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Pocklington's Theorem and Bertrand's Postulate

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Summary. The first four sections of this article include some auxiliary theorems related to number and finite sequence of numbers, in particular a primality test, the Pocklington's theorem (see [19]). The last section presents the formalization of Bertrand's postulate closely following the book [1], pp. 7–9.

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The articles [26], [4], [24], [28], [3], [2], [20], [17], [14], [16], [30], [10], [11], [6], [23], [13], [15], [5], [21], [8], [22], [27], [18], [29], [9], [7], [12], [25], and [31] provide the notation and terminology for this paper.

1. Some Theorems on Real and Natural Numbers

The following propositions are true:

(1) For all real numbers r, s such that $0 \leq r$ and $s \cdot s < r \cdot r$ holds $s < r$.

(2) For all real numbers r, s such that $1 < r$ and $r \cdot r \leq s$ holds $r < s$.

(3) For all natural numbers a, n such that $a > 1$ holds $a^n > n$.

(4) For all natural numbers n, k, m such that $k \leq n$ and $m = \lfloor \frac{n}{2} \rfloor$ holds $\binom{n}{m} \geq \binom{n}{k}$.

(5) For all natural numbers n, m such that $m = \lfloor \frac{n}{2} \rfloor$ and $n \geq 2$ holds $\binom{n}{m} \geq \frac{2^n}{n}$.

(6) For every natural number n holds $\binom{2 \cdot n}{n} \geq \frac{4^n}{2 \cdot n}$.

(7) For all natural numbers n, p such that $p > 0$ and $n \mid p$ and $n \neq 1$ and $n \neq p$ holds $1 < n$ and $n < p$.

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(8) Let p be a natural number. Given a natural number n such that $n \mid p$ and $1 < n$ and $n < p$. Then there exists a natural number n such that $n \mid p$ and $1 < n$ and $n \cdot n \leq p$.

(9) For all natural numbers i, j, k, l such that $i = j \cdot k + l$ and $l < j$ and $0 < l$ holds $j \nmid i$.

(10) For all natural numbers n, q, b such that $\gcd(q, b) = 1$ and $q \neq 0$ and $b \neq 0$ holds $\gcd(q^n, b) = 1$.

(11) For all natural numbers a, b, c holds $a^{2 \cdot b} \bmod c = (a^b \bmod c) \cdot (a^b \bmod c) \bmod c$.

(12) Let p be a natural number. Then p is not prime if and only if one of the following conditions is satisfied:

(i) $p \leq 1$, or

(ii) there exists a natural number n such that $n \mid p$ and $1 < n$ and $n < p$.

(13) Let n, k be natural numbers. Suppose $n \mid k$ and $1 < n$. Then there exists a natural number p such that $p \mid k$ and $p \leq n$ and p is prime.

(14) Let p be a natural number. Then p is prime if and only if the following conditions are satisfied:

(i) $p > 1$, and

(ii) for every natural number n such that $1 < n$ and $n \cdot n \leq p$ and n is prime holds $n \nmid p$.

(15) For all natural numbers a, p, k such that $a^k \bmod p = 1$ and $k \geq 1$ and p is prime holds a and p are relative prime.

(16) Let p be a prime number, a be a natural number, and x be a set. Suppose $a \neq 0$ and $x = p^{p\text{-count}(a)}$. Then there exists a natural number b such that $b = x$ and $1 \leq b$ and $b \leq a$.

(17) For all natural numbers k, q, n, d such that q is prime and $d \mid k \cdot q^{n+1}$ and $d \nmid k \cdot q^n$ holds $q^{n+1} \mid d$.

(18) For all natural numbers q_1, q, n_1 such that $q_1 \mid q^{n_1}$ and q is prime and q_1 is prime and $n_1 > 0$ holds $q = q_1$.

(19) For every prime number p and for every natural number n such that $n < p$ holds $p \nmid n!$.

(20) Let a, b be non empty natural numbers. Suppose that for every natural number p such that p is prime holds $p\text{-count}(a) \leq p\text{-count}(b)$. Then there exists a natural number c such that $b = a \cdot c$.

(21) Let a, b be non empty natural numbers. Suppose that for every natural number p such that p is prime holds $p\text{-count}(a) = p\text{-count}(b)$. Then $a = b$.

(22) For all prime numbers p_1, p_2 and for every non empty natural number m such that ${p_1}^{p_1\text{-count}(m)} = {p_2}^{p_2\text{-count}(m)}$ and $p_1\text{-count}(m) > 0$ holds $p_1 = p_2$.

2. Pocklington's Theorem

One can prove the following propositions:

(23) Let n, k, q, p, n_1, p, a be natural numbers. Suppose $n - 1 = k \cdot q^{n_1}$ and $k > 0$ and $n_1 > 0$ and q is prime and $a^{n -' 1} \bmod n = 1$ and p is prime and $p \mid n$. Then $p \mid a^{(n -' 1) \div q} -' 1$ or $p \bmod q^{n_1} = 1$.

(24) Let n, f, c be natural numbers. Suppose that

(i) $n - 1 = f \cdot c$,

(ii) $f > c$,

(iii) $c > 0$,

(iv) $\gcd(f, c) = 1$, and

(v) for every natural number q such that $q \mid f$ and q is prime there exists a natural number a such that $a^{n -' 1} \bmod n = 1$ and $\gcd(a^{(n -' 1) \div q} -' 1, n) = 1$.

Then n is prime.

(25) Let n, f, d, n_1, a, q be natural numbers. Suppose $n - 1 = q^{n_1} \cdot d$ and $q^{n_1} > d$ and $d > 0$ and $\gcd(q, d) = 1$ and q is prime and $a^{n -' 1} \bmod n = 1$ and $\gcd(a^{(n -' 1) \div q} -' 1, n) = 1$. Then n is prime.

3. Some Prime Numbers

The following propositions are true:

(26) 7 is prime.

(27) 11 is prime.

(28) 13 is prime.

(29) 19 is prime.

(30) 23 is prime.

