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On the Borel Families of Subsets of Topological Spaces[1]

Adam Grabowski
Institute of Mathematics
University of Białystok
Akademicka 2, 15-267 Białystok, Poland

Summary. This is the next Mizar article in a series aiming at complete formalization of "General Topology" [14] by Engelking. We cover the second part of Section 1.3.

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The papers [27], [30], [31], [9], [1], [2], [26], [3], [28], [10], [12], [21], [29], [22], [5], [16], [6], [23], [32], [11], [20], [17], [18], [19], [7], [13], [25], [24], [15], [4], and [8] provide the terminology and notation for this paper.

1. Preliminaries

Let T be a 1-sorted structure. The functor $\mathrm{TotFam}\,T$ yielding a family of subsets of T is defined by:

(Def. 1) $\mathrm{TotFam}\,T = 2^{\text{the carrier of } T}$.

The following proposition is true

(1) For every set T and for every family F of subsets of T holds F is countable iff F^{c} is countable.

Let us note that $\mathbb{Q}$ is countable.

The scheme *FraenCoun11* concerns a unary predicate $\mathcal{P}$, and states that:

$\{\{n\}; n$ ranges over elements of $\mathbb{Q}$: $\mathcal{P}[n]\}$ is countable

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for all values of the parameters.

One can prove the following proposition

(2) For every non empty topological space T and for every subset A of T holds $\operatorname{Der} A = \{x; x$ ranges over points of T: $x \in \overline{A \setminus \{x\}}\}$.

Let us note that every topological structure which is finite is also second-countable.

One can verify that $\mathbb{R}$ is non countable.

One can verify the following observations:

* every set which is non countable is also non finite,
* every set which is non finite is also non trivial, and
* there exists a set which is non countable and non empty.

We adopt the following rules: T is a non empty topological space, A, B are subsets of T, and F, G are families of subsets of T.

One can prove the following propositions:

(3) A is closed iff $\operatorname{Der} A \subseteq A$.

(4) Let T be a non empty topological structure, B be a basis of T, and V be a subset of T. Suppose V is open and $V \neq \emptyset$. Then there exists a subset W of T such that $W \in B$ and $W \subseteq V$ and $W \neq \emptyset$.

2. Regular Formalization: Separable Spaces

The following propositions are true:

(5) $\operatorname{density} T \leq \operatorname{weight} T$.

(6) T is separable iff there exists a subset of T which is dense and countable.

(7) If T is second-countable, then T is separable.

One can check that every non empty topological space which is second-countable is also separable.

The following four propositions are true:

(8) Let T be a non empty topological space and A, B be subsets of T. If A and B are separated, then $\operatorname{Fr}(A \cup B) = \operatorname{Fr} A \cup \operatorname{Fr} B$.

(9) If F is locally finite, then $\operatorname{Fr} \bigcup F \subseteq \bigcup \operatorname{Fr} F$.

(10) For every discrete non empty topological space T holds T is separable iff $\overline{\overline{\Omega_T}} \leq \aleph_0$.

(11) For every discrete non empty topological space T holds T is separable iff T is countable.

3. Families of Subsets Closed for Countable Unions and Complement

Let us consider T, F. We say that F is all-open-containing if and only if:

(Def. 2) For every subset A of T such that A is open holds $A \in F$.

Let us consider T, F. We say that F is all-closed-containing if and only if:

(Def. 3) For every subset A of T such that A is closed holds $A \in F$.

Let T be a set and let F be a family of subsets of T. We say that F is closed for countable unions if and only if:

(Def. 4) For every countable family G of subsets of T such that $G \subseteq F$ holds $\bigcup G \in F$.

Let T be a set. Note that every σ-field of subsets of T is closed for countable unions.

One can prove the following proposition

(12) For every set T and for every family F of subsets of T such that F is closed for countable unions holds $\emptyset \in F$.

Let T be a set. One can verify that every family of subsets of T which is closed for countable unions is also non empty.

Next we state the proposition

(13) Let T be a set and F be a family of subsets of T. Then F is a σ-field of subsets of T if and only if F is closed for complement operator and closed for countable unions.

Let T be a set and let F be a family of subsets of T. We say that F is closed for countable meets if and only if:

(Def. 5) For every countable family G of subsets of T such that $G \subseteq F$ holds $\bigcap G \in F$.

Next we state four propositions:

(14) Let F be a family of subsets of T. Then the following statements are equivalent

(i) F is all-closed-containing and closed for complement operator,

(ii) F is all-open-containing and closed for complement operator.

(15) For every set T and for every family F of subsets of T such that F is closed for complement operator holds $F = F^{c}$.

(16) Let T be a set and F, G be families of subsets of T. If $F \subseteq G$ and G is closed for complement operator, then $F^{c} \subseteq G$.

