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Set Sequences and Monotone Class

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Summary. In this paper we first defined the partial-union sequence, the partial-intersection sequence, and the partial-difference-union sequence of given sequence of subsets, and then proved the additive theorem of infinite sequences and sub-additive theorem of finite sequences for probability. Further, we defined the monotone class of families of subsets, and discussed the relations between the monotone class and the σ-field which are generated by the field of subsets of a given set.

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The articles [4], [3], [2], [20], [23], [19], [9], [21], [22], [18], [16], [6], [1], [13], [11], [24], [7], [8], [15], [14], [10], [12], [26], [25], [17], and [5] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: n, m, k are natural numbers, g is a real number, x, X, Y, Z are sets, A_1 is a sequence of subsets of X, F_1 is a finite sequence of elements of 2^X, R_1 is a finite sequence of elements of $\mathbb{R}$, S_1 is a σ-field of subsets of X, O_1 is a non empty set, S_2 is a σ-field of subsets of O_1, A_2, B_1 are sequences of subsets of S_2, and P is a probability on S_2.

One can prove the following propositions:

(1) For every finite sequence f holds $0 \notin \operatorname{dom} f$.

(2) For every finite sequence f holds $n \in \operatorname{dom} f$ iff $n \neq 0$ and $n \leq \operatorname{len} f$.

(3) Let f be a sequence of real numbers. Given k such that let given n. If $k \leq n$, then $f(n) = g$. Then f is convergent and $\lim f = g$.

(4) $(P \cdot A_2)(n) \geq 0$.

(5) If $A_2(n) \subseteq B_1(n)$, then $(P \cdot A_2)(n) \leq (P \cdot B_1)(n)$.

(6) If A_2 is non-decreasing, then $P \cdot A_2$ is non-decreasing.

(7) If A_2 is non-increasing, then $P \cdot A_2$ is non-increasing.

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Let A_1 be a function. The partial intersections of A_1 constitute a function defined by the conditions (Def. 1).

(Def. 1)(i) dom (the partial intersections of A_1) $= \mathbb{N}$,

(ii) (the partial intersections of A_1)$(0) = A_1(0)$, and

(iii) for every natural number n holds (the partial intersections of A_1)$(n + 1) =$ (the partial intersections of A_1)$(n) \cap A_1(n + 1)$.

Let X be a set and let A_1 be a sequence of subsets of X. Then the partial intersections of A_1 is a sequence of subsets of X.

Let A_1 be a function. The partial unions of A_1 constitute a function defined by the conditions (Def. 2).

(Def. 2)(i) dom (the partial unions of A_1) $= \mathbb{N}$,

(ii) (the partial unions of A_1)$(0) = A_1(0)$, and

(iii) for every natural number n holds (the partial unions of A_1)$(n+1) =$ (the partial unions of A_1)$(n) \cup A_1(n + 1)$.

Let X be a set and let A_1 be a sequence of subsets of X. Then the partial unions of A_1 is a sequence of subsets of X.

The following propositions are true:

(8) (The partial intersections of A_1)$(n) \subseteq A_1(n)$.

(9) $A_1(n) \subseteq$ (the partial unions of A_1)(n).

(10) The partial intersections of A_1 are non-increasing.

(11) The partial unions of A_1 are non-decreasing.

(12) $x \in$ (the partial intersections of A_1)(n) iff for every k such that $k \leq n$ holds $x \in A_1(k)$.

(13) $x \in$ (the partial unions of A_1)(n) iff there exists k such that $k \leq n$ and $x \in A_1(k)$.

(14) Intersection (the partial intersections of A_1) $=$ Intersection A_1.

(15) $\bigcup$(the partial unions of A_1) $= \bigcup A_1$.

Let A_1 be a function. The partial diff-unions of A_1 constitute a function defined by the conditions (Def. 3).