(31) 37 is prime.

(32) 43 is prime.

(33) 83 is prime.

(34) 139 is prime.

(35) 163 is prime.

(36) 317 is prime.

(37) 631 is prime.

(38) 1259 is prime.

(39) 2503 is prime.

(40) 4001 is prime.

4. Some Theorems on Finite Sequence of Numbers

One can prove the following propositions:

(41) For all finite sequences f, f_0, f_1 of elements of $\mathbb{R}$ such that $f = f_0 + f_1$ holds $\operatorname{dom} f = \operatorname{dom} f_0 \cap \operatorname{dom} f_1$.

(42) Let F be a finite sequence of elements of $\mathbb{R}$. If for every natural number k such that $k \in \operatorname{dom} F$ holds $F(k) > 0$, then $\prod F > 0$.

(43) For every set X_1 and for every finite set X_2 such that $X_1 \subseteq X_2$ and $X_2 \subseteq \mathbb{N}$ and $\emptyset \notin X_2$ holds $\prod \operatorname{Sgm} X_1 \leq \prod \operatorname{Sgm} X_2$.

(44) Let a, k be natural numbers, X be a set, F be a finite sequence of elements of Prime, and p be a prime number such that $X \subseteq$ Prime and $X \subseteq \operatorname{Seg} k$ and $F = \operatorname{Sgm} X$ and $a = \prod F$. Then

(i) if $p \in \operatorname{rng} F$, then $p\text{-count}(a) = 1$, and

(ii) if $p \notin \operatorname{rng} F$, then $p\text{-count}(a) = 0$.

(45) For every natural number n holds $\prod \operatorname{Sgm}\{p; p$ ranges over prime numbers: $p \leq n+1\} \leq 4^n$.

(46) For every real number x such that $x \geq 2$ holds $\prod \operatorname{Sgm}\{p; p$ ranges over prime numbers: $p \leq x\} \leq 4^{x-1}$.

(47) Let n be a natural number and p be a prime number. Suppose $n \neq 0$. Then there exists a finite sequence f of elements of $\mathbb{N}$ such that

(i) $\operatorname{len} f = n$,

(ii) for every natural number k such that $k \in \operatorname{dom} f$ holds $f(k) = 1$ iff $p^k \mid n$ and $f(k) = 0$ iff $p^k \nmid n$, and

(iii) $p\text{-count}(n) = \sum f$.

(48) Let n be a natural number and p be a prime number. Then there exists a finite sequence f of elements of $\mathbb{N}$ such that $\operatorname{len} f = n$ and for every natural number k such that $k \in \operatorname{dom} f$ holds $f(k) = \lfloor \frac{n}{p^k} \rfloor$ and $p\text{-count}(n!) = \sum f$.

(49) Let n be a natural number and p be a prime number. Then there exists a finite sequence f of elements of $\mathbb{R}$ such that $\operatorname{len} f = 2 \cdot n$ and for every natural number k such that $k \in \operatorname{dom} f$ holds $f(k) = \lfloor \frac{2 \cdot n}{p^k} \rfloor - 2 \cdot \lfloor \frac{n}{p^k} \rfloor$ and $p\text{-count}(\binom{2 \cdot n}{n}) = \sum f$.

Let f be a finite sequence of elements of $\mathbb{N}$ and let p be a prime number. The functor $p\text{-count}(f)$ yielding a finite sequence of elements of $\mathbb{N}$ is defined by:

(Def. 1) $\operatorname{len}(p\text{-count}(f)) = \operatorname{len} f$ and for every set i such that $i \in \operatorname{dom}(p\text{-count}(f))$ holds $(p\text{-count}(f))(i) = p\text{-count}(f(i))$.

One can prove the following propositions:

(50) For every prime number p and for every finite sequence f of elements of $\mathbb{N}$ such that $f = \emptyset$ holds $p\text{-count}(f) = \emptyset$.

(51) For every prime number p and for all finite sequences f_1, f_2 of elements of $\mathbb{N}$ holds $p\text{-count}(f_1 \frown f_2) = (p\text{-count}(f_1)) \frown (p\text{-count}(f_2))$.

(52) For every prime number p and for every non empty natural number n holds $p\text{-count}(\langle n \rangle) = \langle p\text{-count}(n) \rangle$.

(53) For every finite sequence f of elements of $\mathbb{N}$ and for every prime number p such that $\prod f \neq 0$ holds $p\text{-count}(\prod f) = \sum(p\text{-count}(f))$.

(54) Let f_1, f_2 be finite sequences of elements of $\mathbb{R}$. Suppose $\operatorname{len} f_1 = \operatorname{len} f_2$ and for every natural number k such that $k \in \operatorname{dom} f_1$ holds $f_1(k) \leq f_2(k)$ and $f_1(k) > 0$. Then $\prod f_1 \leq \prod f_2$.

(55) For every natural number n and for every real number r such that $r > 0$ holds $\prod(n \mapsto r) = r^n$.

In this article we present several logical schemes. The scheme *scheme1* concerns a ternary predicate $\mathcal{P}$, and states that:

> Let p be a prime number, n be a natural number, m be a non empty natural number, and X be a set. If $X = \{p'^{p'\text{-count}(m)}; p'$ ranges over prime numbers: $\mathcal{P}[n, m, p']\}$, then $\prod \operatorname{Sgm} X > 0$

for all values of the parameters.

The scheme *scheme2* concerns a ternary predicate $\mathcal{P}$, and states that:

> Let p be a prime number, n be a natural number, m be a non empty natural number, and X be a set. If $X = \{p'^{p'\text{-count}(m)}; p'$ ranges over prime numbers: $\mathcal{P}[n, m, p']\}$ and $p^{p\text{-count}(m)} \notin X$, then $p\text{-count}(\prod \operatorname{Sgm} X) = 0$

for all values of the parameters.

