FORMALIZED MATHEMATICS
Volume 13, Number 4, Pages 463–465
University of Białystok, 2005

Linearity of Lebesgue Integral of Simple Valued Function[1]

Noboru Endou
Gifu National College of Technology
Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article the authors prove linearity of the Lebesgue integral of simple valued function.

MML identifier: `MESFUNC4`, version: `7.5.01 4.39.921`

The notation and terminology used here are introduced in the following papers: [16], [17], [1], [15], [2], [18], [7], [9], [8], [3], [4], [5], [6], [10], [11], [12], [14], and [13].

One can prove the following propositions:

(1) Let F, G, H be finite sequences of elements of $\overline{\mathbb{R}}$. Suppose that

(i) for every natural number i such that $i \in \operatorname{dom} F$ holds $0_{\overline{\mathbb{R}}} \leq F(i)$,

(ii) for every natural number i such that $i \in \operatorname{dom} G$ holds $0_{\overline{\mathbb{R}}} \leq G(i)$,

(iii) $\operatorname{dom} F = \operatorname{dom} G$, and

(iv) $H = F + G$.

Then $\sum H = \sum F + \sum G$.

(2) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, n be a natural number, f be a partial function from X to $\overline{\mathbb{R}}$, F be a finite sequence of separated subsets of S, and a, x be finite sequences of elements of $\overline{\mathbb{R}}$. Suppose that f is simple function in S and $\operatorname{dom} f \neq \emptyset$ and for every set x such that $x \in \operatorname{dom} f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$ and F and a are representation of f and $\operatorname{dom} x = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} x$ holds $x(i) = a(i) \cdot (M \cdot F)(i)$ and $\operatorname{len} F = n$. Then $\int_X f \, \mathrm{d}M = \sum x$.

[1]This work has been partially supported by the MEXT grant Grant-in-Aid for Young Scientists (B)16700156.

ISSN 1426–2630

(3) Let X be a non empty set, S be a σ-field of subsets of X, f be a partial function from X to $\overline{\mathbb{R}}$, M be a σ-measure on S, F be a finite sequence of separated subsets of S, and a, x be finite sequences of elements of $\overline{\mathbb{R}}$. Suppose that

(i) f is simple function in S,

(ii) $\mathrm{dom}\, f \neq \emptyset$,

(iii) for every set x such that $x \in \mathrm{dom}\, f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$,

(iv) F and a are representation of f,

(v) $\mathrm{dom}\, x = \mathrm{dom}\, F$, and

(vi) for every natural number n such that $n \in \mathrm{dom}\, x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$.

Then $\int_X f \, \mathrm{d}M = \sum x$.

(4) Let X be a non empty set, S be a σ-field of subsets of X, f be a partial function from X to $\overline{\mathbb{R}}$, and M be a σ-measure on S. Suppose f is simple function in S and $\mathrm{dom}\, f \neq \emptyset$ and for every set x such that $x \in \mathrm{dom}\, f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$. Then there exists a finite sequence F of separated subsets of S and there exist finite sequences a, x of elements of $\overline{\mathbb{R}}$ such that

(i) F and a are representation of f,

(ii) $\mathrm{dom}\, x = \mathrm{dom}\, F$,

(iii) for every natural number n such that $n \in \mathrm{dom}\, x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$, and

(iv) $\int_X f \, \mathrm{d}M = \sum x$.

(5) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) f is simple function in S,

(ii) $\mathrm{dom}\, f \neq \emptyset$,

(iii) for every set x such that $x \in \mathrm{dom}\, f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$,

(iv) g is simple function in S,

(v) $\mathrm{dom}\, g = \mathrm{dom}\, f$, and

(vi) for every set x such that $x \in \mathrm{dom}\, g$ holds $0_{\overline{\mathbb{R}}} \leq g(x)$.

Then

(vii) $f + g$ is simple function in S,

(viii) $\mathrm{dom}(f + g) \neq \emptyset$,

(ix) for every set x such that $x \in \mathrm{dom}(f + g)$ holds $0_{\overline{\mathbb{R}}} \leq (f + g)(x)$, and

(x) $\int_X f + g \, \mathrm{d}M = \int_X f \, \mathrm{d}M + \int_X g \, \mathrm{d}M$.

(6) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f, g be partial functions from X to $\overline{\mathbb{R}}$, and c be an extended real number. Suppose that f is simple function in S and $\mathrm{dom}\, f \neq \emptyset$ and for every set x such that $x \in \mathrm{dom}\, f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$ and $0_{\overline{\mathbb{R}}} \leq c$ and

$c < +\infty$ and $\mathrm{dom}\, g = \mathrm{dom}\, f$ and for every set x such that $x \in \mathrm{dom}\, g$ holds $g(x) = c \cdot f(x)$. Then $\int\limits_X g\, \mathrm{d}M = c \cdot \int\limits_X f\, \mathrm{d}M$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):163–171, 1991.
[5] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.
[6] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[10] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Basic properties of extended real numbers. *Formalized Mathematics*, 9(**3**):491–494, 2001.
[11] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Formalized Mathematics*, 9(**3**):495–500, 2001.
[12] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. The measurability of extended real valued functions. *Formalized Mathematics*, 9(**3**):525–529, 2001.
[13] Grigory E. Ivanov. Definition of convex function and Jensen's inequality. *Formalized Mathematics*, 11(**4**):349–354, 2003.
[14] Yasunari Shidama and Noboru Endou. Lebesgue integral of simple valued function. *Formalized Mathematics*, 13(**1**):67–71, 2005.
[15] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received September 14, 2005