$, then $\sinh' x = \cosh_2' \sqrt{x^2 + 1}$.

(35) $\sinh' x = \tanh'(\frac{x}{\sqrt{x^2+1}})$.

(36) If $x \geq 1$, then $\cosh_1' x = \sinh' \sqrt{x^2 - 1}$.

(37) If $x > 1$, then $\cosh_1' x = \tanh'(\frac{\sqrt{x^2-1}}{x})$.

(38) If $x \geq 1$, then $\cosh_1' x = 2 \cdot \cosh_1' \sqrt{\frac{x+1}{2}}$.

(39) If $x \geq 1$, then $\cosh_2' x = 2 \cdot \cosh_2' \sqrt{\frac{x+1}{2}}$.

(40) If $x \geq 1$, then $\cosh_1' x = 2 \cdot \sinh' \sqrt{\frac{x-1}{2}}$.

(41) If $x^2 < 1$, then $\tanh' x = \sinh'(\frac{x}{\sqrt{1-x^2}})$.

(42) If $0 < x$ and $x < 1$, then $\tanh' x = \cosh_1'(\frac{1}{\sqrt{1-x^2}})$.

(43) If $x^2 < 1$, then $\tanh' x = \frac{1}{2} \cdot \sinh'(\frac{2 \cdot x}{1-x^2})$.

(44) If $x > 0$ and $x < 1$, then $\tanh' x = \frac{1}{2} \cdot \cosh_1'(\frac{1+x^2}{1-x^2})$.

(45) If $x^2 < 1$, then $\tanh' x = \frac{1}{2} \cdot \tanh'(\frac{2 \cdot x}{1+x^2})$.

(46) If $x^{\mathbf{2}} > 1$, then $\coth' x = \tanh'(\frac{1}{x})$.

(47) If $x > 0$ and $x \leq 1$, then $\mathrm{sech}'_1 x = \cosh'_1(\frac{1}{x})$.

(48) If $x > 0$ and $x \leq 1$, then $\mathrm{sech}'_2 x = \cosh'_2(\frac{1}{x})$.

(49) If $x > 0$, then $\mathrm{csch}' x = \sinh'(\frac{1}{x})$.

(50) If $x \cdot y + \sqrt{x^{\mathbf{2}} + 1} \cdot \sqrt{y^{\mathbf{2}} + 1} \geq 0$, then $\sinh' x + \sinh' y = \sinh'(x \cdot \sqrt{1 + y^{\mathbf{2}}} + y \cdot \sqrt{1 + x^{\mathbf{2}}})$.

(51) $\sinh' x - \sinh' y = \sinh'(x \cdot \sqrt{1 + y^{\mathbf{2}}} - y \cdot \sqrt{1 + x^{\mathbf{2}}})$.

(52) If $1 \leq x$ and $1 \leq y$, then $\cosh'_1 x + \cosh'_1 y = \cosh'_1(x \cdot y + \sqrt{(x^{\mathbf{2}} - 1) \cdot (y^{\mathbf{2}} - 1)})$.

(53) If $1 \leq x$ and $1 \leq y$, then $\cosh'_2 x + \cosh'_2 y = \cosh'_2(x \cdot y + \sqrt{(x^{\mathbf{2}} - 1) \cdot (y^{\mathbf{2}} - 1)})$.

(54) If $1 \leq x$ and $1 \leq y$ and $|y| \leq |x|$, then $\cosh'_1 x - \cosh'_1 y = \cosh'_1(x \cdot y - \sqrt{(x^{\mathbf{2}} - 1) \cdot (y^{\mathbf{2}} - 1)})$.

(55) If $1 \leq x$ and $1 \leq y$ and $|y| \leq |x|$, then $\cosh'_2 x - \cosh'_2 y = \cosh'_2(x \cdot y - \sqrt{(x^{\mathbf{2}} - 1) \cdot (y^{\mathbf{2}} - 1)})$.