The scheme *scheme3* concerns a ternary predicate $\mathcal{P}$, and states that:

> Let p be a prime number, n be a natural number, m be a non empty natural number, and X be a set. If $X = \{p'^{p'\text{-count}(m)}; p'$ ranges over prime numbers: $\mathcal{P}[n, m, p']\}$ and $p^{p\text{-count}(m)} \in X$, then $p\text{-count}(\prod \operatorname{Sgm} X) = p\text{-count}(m)$

for all values of the parameters.

The scheme *scheme4* deals with a binary functor $\mathcal{F}$ yielding a set and a binary predicate $\mathcal{P}$, and states that:

> Let n, m be natural numbers, r be a real number, and X be a finite set. If $X = \{\mathcal{F}(p, m); p$ ranges over prime numbers: $p \leq r \ \wedge \ \mathcal{P}[p, m]\}$ and $r \geq 0$, then $\operatorname{card} X \leq \lfloor r \rfloor$

for all values of the parameters.

5. Bertrand's Postulate

The following proposition is true

(56) For every natural number n such that $n \geq 1$ there exists a prime number p such that $n < p$ and $p \leq 2 \cdot n$.

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Integral of Measurable Function[1]

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Summary. In this paper we construct integral of measurable function.

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The terminology and notation used here are introduced in the following articles: [29], [12], [32], [1], [27], [18], [33], [9], [2], [34], [13], [11], [10], [28], [31], [20], [30], [3], [4], [5], [14], [7], [17], [15], [16], [26], [8], [19], [21], [24], [23], [6], [22], and [25].

1. Lemmas for Extended Real Numbers

One can prove the following propositions:

(1) For all extended real numbers x, y holds $|x-y| = |y-x|$.

(2) For all extended real numbers x, y holds $y - x \leq |x-y|$.

(3) Let x, y be extended real numbers and e be a real number. Suppose $|x-y| < e$ and $x \neq +\infty$ or $y \neq +\infty$ but $x \neq -\infty$ or $y \neq -\infty$. Then $x \neq +\infty$ and $x \neq -\infty$ and $y \neq +\infty$ and $y \neq -\infty$.

(4) For all extended real numbers x, y such that for every real number e such that $0 < e$ holds $x < y + \overline{\mathbb{R}}(e)$ holds $x \leq y$.

(5) For all extended real numbers x, y, t such that $t \neq -\infty$ and $t \neq +\infty$ and $x < y$ holds $x + t < y + t$.

(6) For all extended real numbers x, y, t such that $t \neq -\infty$ and $t \neq +\infty$ and $x < y$ holds $x - t < y - t$.

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(7) For all real numbers a, b holds $\overline{\mathbb{R}}(a) + \overline{\mathbb{R}}(b) = a + b$ and $-\overline{\mathbb{R}}(a) = -a$.

(8) Let n be a natural number and p be an extended real number. Suppose $0 \leq p$ and $p < n$. Then there exists a natural number k such that $1 \leq k$ and $k \leq 2^n \cdot n$ and $\frac{k-1}{2^n} \leq p$ and $p < \frac{k}{2^n}$.

(9) Let n, k be natural numbers and p be an extended real number. If $1 \leq k$ and $k \leq 2^n \cdot n$ and $n \leq p$ and $\frac{k-1}{2^n} \leq p$, then $\frac{k}{2^n} \leq p$.

(10) For all extended real numbers x, y, w, z such that $-\infty < w$ holds if $x < y$ and $w < z$, then $x + w < y + z$.

(11) For all extended real numbers x, y, k such that $0 \leq k$ holds $k \cdot \max(x, y) = \max(k \cdot x, k \cdot y)$ and $k \cdot \min(x, y) = \min(k \cdot x, k \cdot y)$.

(12) For all extended real numbers x, y, k such that $k \leq 0$ holds $k \cdot \min(x, y) = \max(k \cdot x, k \cdot y)$ and $k \cdot \max(x, y) = \min(k \cdot x, k \cdot y)$.

(13) For all extended real numbers x, y, z such that $0 \leq x$ and $0 \leq z$ and $z + x \leq y$ holds $z \leq y$.

2. Lemmas for Partial Function of Non-empty Set, Extended Real Numbers

Let I_1 be a set. We say that I_1 is non-positive if and only if:

(Def. 1) For every extended real number x such that $x \in I_1$ holds $x \leq 0$.

Let R be a binary relation. We say that R is non-positive if and only if:

(Def. 2) $\operatorname{rng} R$ is non-positive.

The following propositions are true:

(14) Let X be a set and F be a partial function from X to $\overline{\mathbb{R}}$. Then F is non-positive if and only if for every set n holds $F(n) \leq 0_{\overline{\mathbb{R}}}$.

(15) Let X be a set and F be a partial function from X to $\overline{\mathbb{R}}$. If for every set n such that $n \in \operatorname{dom} F$ holds $F(n) \leq 0_{\overline{\mathbb{R}}}$, then F is non-positive.

Let R be a binary relation. We say that R is without $-\infty$ if and only if:

(Def. 3) $-\infty \notin \operatorname{rng} R$.

We say that R is without $+\infty$ if and only if:

(Def. 4) $+\infty \notin \operatorname{rng} R$.

Let X be a non empty set and let f be a partial function from X to $\overline{\mathbb{R}}$. Let us observe that f is without $-\infty$ if and only if:

(Def. 5) For every set x holds $-\infty < f(x)$.

Let us observe that f is without $+\infty$ if and only if:

(Def. 6) For every set x holds $f(x) < +\infty$.

Next we state four propositions:

(16) Let X be a non empty set and f be a partial function from X to $\overline{\mathbb{R}}$. Then for every set x such that $x \in \operatorname{dom} f$ holds $-\infty < f(x)$ if and only if f is without $-\infty$.

(17) Let X be a non empty set and f be a partial function from X to $\overline{\mathbb{R}}$. Then for every set x such that $x \in \operatorname{dom} f$ holds $f(x) < +\infty$ if and only if f is without $+\infty$.

(18) Let X be a non empty set and f be a partial function from X to $\overline{\mathbb{R}}$. If f is non-negative, then f is without $-\infty$.