(Def. 3)(i) dom (the partial diff-unions of A_1) $= \mathbb{N}$,

(ii) (the partial diff-unions of A_1)$(0) = A_1(0)$, and

(iii) for every natural number n holds (the partial diff-unions of A_1)$(n+1) =$ $A_1(n + 1) \setminus$ (the partial unions of A_1)(n).

Let X be a set and let A_1 be a sequence of subsets of X. Then the partial diff-unions of A_1 is a sequence of subsets of X.

One can prove the following propositions:

(16) $x \in$ (the partial diff-unions of A_1)(n) iff $x \in A_1(n)$ and for every k such that $k < n$ holds $x \notin A_1(k)$.

(17) (The partial diff-unions of A_1)$(n) \subseteq A_1(n)$.

(18) (The partial diff-unions of A_1)$(n) \subseteq$ (the partial unions of A_1)(n).

(19) The partial unions of the partial diff-unions of A_1 = the partial unions of A_1.

(20) $\bigcup$(the partial diff-unions of A_1) $= \bigcup A_1$.

Let us consider X, A_1. Let us observe that A_1 is disjoint valued if and only if:

(Def. 4) For all m, n such that $m \neq n$ holds $A_1(m)$ misses $A_1(n)$.

We now state the proposition

(21) The partial diff-unions of A_1 are disjoint valued.

Let X be a set, let S_1 be a σ-field of subsets of X, and let X_1 be a sequence of subsets of S_1. Then the partial intersections of X_1 is a sequence of subsets of S_1.

Let X be a set, let S_1 be a σ-field of subsets of X, and let X_1 be a sequence of subsets of S_1. Then the partial unions of X_1 is a sequence of subsets of S_1.

Let X be a set, let S_1 be a σ-field of subsets of X, and let X_1 be a sequence of subsets of S_1. Then the partial diff-unions of X_1 is a sequence of subsets of S_1.

Next we state a number of propositions:

(22) $P \cdot$ the partial unions of A_2 is non-decreasing.

(23) $P \cdot$ the partial intersections of A_2 is non-increasing.

(24) $(\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing.

(25) $(P \cdot$ the partial unions of $A_2)(0) = (\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa\in\mathbb{N}}(0)$.

(26)(i) $P \cdot$ the partial unions of A_2 is convergent,

(ii) $\lim(P \cdot$ the partial unions of $A_2) = \sup(P \cdot$ the partial unions of $A_2)$, and

(iii) $\lim(P \cdot$ the partial unions of $A_2) = P(\bigcup A_2)$.

(27) If A_2 is disjoint valued, then for all n, m such that $n < m$ holds (the partial unions of $A_2)(n)$ misses $A_2(m)$.

(28) If A_2 is disjoint valued, then $(P \cdot$the partial unions of $A_2)(n) = (\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(29) If A_2 is disjoint valued, then $P \cdot$ the partial unions of $A_2 = (\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa\in\mathbb{N}}$.

(30) If A_2 is disjoint valued, then $(\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent and $\lim((\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa\in\mathbb{N}}) = \sup((\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa\in\mathbb{N}})$ and $\lim((\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa\in\mathbb{N}}) = P(\bigcup A_2)$.

(31) If A_2 is disjoint valued, then $P(\bigcup A_2) = \sum(P \cdot A_2)$.

Let us consider X, F_1, n. Then $F_1(n)$ is a subset of X.

One can prove the following two propositions:

(32) There exists a finite sequence F_1 of elements of 2^X such that for every k such that $k \in \operatorname{dom} F_1$ holds $F_1(k) = X$.

(33) For every finite sequence F_1 of elements of 2^X holds $\bigcup \operatorname{rng} F_1$ is a subset of X.

Let X be a set and let F_1 be a finite sequence of elements of 2^X. Then $\bigcup F_1$ is a subset of X.

We now state the proposition

(34) $x \in \bigcup F_1$ iff there exists k such that $k \in \operatorname{dom} F_1$ and $x \in F_1(k)$.