(17) Let T be a set and F be a family of subsets of T. Then the following statements are equivalent

(i) F is closed for countable meets and closed for complement operator,

(ii) F is closed for countable unions and closed for complement operator.

Let us consider T. One can verify that every family of subsets of T which is all-open-containing, closed for complement operator, and closed for countable unions is also all-closed-containing and closed for countable meets and every family of subsets of T which is all-closed-containing, closed for complement operator, and closed for countable meets is also all-open-containing and closed for countable unions.

4. On the Families of Subsets

Let T be a set and let F be a countable family of subsets of T. Note that F^{c} is countable.

Let us consider T. Note that every family of subsets of T which is empty is also open and closed.

Let us consider T. One can check that there exists a family of subsets of T which is countable, open, and closed.

We now state the proposition

(18) For every set T holds $\emptyset$ is an empty family of subsets of T.

Let us observe that every set which is empty is also countable.

5. Collective Properties of Families

One can prove the following two propositions:

(19) If $F = \{A\}$, then A is open iff F is open.

(20) If $F = \{A\}$, then A is closed iff F is closed.

Let T be a set and let F, G be families of subsets of T. Then $F \Cap G$ is a family of subsets of T. Then $F \Cup G$ is a family of subsets of T.

Next we state a number of propositions:

(21) If F is closed and G is closed, then $F \Cap G$ is closed.

(22) If F is closed and G is closed, then $F \Cup G$ is closed.

(23) If F is open and G is open, then $F \Cap G$ is open.

(24) If F is open and G is open, then $F \Cup G$ is open.

(25) For every set T and for all families F, G of subsets of T holds $\overline{\overline{F \Cap G}} \leq \overline{\overline{[: F, G :]}}$.

(26) For every set T and for all families F, G of subsets of T holds $\overline{\overline{F \Cup G}} \leq \overline{\overline{[: F, G :]}}$.

(27) For all sets F, G holds $\bigcup(F \Cup G) \subseteq \bigcup F \cup \bigcup G$.

(28) For all sets F, G such that $F \neq \emptyset$ and $G \neq \emptyset$ holds $\bigcup F \cup \bigcup G = \bigcup(F \Cup G)$.

(29) For every set F holds $\emptyset \Cup F = \emptyset$.

(30) For all sets F, G such that $F \Cup G = \emptyset$ holds $F = \emptyset$ or $G = \emptyset$.

(31) For all sets F, G such that $F \Cap G = \emptyset$ holds $F = \emptyset$ or $G = \emptyset$.

(32) For all sets F, G holds $\bigcap(F \Cup G) \subseteq \bigcap F \cup \bigcap G$.

(33) For all sets F, G such that $F \neq \emptyset$ and $G \neq \emptyset$ holds $\bigcap(F \Cup G) = \bigcap F \cup \bigcap G$.

(34) For all sets F, G such that $F \neq \emptyset$ and $G \neq \emptyset$ holds $\bigcap F \cap \bigcap G = \bigcap(F \Cap G)$.

6. F_σ and G_δ Types of Subsets

Let us consider T, A. We say that A is F_σ if and only if:

(Def. 6) There exists a closed countable family F of subsets of T such that $A = \bigcup F$.

Let us consider T, A. We say that A is G_δ if and only if:

(Def. 7) There exists an open countable family F of subsets of T such that $A = \bigcap F$.

The following propositions are true:

(35) $\emptyset_T$ is F_σ.

(36) $\emptyset_T$ is G_δ.

Let us consider T. Note that $\emptyset_T$ is F_σ and G_δ.

Next we state two propositions:

(37) Ω_T is F_σ.

(38) Ω_T is G_δ.

Let us consider T. One can verify that Ω_T is F_σ and G_δ.

One can prove the following propositions:

(39) If A is F_σ, then A^{c} is G_δ.

(40) If A is G_δ, then A^{c} is F_σ.

(41) If A is F_σ and B is F_σ, then $A \cap B$ is F_σ.

(42) If A is F_σ and B is F_σ, then $A \cup B$ is F_σ.

(43) If A is G_δ and B is G_δ, then $A \cup B$ is G_δ.

(44) If A is G_δ and B is G_δ, then $A \cap B$ is G_δ.

(45) For every subset A of T such that A is closed holds A is F_σ.

(46) For every subset A of T such that A is open holds A is G_δ.

(47) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{Q}$ holds A is F_σ.

7. $T_{1/2}$ Topological Spaces

Let T be a topological space. We say that T is $T_{1/2}$ if and only if:

(Def. 8) For every subset A of T holds $\operatorname{Der} A$ is closed.

We now state three propositions:

(48) For every topological space T such that T is T_1 holds T is $T_{1/2}$.