(19) Let X be a non empty set and f be a partial function from X to $\overline{\mathbb{R}}$. If f is non-positive, then f is without $+\infty$.

Let X be a non empty set. Note that every partial function from X to $\overline{\mathbb{R}}$ which is non-negative is also without $-\infty$ and every partial function from X to $\overline{\mathbb{R}}$ which is non-positive is also without $+\infty$.

The following propositions are true:

(20) Let X be a non empty set, S be a σ-field of subsets of X, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S. Then f is without $+\infty$ and without $-\infty$.

(21) Let X be a non empty set, Y be a set, and f be a partial function from X to $\overline{\mathbb{R}}$. If f is non-negative, then $f{\restriction}Y$ is non-negative.

(22) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and g is without $-\infty$. Then $\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$.

(23) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and g is without $+\infty$. Then $\operatorname{dom}(f-g) = \operatorname{dom} f \cap \operatorname{dom} g$.

(24) Let X be a non empty set, S be a σ-field of subsets of X, f, g be partial functions from X to $\overline{\mathbb{R}}$, F be a function from $\mathbb{Q}$ into S, r be a real number, and A be an element of S. Suppose f is without $-\infty$ and g is without $-\infty$ and for every rational number p holds $F(p) = A \cap \text{LE-dom}(f, \overline{\mathbb{R}}(p)) \cap (A \cap \text{LE-dom}(g, \overline{\mathbb{R}}(r-p)))$. Then $A \cap \text{LE-dom}(f+g, \overline{\mathbb{R}}(r)) = \bigcup \operatorname{rng} F$.

Let X be a non empty set and let f be a partial function from X to $\mathbb{R}$. The functor $\overline{\mathbb{R}}(f)$ yielding a partial function from X to $\overline{\mathbb{R}}$ is defined as follows:

(Def. 7) $\overline{\mathbb{R}}(f) = f$.

Next we state a number of propositions:

(25) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. If f is non-negative and g is non-negative, then $f+g$ is non-negative.

(26) Let X be a non empty set, f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number such that f is non-negative. Then

(i) if $0 \leq c$, then $c\,f$ is non-negative, and

(ii) if $c \leq 0$, then $c\,f$ is non-positive.

(27) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that for every set x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $g(x) \leq f(x)$ and $-\infty < g(x)$ and $f(x) < +\infty$. Then $f - g$ is non-negative.

(28) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is non-negative and g is non-negative. Then $\operatorname{dom}(f + g) = \operatorname{dom} f \cap \operatorname{dom} g$ and $f + g$ is non-negative.

(29) Let X be a non empty set and f, g, h be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is non-negative and g is non-negative and h is non-negative. Then $\operatorname{dom}(f + g + h) = \operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} h$ and $f + g + h$ is non-negative and for every set x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} h$ holds $(f + g + h)(x) = f(x) + g(x) + h(x)$.

(30) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and g is without $-\infty$. Then

(i) $\operatorname{dom}(\max_+(f + g) + \max_-(f)) = \operatorname{dom} f \cap \operatorname{dom} g$,

(ii) $\operatorname{dom}(\max_-(f + g) + \max_+(f)) = \operatorname{dom} f \cap \operatorname{dom} g$,

(iii) $\operatorname{dom}(\max_+(f + g) + \max_-(f) + \max_-(g)) = \operatorname{dom} f \cap \operatorname{dom} g$,

(iv) $\operatorname{dom}(\max_-(f + g) + \max_+(f) + \max_+(g)) = \operatorname{dom} f \cap \operatorname{dom} g$,

(v) $\max_+(f + g) + \max_-(f)$ is non-negative, and

(vi) $\max_-(f + g) + \max_+(f)$ is non-negative.

(31) Let X be a non empty set and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and without $+\infty$ and g is without $-\infty$ and without $+\infty$. Then $\max_+(f + g) + \max_-(f) + \max_-(g) = \max_-(f + g) + \max_+(f) + \max_+(g)$.

(32) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and c be a real number. If $0 \leq c$, then $\max_+(c\,f) = c \max_+(f)$ and $\max_-(c\,f) = c \max_-(f)$.

(33) Let C be a non empty set, f be a partial function from C to $\overline{\mathbb{R}}$, and c be a real number. If $0 \leq c$, then $\max_+((-c)\,f) = c \max_-(f)$ and $\max_-((-c)\,f) = c \max_+(f)$.

(34) Let X be a non empty set, f be a partial function from X to $\overline{\mathbb{R}}$, and A be a set. Then $\max_+(f{\restriction}A) = \max_+(f){\restriction}A$ and $\max_-(f{\restriction}A) = \max_-(f){\restriction}A$.

(35) Let X be a non empty set, f, g be partial functions from X to $\overline{\mathbb{R}}$, and B be a set. If $B \subseteq \operatorname{dom}(f + g)$, then $\operatorname{dom}((f + g){\restriction}B) = B$ and $\operatorname{dom}(f{\restriction}B + g{\restriction}B) = B$ and $(f + g){\restriction}B = f{\restriction}B + g{\restriction}B$.

(36) Let X be a non empty set, f be a partial function from X to $\overline{\mathbb{R}}$, and a be an extended real number. Then $\text{EQ-dom}(f, a) = f^{-1}(\{a\})$.

3. Lemmas for Measurable Function and Simple Valued Function

The following propositions are true:

(37) Let X be a non empty set, S be a σ-field of subsets of X, f, g be partial functions from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose f is without $-\infty$ and g is without $-\infty$ and f is measurable on A and g is measurable on A. Then $f+g$ is measurable on A.

(38) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and $\operatorname{dom} f = \emptyset$. Then there exists a finite sequence F of separated subsets of S and there exist finite sequences a, x of elements of $\overline{\mathbb{R}}$ such that

(i) F and a are representation of f,

(ii) $a(1) = 0$,

(iii) for every natural number n such that $2 \leq n$ and $n \in \operatorname{dom} a$ holds $0 < a(n)$ and $a(n) < +\infty$,

(iv) $\operatorname{dom} x = \operatorname{dom} F$,

(v) for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$, and

(vi) $\sum x = 0$.