Let us consider X, F_1. The functor Complement F_1 yields a finite sequence of elements of 2^X and is defined by:

(Def. 5) $\operatorname{len} \text{Complement } F_1 = \operatorname{len} F_1$ and for every n such that $n \in \operatorname{dom} \text{Complement } F_1$ holds $(\text{Complement } F_1)(n) = F_1(n)^{\mathrm{c}}$.

Let us consider X, F_1. The functor Intersection F_1 yields a subset of X and is defined by:

(Def. 6) $$\text{Intersection } F_1 = \begin{cases} (\bigcup \text{Complement } F_1)^{\mathrm{c}}, & \text{if } F_1 \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Next we state several propositions:

(35) $\operatorname{dom} \text{Complement } F_1 = \operatorname{dom} F_1$.

(36) If $F_1 \neq \emptyset$, then $x \in \text{Intersection } F_1$ iff for every k such that $k \in \operatorname{dom} F_1$ holds $x \in F_1(k)$.

(37) If $F_1 \neq \emptyset$, then $x \in \bigcap \operatorname{rng} F_1$ iff for every n such that $n \in \operatorname{dom} F_1$ holds $x \in F_1(n)$.

(38) $\text{Intersection } F_1 = \bigcap \operatorname{rng} F_1$.

(39) Let F_1 be a finite sequence of elements of 2^X. Then there exists a sequence A_1 of subsets of X such that for every k such that $k \in \operatorname{dom} F_1$ holds $A_1(k) = F_1(k)$ and for every k such that $k \notin \operatorname{dom} F_1$ holds $A_1(k) = \emptyset$.

(40) Let F_1 be a finite sequence of elements of 2^X and A_1 be a sequence of subsets of X. Suppose for every k such that $k \in \operatorname{dom} F_1$ holds $A_1(k) = F_1(k)$ and for every k such that $k \notin \operatorname{dom} F_1$ holds $A_1(k) = \emptyset$. Then $A_1(0) = \emptyset$ and $\bigcup A_1 = \bigcup F_1$.

Let X be a set and let S_1 be a σ-field of subsets of X. A finite sequence of elements of 2^X is said to be a finite sequence of elements of S_1 if:

(Def. 7) For every k such that $k \in \operatorname{dom} \text{it}$ holds $\text{it}(k) \in S_1$.

Let X be a set, let S_1 be a σ-field of subsets of X, let F_2 be a finite sequence of elements of S_1, and let us consider n. Then $F_2(n)$ is an event of S_1.

We now state two propositions:

(41) Let F_2 be a finite sequence of elements of S_1. Then there exists a sequence A_2 of subsets of S_1 such that for every k such that $k \in \operatorname{dom} F_2$ holds $A_2(k) = F_2(k)$ and for every k such that $k \notin \operatorname{dom} F_2$ holds $A_2(k) = \emptyset$.

(42) For every finite sequence F_2 of elements of S_1 holds $\bigcup F_2 \in S_1$.

Let X be a set, let S be a σ-field of subsets of X, and let F be a finite sequence of elements of S. The functor $F^{\mathbf{c}}$ yielding a finite sequence of elements of S is defined as follows:

(Def. 8) $F^{\mathbf{c}} = \text{Complement } F$.

We now state the proposition

(43) For every finite sequence F_2 of elements of S_1 holds Intersection $F_2 \in S_1$.

In the sequel F_3 denotes a finite sequence of elements of S_2.

The following two propositions are true:

(44) $\operatorname{dom}(P \cdot F_3) = \operatorname{dom} F_3$.

(45) $P \cdot F_3$ is a finite sequence of elements of $\mathbb{R}$.

Let us consider O_1, S_2, F_3, P. Then $P \cdot F_3$ is a finite sequence of elements of $\mathbb{R}$.

Next we state several propositions:

(46) $\operatorname{len}(P \cdot F_3) = \operatorname{len} F_3$.

(47) If $\operatorname{len} R_1 = 0$, then $\sum R_1 = 0$.