(56) If $x^{\mathbf{2}} < 1$ and $y^{\mathbf{2}} < 1$, then $\tanh' x + \tanh' y = \tanh'(\frac{x+y}{1+x \cdot y})$.

(57) If $x^{\mathbf{2}} < 1$ and $y^{\mathbf{2}} < 1$, then $\tanh' x - \tanh' y = \tanh'(\frac{x-y}{1-x \cdot y})$.

(58) If $0 < x$ and $(\frac{x-1}{x+1})^{\mathbf{2}} < 1$, then $\log_e x = 2 \cdot \tanh'(\frac{x-1}{x+1})$.

(59) If $0 < x$ and $(\frac{x^{\mathbf{2}}-1}{x^{\mathbf{2}}+1})^{\mathbf{2}} < 1$, then $\log_e x = \tanh'(\frac{x^{\mathbf{2}}-1}{x^{\mathbf{2}}+1})$.

(60) If $1 < x$ and $1 \leq \frac{x^{\mathbf{2}}+1}{2 \cdot x}$, then $\log_e x = \cosh'_1(\frac{x^{\mathbf{2}}+1}{2 \cdot x})$.

(61) If $0 < x$ and $x < 1$ and $1 \leq \frac{x^{\mathbf{2}}+1}{2 \cdot x}$, then $\log_e x = \cosh'_2(\frac{x^{\mathbf{2}}+1}{2 \cdot x})$.

(62) If $0 < x$, then $\log_e x = \sinh'(\frac{x^{\mathbf{2}}-1}{2 \cdot x})$.

(63) If $y = \frac{1}{2} \cdot (\exp x - \exp(-x))$, then $x = \log_e(y + \sqrt{y^{\mathbf{2}} + 1})$.

(64) If $y = \frac{1}{2} \cdot (\exp x + \exp(-x))$ and $1 \leq y$, then $x = \log_e(y + \sqrt{y^{\mathbf{2}} - 1})$ or $x = -\log_e(y + \sqrt{y^{\mathbf{2}} - 1})$.

(65) If $y = \frac{\exp x - \exp(-x)}{\exp x + \exp(-x)}$, then $x = \frac{1}{2} \cdot \log_e(\frac{1+y}{1-y})$.

(66) If $y = \frac{\exp x + \exp(-x)}{\exp x - \exp(-x)}$ and $x \neq 0$, then $x = \frac{1}{2} \cdot \log_e(\frac{y+1}{y-1})$.

(67) If $y = \frac{1}{\frac{\exp x + \exp(-x)}{2}}$, then $x = \log_e(\frac{1+\sqrt{1-y^{\mathbf{2}}}}{y})$ or $x = -\log_e(\frac{1+\sqrt{1-y^{\mathbf{2}}}}{y})$.

(68) If $y = \frac{1}{\frac{\exp x - \exp(-x)}{2}}$ and $x \neq 0$, then $x = \log_e(\frac{1+\sqrt{1+y^{\mathbf{2}}}}{y})$ or $x = \log_e(\frac{1-\sqrt{1+y^{\mathbf{2}}}}{y})$.

(69) (The function cosh)$(2 \cdot x) = 1 + 2 \cdot$ (the function sinh)$(x)^{\mathbf{2}}$.

(70) (The function cosh)$(x)^{\mathbf{2}} = 1 +$ (the function sinh)$(x)^{\mathbf{2}}$.

(71) (The function sinh)$(x)^{\mathbf{2}} =$ (the function cosh)$(x)^{\mathbf{2}} - 1$.

(72) $\sinh(5 \cdot x) = 5 \cdot \sinh x + 20 \cdot (\sinh x)^3 + 16 \cdot (\sinh x)^5$.

(73) $\cosh(5 \cdot x) = (5 \cdot \cosh x - 20 \cdot (\cosh x)^3) + 16 \cdot (\cosh x)^5$.

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