(39) Let X be a non empty set, S be a σ-field of subsets of X, f be a partial function from X to $\overline{\mathbb{R}}$, A be an element of S, and r, s be real numbers. Suppose f is measurable on A and $A \subseteq \operatorname{dom} f$. Then $A \cap$ GTE-dom$(f, \overline{\mathbb{R}}(r)) \cap$ LE-dom$(f, \overline{\mathbb{R}}(s))$ is measurable on S.

(40) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. If f is simple function in S, then $f \restriction A$ is simple function in S.

(41) Let X be a non empty set, S be a σ-field of subsets of X, A be an element of S, F be a finite sequence of separated subsets of S, and G be a finite sequence. Suppose $\operatorname{dom} F = \operatorname{dom} G$ and for every natural number n such that $n \in \operatorname{dom} F$ holds $G(n) = F(n) \cap A$. Then G is a finite sequence of separated subsets of S.

(42) Let X be a non empty set, S be a σ-field of subsets of X, f be a partial function from X to $\overline{\mathbb{R}}$, A be an element of S, F, G be finite sequences of separated subsets of S, and a be a finite sequence of elements of $\overline{\mathbb{R}}$. Suppose $\operatorname{dom} F = \operatorname{dom} G$ and for every natural number n such that $n \in \operatorname{dom} F$ holds $G(n) = F(n) \cap A$ and F and a are representation of f. Then G and a are representation of $f \restriction A$.

(43) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. If f is simple function in S, then $\operatorname{dom} f$ is an element of S.

(44) Let X be a non empty set, S be a σ-field of subsets of X, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and g is simple function in S. Then $f + g$ is simple function in S.

(45) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number. If f is simple function in S, then $c\,f$ is simple function in S.

(46) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) f is simple function in S,
- (ii) g is simple function in S, and
- (iii) for every set x such that $x \in \operatorname{dom}(f - g)$ holds $g(x) \leq f(x)$.

Then $f - g$ is non-negative.

(47) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, and c be an extended real number. Suppose $c \neq +\infty$ and $c \neq -\infty$. Then there exists a partial function f from X to $\overline{\mathbb{R}}$ such that f is simple function in S and $\operatorname{dom} f = A$ and for every set x such that $x \in A$ holds $f(x) = c$.

(48) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and B, B_1 be elements of S. Suppose f is measurable on B and $B_1 = \operatorname{dom} f \cap B$. Then $f{\restriction}B$ is measurable on B_1.

(49) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) $A \subseteq \operatorname{dom} f$,
- (ii) f is measurable on A,
- (iii) g is measurable on A,
- (iv) f is without $-\infty$, and
- (v) g is without $-\infty$.

Then $\max_{+}(f + g) + \max_{-}(f)$ is measurable on A.

(50) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
- (i) $A \subseteq \operatorname{dom} f \cap \operatorname{dom} g$,
- (ii) f is measurable on A,
- (iii) g is measurable on A,
- (iv) f is without $-\infty$, and
- (v) g is without $-\infty$.

Then $\max_{-}(f + g) + \max_{+}(f)$ is measurable on A.

(51) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and A be a set. If $A \in S$, then $0 \leq M(A)$.

(52) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) there exists an element E_1 of S such that $E_1 = \operatorname{dom} f$ and f is measurable on E_1,

(ii) there exists an element E_2 of S such that $E_2 = \operatorname{dom} g$ and g is measurable on E_2,

(iii) $f^{-1}(\{+\infty\}) \in S$,

(iv) $f^{-1}(\{-\infty\}) \in S$,

(v) $g^{-1}(\{+\infty\}) \in S$, and

(vi) $g^{-1}(\{-\infty\}) \in S$.

Then $\operatorname{dom}(f + g) \in S$.

(53) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) there exists an element E_1 of S such that $E_1 = \operatorname{dom} f$ and f is measurable on E_1, and

(ii) there exists an element E_2 of S such that $E_2 = \operatorname{dom} g$ and g is measurable on E_2.

Then there exists an element E of S such that $E = \operatorname{dom}(f + g)$ and $f + g$ is measurable on E.

(54) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose $\operatorname{dom} f = A$. Then f is measurable on B if and only if f is measurable on $A \cap B$.

(55) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Given an element A of S such that $\operatorname{dom} f = A$. Let c be a real number and B be an element of S. If f is measurable on B, then $c\,f$ is measurable on B.

4. Sequence of Extended Real Numbers

A sequence of extended reals is a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$.

Let s_1 be a sequence of extended reals. We say that s_1 is convergent to finite number if and only if the condition (Def. 8) is satisfied.

(Def. 8) There exists a real number g such that for every real number p if $0 < p$, then there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|s_1(m) - \overline{\mathbb{R}}(g)| < p$.

Let s_1 be a sequence of extended reals. We say that s_1 is convergent to $+\infty$ if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let g be a real number. Suppose $0 < g$. Then there exists a natural number n such that for every natural number m such that $n \leq m$ holds $g \leq s_1(m)$.

Let s_1 be a sequence of extended reals. We say that s_1 is convergent to $-\infty$ if and only if the condition (Def. 10) is satisfied.

(Def. 10) Let g be a real number. Suppose $g < 0$. Then there exists a natural number n such that for every natural number m such that $n \leq m$ holds $s_1(m) \leq g$.

We now state two propositions:

(56) Let s_1 be a sequence of extended reals. Suppose s_1 is convergent to $+\infty$. Then s_1 is not convergent to $-\infty$ and s_1 is not convergent to finite number.

(57) Let s_1 be a sequence of extended reals. Suppose s_1 is convergent to $-\infty$. Then s_1 is not convergent to $+\infty$ and s_1 is not convergent to finite number.

Let s_1 be a sequence of extended reals. We say that s_1 is convergent if and only if:

(Def. 11) s_1 is convergent to finite number, or convergent to $+\infty$, or convergent to $-\infty$.