(48) Suppose $\operatorname{len} R_1 \geq 1$. Then there exists a sequence f of real numbers such that $f(1) = R_1(1)$ and for every n such that $0 \neq n$ and $n < \operatorname{len} R_1$ holds $f(n+1) = f(n) + R_1(n+1)$ and $\sum R_1 = f(\operatorname{len} R_1)$.

(49) Let F_3 be a finite sequence of elements of S_2 and A_2 be a sequence of subsets of S_2. Suppose for every k such that $k \in \operatorname{dom} F_3$ holds $A_2(k) = F_3(k)$ and for every k such that $k \notin \operatorname{dom} F_3$ holds $A_2(k) = \emptyset$. Then $(\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent and $\sum(P \cdot A_2) = (\sum_{\alpha=0}^{\kappa}(P \cdot A_2)(\alpha))_{\kappa \in \mathbb{N}}(\operatorname{len} F_3)$ and $P(\bigcup A_2) \leq \sum(P \cdot A_2)$ and $\sum(P \cdot F_3) = \sum(P \cdot A_2)$.

(50) $P(\bigcup F_3) \leq \sum(P \cdot F_3)$ and if F_3 is disjoint valued, then $P(\bigcup F_3) = \sum(P \cdot F_3)$.

Let us consider X and let I_1 be a family of subsets of X. We say that I_1 is non-decreasing-union-closed if and only if:

(Def. 9) For every sequence A_1 of subsets of X such that A_1 is non-decreasing and for every n holds $A_1(n) \in I_1$ holds $\bigcup A_1 \in I_1$.

We say that I_1 is non-increasing-intersection-closed if and only if:

(Def. 10) For every sequence A_1 of subsets of X such that A_1 is non-increasing and for every n holds $A_1(n) \in I_1$ holds Intersection $A_1 \in I_1$.

We now state three propositions:

(51) Let I_1 be a family of subsets of X. Then I_1 is non-decreasing-union-closed if and only if for every sequence A_1 of subsets of X such that A_1 is non-decreasing and for every n holds $A_1(n) \in I_1$ holds $\lim A_1 \in I_1$.

(52) Let I_1 be a family of subsets of X. Then I_1 is non-increasing-intersection-closed if and only if for every sequence A_1 of subsets of X such that A_1 is

non-increasing and for every n holds $A_1(n) \in I_1$ holds $\lim A_1 \in I_1$.

(53) 2^X is non-decreasing-union-closed and 2^X is non-increasing-intersection-closed.

Let us consider X. A family of subsets of X is said to be a monotone class of X if:

(Def. 11) It is non-decreasing-union-closed and it is non-increasing-intersection-closed.

Next we state four propositions:

(54) Z is a monotone class of X if and only if the following conditions are satisfied:

(i) $Z \subseteq 2^X$, and

(ii) for every sequence A_1 of subsets of X such that A_1 is monotone and for every n holds $A_1(n) \in Z$ holds $\lim A_1 \in Z$.

(55) Let F be a field of subsets of X. Then F is a σ-field of subsets of X if and only if F is a monotone class of X.

(56) 2^{O_1} is a monotone class of O_1.

(57) Let X be a family of subsets of O_1. Then there exists a monotone class Y of O_1 such that $X \subseteq Y$ and for every Z such that $X \subseteq Z$ and Z is a monotone class of O_1 holds $Y \subseteq Z$.

Let us consider O_1 and let X be a family of subsets of O_1. The functor monotone-class(X) yielding a monotone class of O_1 is defined as follows:

(Def. 12) $X \subseteq$ monotone-class(X) and for every Z such that $X \subseteq Z$ and Z is a monotone class of O_1 holds monotone-class(X) $\subseteq Z$.

We now state two propositions:

(58) For every field Z of subsets of O_1 holds monotone-class(Z) is a field of subsets of O_1.

(59) For every field Z of subsets of O_1 holds $\sigma(Z) =$ monotone-class(Z).

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