(49) For every non empty topological space T such that T is $T_{1/2}$ holds T is T_0.

(50) For every non empty topological space T holds every point p of T is isolated in Ω_T or an accumulation point of Ω_T.

Let us note that every topological space which is $T_{1/2}$ is also T_0 and every topological space which is T_1 is also $T_{1/2}$.

8. Condensation Points

Let us consider T, A and let x be a point of T. We say that x is a condensation point of A if and only if:

(Def. 9) For every neighbourhood N of x holds $N \cap A$ is not countable.

In the sequel x denotes a point of T.

One can prove the following proposition

(51) If x is a condensation point of A and $A \subseteq B$, then x is a condensation point of B.

Let us consider T, A. The functor A^0 yielding a subset of T is defined as follows:

(Def. 10) For every point x of T holds $x \in A^0$ iff x is a condensation point of A.

The following propositions are true:

(52) For every point p of T such that p is a condensation point of A holds p is an accumulation point of A.

(53) $A^0 \subseteq \operatorname{Der} A$.

(54) $A^0 = \overline{A^0}$.

(55) If $A \subseteq B$, then $A^0 \subseteq B^0$.

(56) If x is a condensation point of $A \cup B$, then x is a condensation point of A or a condensation point of B.

(57) $A \cup B^0 = A^0 \cup B^0$.

(58) If A is countable, then there exists no point of T which is a condensation point of A.

(59) If A is countable, then $A^0 = \emptyset$.

Let us consider T and let A be a countable subset of T. Note that A^0 is empty.

The following proposition is true

(60) If T is second-countable, then there exists a basis of T which is countable.

Let us mention that there exists a topological space which is second-countable and non empty.

9. Borel Families of Subsets

Let us consider T. Observe that TotFam T is non empty, all-open-containing, closed for complement operator, and closed for countable unions.

We now state four propositions:

(61) For every set T and for every sequence A of subsets of T holds rng A is a countable non empty family of subsets of T.

(62) Let T, F be sets. Then F is a σ-field of subsets of T if and only if F is a closed for complement operator σ-field of subsets-like non empty family of subsets of T.

(63) For all families F, G of subsets of T such that F is all-open-containing and $F \subseteq G$ holds G is all-open-containing.

(64) Let F, G be families of subsets of T. Suppose F is all-closed-containing and $F \subseteq G$. Then G is all-closed-containing.

Let T be a 1-sorted structure. A σ-field of subsets of T is a σ-field of subsets of the carrier of T.

Let T be a non empty topological space. Note that there exists a family of subsets of T which is closed for complement operator, closed for countable unions, closed for countable meets, all-closed-containing, and all-open-containing.

We now state the proposition

(65) σ(TotFam T) is all-open-containing, closed for complement operator, and closed for countable unions.

Let us consider T. One can verify that σ(TotFam T) is all-open-containing, closed for complement operator, and closed for countable unions.

Let T be a non empty 1-sorted structure. Note that there exists a family of subsets of T which is σ-field of subsets-like, closed for complement operator, closed for countable unions, and non empty.

Let T be a non empty topological space. One can verify that every σ-field of subsets of T is closed for countable unions.

We now state the proposition

(66) Let T be a non empty topological space and F be a family of subsets of T. Suppose F is closed for complement operator and closed for countable unions. Then F is a σ-field of subsets of T.

Let T be a non empty topological space. Note that there exists a σ-field of subsets of T which is all-open-containing.

Let T be a non empty topological space. Note that $\mathrm{Topology}(T)$ is open and all-open-containing.

We now state the proposition

(67) Let X be a family of subsets of T. Then there exists an all-open-containing closed for complement operator closed for countable unions family Y of subsets of T such that

(i) $X \subseteq Y$, and

(ii) for every all-open-containing closed for complement operator closed for countable unions family Z of subsets of T such that $X \subseteq Z$ holds $Y \subseteq Z$.

Let us consider T. The functor BorelSets T yields an all-open-containing closed for complement operator closed for countable unions family of subsets of T and is defined by the condition (Def. 11).

(Def. 11) Let G be an all-open-containing closed for complement operator closed for countable unions family of subsets of T. Then BorelSets $T \subseteq G$.

Next we state three propositions:

(68) For every closed family F of subsets of T holds $F \subseteq$ BorelSets T.

(69) For every open family F of subsets of T holds $F \subseteq$ BorelSets T.

(70) BorelSets $T = \sigma(\mathrm{Topology}(T))$.

Let us consider T, A. We say that A is Borel if and only if:

(Def. 12) $A \in$ BorelSets T.

Let us consider T. Note that every subset of T which is F_σ is also Borel.

Let us consider T. Note that every subset of T which is G_δ is also Borel.

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