Let s_1 be a sequence of extended reals. Let us assume that s_1 is convergent. The functor $\lim s_1$ yields an extended real number and is defined by the conditions (Def. 12).

(Def. 12)(i) There exists a real number g such that $\lim s_1 = g$ and for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|s_1(m) - \lim s_1| < p$ and s_1 is convergent to finite number, or

(ii) $\lim s_1 = +\infty$ and s_1 is convergent to $+\infty$, or

(iii) $\lim s_1 = -\infty$ and s_1 is convergent to $-\infty$.

We now state a number of propositions:

(58) Let s_1 be a sequence of extended reals and r be a real number. Suppose that for every natural number n holds $s_1(n) = r$. Then s_1 is convergent to finite number and $\lim s_1 = r$.

(59) Let F be a finite sequence of elements of $\overline{\mathbb{R}}$. If for every natural number n such that $n \in \operatorname{dom} F$ holds $0 \leq F(n)$, then $0 \leq \sum F$.

(60) Let L be a sequence of extended reals. Suppose that for all natural numbers n, m such that $n \leq m$ holds $L(n) \leq L(m)$. Then L is convergent and $\lim L = \sup \operatorname{rng} L$.

(61) For all sequences L, G of extended reals such that for every natural number n holds $L(n) \leq G(n)$ holds $\sup \operatorname{rng} L \leq \sup \operatorname{rng} G$.

(62) For every sequence L of extended reals and for every natural number n holds $L(n) \leq \sup \operatorname{rng} L$.

(63) Let L be a sequence of extended reals and K be an extended real number. If for every natural number n holds $L(n) \leq K$, then $\sup \operatorname{rng} L \leq K$.

(64) Let L be a sequence of extended reals and K be an extended real number. If $K \neq +\infty$ and for every natural number n holds $L(n) \leq K$, then $\sup \mathrm{rng}\, L < +\infty$.

(65) Let L be a sequence of extended reals. Suppose L is without $-\infty$. Then $\sup \mathrm{rng}\, L \neq +\infty$ if and only if there exists a real number K such that $0 < K$ and for every natural number n holds $L(n) \leq K$.

(66) Let L be a sequence of extended reals and c be an extended real number. Suppose that for every natural number n holds $L(n) = c$. Then L is convergent and $\lim L = c$ and $\lim L = \sup \mathrm{rng}\, L$.

(67) Let J, K, L be sequences of extended reals. Suppose that
- (i) for all natural numbers n, m such that $n \leq m$ holds $J(n) \leq J(m)$,
- (ii) for all natural numbers n, m such that $n \leq m$ holds $K(n) \leq K(m)$,
- (iii) J is without $-\infty$,
- (iv) K is without $-\infty$, and
- (v) for every natural number n holds $J(n) + K(n) = L(n)$.

Then L is convergent and $\lim L = \sup \mathrm{rng}\, L$ and $\lim L = \lim J + \lim K$ and $\sup \mathrm{rng}\, L = \sup \mathrm{rng}\, K + \sup \mathrm{rng}\, J$.

(68) Let L, K be sequences of extended reals and c be a real number. Suppose $0 \leq c$ and L is without $-\infty$ and for every natural number n holds $K(n) = \overline{\mathbb{R}}(c) \cdot L(n)$. Then $\sup \mathrm{rng}\, K = \overline{\mathbb{R}}(c) \cdot \sup \mathrm{rng}\, L$ and K is without $-\infty$.

(69) Let L, K be sequences of extended reals and c be a real number. Suppose that
- (i) $0 \leq c$,
- (ii) for all natural numbers n, m such that $n \leq m$ holds $L(n) \leq L(m)$,
- (iii) for every natural number n holds $K(n) = \overline{\mathbb{R}}(c) \cdot L(n)$, and
- (iv) L is without $-\infty$.

Then
- (v) for all natural numbers n, m such that $n \leq m$ holds $K(n) \leq K(m)$,
- (vi) K is without $-\infty$ and convergent,
- (vii) $\lim K = \sup \mathrm{rng}\, K$, and
- (viii) $\lim K = \overline{\mathbb{R}}(c) \cdot \lim L$.

5. Sequence of Extended Real Valued Functions

Let X be a non empty set, let H be a sequence of partial functions from X into $\overline{\mathbb{R}}$, and let x be an element of X. The functor $H\#x$ yields a sequence of extended reals and is defined as follows:

(Def. 13) For every natural number n holds $(H\#x)(n) = H(n)(x)$.

Let D_1, D_2 be sets, let F be a function from $\mathbb{N}$ into $D_1 \dot{\rightarrow} D_2$, and let n be a natural number. Then $F(n)$ is a partial function from D_1 to D_2.

Next we state the proposition

(70) Let X be a non empty set, S be a σ-field of subsets of X, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative. Then there exists a sequence F of partial functions from X into $\overline{\mathbb{R}}$ such that

(i) for every natural number n holds $F(n)$ is simple function in S and $\operatorname{dom} F(n) = \operatorname{dom} f$,

(ii) for every natural number n holds $F(n)$ is non-negative,

(iii) for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in \operatorname{dom} f$ holds $F(n)(x) \leq F(m)(x)$, and

(iv) for every element x of X such that $x \in \operatorname{dom} f$ holds $F\#x$ is convergent and $\lim(F\#x) = f(x)$.

6. Integral of Non Negative Simple Valued Function

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\overline{\mathbb{R}}$. The functor $\int' f \,\mathrm{d}M$ yielding an element of $\overline{\mathbb{R}}$ is defined as follows:

(Def. 14) $$\int' f \,\mathrm{d}M = \begin{cases} \int\limits_X f \,\mathrm{d}M, & \text{if } \operatorname{dom} f \neq \emptyset, \\ 0_{\overline{\mathbb{R}}}, & \text{otherwise.} \end{cases}$$

The following propositions are true:

(71) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and g is simple function in S and f is non-negative and g is non-negative. Then $\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int' f+g \,\mathrm{d}M = \int' f{\restriction}\operatorname{dom}(f+g) \,\mathrm{d}M + \int' g{\restriction}\operatorname{dom}(f+g) \,\mathrm{d}M$.

(72) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number. Suppose f is simple function in S and f is non-negative and $0 \leq c$. Then $\int' c\, f \,\mathrm{d}M = \overline{\mathbb{R}}(c) \cdot \int' f \,\mathrm{d}M$.

(73) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose f is simple function in S and f is non-negative and A misses B. Then $\int' f{\restriction}(A \cup B) \,\mathrm{d}M = \int' f{\restriction}A \,\mathrm{d}M + \int' f{\restriction}B \,\mathrm{d}M$.

(74) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. If f is simple function in S and f is non-negative, then $0 \leq \int' f \,\mathrm{d}M$.

(75) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) f is simple function in S,

(ii) f is non-negative,
(iii) g is simple function in S,
(iv) g is non-negative, and
(v) for every set x such that $x \in \operatorname{dom}(f - g)$ holds $g(x) \leq f(x)$.

Then $\operatorname{dom}(f - g) = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int' f \restriction \operatorname{dom}(f - g)\, \mathrm{d}M = \int' f - g\, \mathrm{d}M + \int' g \restriction \operatorname{dom}(f - g)\, \mathrm{d}M$.

(76) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that
(i) f is simple function in S,
(ii) g is simple function in S,
(iii) f is non-negative,
(iv) g is non-negative, and
(v) for every set x such that $x \in \operatorname{dom}(f - g)$ holds $g(x) \leq f(x)$.

Then $\int' g \restriction \operatorname{dom}(f - g)\, \mathrm{d}M \leq \int' f \restriction \operatorname{dom}(f - g)\, \mathrm{d}M$.

(77) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and c be an extended real number. Suppose $0 \leq c$ and f is simple function in S and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) = c$. Then $\int' f\, \mathrm{d}M = c \cdot M(\operatorname{dom} f)$.

(78) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and f is non-negative. Then $\int' f \restriction \text{EQ-dom}(f, \overline{\mathbb{R}}(0))\, \mathrm{d}M = 0$.

(79) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, B be an element of S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and $M(B) = 0$ and f is non-negative. Then $\int' f \restriction B\, \mathrm{d}M = 0$.

(80) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, g be a partial function from X to $\overline{\mathbb{R}}$, F be a sequence of partial functions from X into $\overline{\mathbb{R}}$, and L be a sequence of extended reals. Suppose that g is simple function in S and for every set x such that $x \in \operatorname{dom} g$ holds $0 < g(x)$ and for every natural number n holds $F(n)$ is simple function in S and for every natural number n holds $\operatorname{dom} F(n) = \operatorname{dom} g$ and for every natural number n holds $F(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in \operatorname{dom} g$ holds $F(n)(x) \leq F(m)(x)$ and for every element x of X such that $x \in \operatorname{dom} g$ holds $F\#x$ is convergent and $g(x) \leq \lim(F\#x)$ and for every natural number n holds $L(n) = \int' F(n)\, \mathrm{d}M$. Then L is convergent and $\int' g\, \mathrm{d}M \leq \lim L$.

(81) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, g be a partial function from X to $\overline{\mathbb{R}}$, and F be a sequence of partial functions from X into $\overline{\mathbb{R}}$. Suppose that g is simple function in S and g is non-negative and for every natural number n holds $F(n)$ is simple

function in S and for every natural number n holds $\operatorname{dom} F(n) = \operatorname{dom} g$ and for every natural number n holds $F(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in \operatorname{dom} g$ holds $F(n)(x) \leq F(m)(x)$ and for every element x of X such that $x \in \operatorname{dom} g$ holds $F\#x$ is convergent and $g(x) \leq \lim(F\#x)$. Then there exists a sequence G of extended reals such that for every natural number n holds $G(n) = \int' F(n)\,\mathrm{d}M$ and G is convergent and $\sup \operatorname{rng} G = \lim G$ and $\int' g\,\mathrm{d}M \leq \lim G$.

(82) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, F, G be sequences of partial functions from X into $\overline{\mathbb{R}}$, and K, L be sequences of extended reals. Suppose that for every natural number n holds $F(n)$ is simple function in S and $\operatorname{dom} F(n) = A$ and for every natural number n holds $F(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in A$ holds $F(n)(x) \leq F(m)(x)$ and for every natural number n holds $G(n)$ is simple function in S and $\operatorname{dom} G(n) = A$ and for every natural number n holds $G(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in A$ holds $G(n)(x) \leq G(m)(x)$ and for every element x of X such that $x \in A$ holds $F\#x$ is convergent and $G\#x$ is convergent and $\lim(F\#x) = \lim(G\#x)$ and for every natural number n holds $K(n) = \int' F(n)\,\mathrm{d}M$ and $L(n) = \int' G(n)\,\mathrm{d}M$. Then K is convergent and L is convergent and $\lim K = \lim L$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\overline{\mathbb{R}}$. Let us assume that there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative. The functor $\int^+ f\,\mathrm{d}M$ yielding an element of $\overline{\mathbb{R}}$ is defined by the condition (Def. 15).

(Def. 15) There exists a sequence F of partial functions from X into $\overline{\mathbb{R}}$ and there exists a sequence K of extended reals such that
for every natural number n holds $F(n)$ is simple function in S and $\operatorname{dom} F(n) = \operatorname{dom} f$ and for every natural number n holds $F(n)$ is non-negative and for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in \operatorname{dom} f$ holds $F(n)(x) \leq F(m)(x)$ and for every element x of X such that $x \in \operatorname{dom} f$ holds $F\#x$ is convergent and $\lim(F\#x) = f(x)$ and for every natural number n holds $K(n) = \int' F(n)\,\mathrm{d}M$ and K is convergent and $\int^+ f\,\mathrm{d}M = \lim K$.

The following propositions are true:

(83) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. If f is simple function in S and f is non-negative, then $\int^+ f\,\mathrm{d}M = \int' f\,\mathrm{d}M$.

(84) Let X be a non empty set, S be a σ-field of subsets of X, M be a

σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A,

(ii) there exists an element B of S such that $B = \operatorname{dom} g$ and g is measurable on B,

(iii) f is non-negative, and

(iv) g is non-negative.

Then there exists an element C of S such that $C = \operatorname{dom}(f + g)$ and $\int^{+} f + g \, \mathrm{d}M = \int^{+} f{\restriction}C \, \mathrm{d}M + \int^{+} g{\restriction}C \, \mathrm{d}M$.

(85) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative. Then $0 \leq \int^{+} f \, \mathrm{d}M$.

(86) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative. Then $0 \leq \int^{+} f{\restriction}A \, \mathrm{d}M$.

(87) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and A misses B. Then $\int^{+} f{\restriction}(A \cup B) \, \mathrm{d}M = \int^{+} f{\restriction}A \, \mathrm{d}M + \int^{+} f{\restriction}B \, \mathrm{d}M$.

(88) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and $M(A) = 0$. Then $\int^{+} f{\restriction}A \, \mathrm{d}M = 0$.

(89) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and $A \subseteq B$. Then $\int^{+} f{\restriction}A \, \mathrm{d}M \leq \int^{+} f{\restriction}B \, \mathrm{d}M$.

(90) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and E, A be elements of S. Suppose f is non-negative and $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$. Then $\int^{+} f{\restriction}(E \setminus A) \, \mathrm{d}M = \int^{+} f \, \mathrm{d}M$.

(91) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) there exists an element E of S such that $E = \operatorname{dom} f$ and $E = \operatorname{dom} g$ and f is measurable on E and g is measurable on E,

(ii) f is non-negative,

(iii) g is non-negative, and

(iv) for every element x of X such that $x \in \mathrm{dom}\, g$ holds $g(x) \leq f(x)$.

Then $\int^+ g \,\mathrm{d}M \leq \int^+ f \,\mathrm{d}M$.

(92) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number. Suppose $0 \leq c$ and there exists an element A of S such that $A = \mathrm{dom}\, f$ and f is measurable on A and f is non-negative. Then $\int^+ c\, f \,\mathrm{d}M = \overline{\mathbb{R}}(c) \cdot \int^+ f \,\mathrm{d}M$.

(93) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose that

(i) there exists an element A of S such that $A = \mathrm{dom}\, f$ and f is measurable on A, and

(ii) for every element x of X such that $x \in \mathrm{dom}\, f$ holds $0 = f(x)$.

Then $\int^+ f \,\mathrm{d}M = 0$.

7. Integral of Measurable Function

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\overline{\mathbb{R}}$. The functor $\int f \,\mathrm{d}M$ yielding an element of $\overline{\mathbb{R}}$ is defined as follows:

(Def. 16) $\int f \,\mathrm{d}M = \int^+ \max_+(f) \,\mathrm{d}M - \int^+ \max_-(f) \,\mathrm{d}M$.

We now state several propositions:

(94) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \mathrm{dom}\, f$ and f is measurable on A and f is non-negative. Then $\int f \,\mathrm{d}M = \int^+ f \,\mathrm{d}M$.

(95) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and f is non-negative. Then $\int f \,\mathrm{d}M = \int^+ f \,\mathrm{d}M$ and $\int f \,\mathrm{d}M = \int' f \,\mathrm{d}M$.

(96) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \mathrm{dom}\, f$ and f is measurable on A and f is non-negative. Then $0 \leq \int f \,\mathrm{d}M$.

(97) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose there exists an element E of S such that $E = \mathrm{dom}\, f$ and f is measurable on E and f is non-negative and A misses B. Then $\int f{\restriction}(A \cup B) \,\mathrm{d}M = \int f{\restriction}A \,\mathrm{d}M + \int f{\restriction}B \,\mathrm{d}M$.

(98) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative. Then $0 \leq \int f{\restriction}A \, \mathrm{d}M$.

(99) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and $A \subseteq B$. Then $\int f{\restriction}A \, \mathrm{d}M \leq \int f{\restriction}B \, \mathrm{d}M$.

(100) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$. Then $\int f{\restriction}A \, \mathrm{d}M = 0$.

(101) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and E, A be elements of S. If $E = \operatorname{dom} f$ and f is measurable on E and $M(A) = 0$, then $\int f{\restriction}(E \setminus A) \, \mathrm{d}M = \int f \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\overline{\mathbb{R}}$. We say that f is integrable on M if and only if:

(Def. 17) There exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and $\int^{+} \max_{+}(f) \, \mathrm{d}M < +\infty$ and $\int^{+} \max_{-}(f) \, \mathrm{d}M < +\infty$.

One can prove the following propositions:

(102) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M. Then $0 \leq \int^{+} \max_{+}(f) \, \mathrm{d}M$ and $0 \leq \int^{+} \max_{-}(f) \, \mathrm{d}M$ and $-\infty < \int f \, \mathrm{d}M$ and $\int f \, \mathrm{d}M < +\infty$.

(103) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose f is integrable on M. Then $\int^{+} \max_{+}(f{\restriction}A) \, \mathrm{d}M \leq \int^{+} \max_{+}(f) \, \mathrm{d}M$ and $\int^{+} \max_{-}(f{\restriction}A) \, \mathrm{d}M \leq \int^{+} \max_{-}(f) \, \mathrm{d}M$ and $f{\restriction}A$ is integrable on M.

(104) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose f is integrable on M and A misses B. Then $\int f{\restriction}(A \cup B) \, \mathrm{d}M = \int f{\restriction}A \, \mathrm{d}M + \int f{\restriction}B \, \mathrm{d}M$.

(105) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A, B be elements of S. Suppose f is integrable on M and $B = \operatorname{dom} f \setminus A$. Then $f{\restriction}A$ is integrable on M and $\int f \, \mathrm{d}M = \int f{\restriction}A \, \mathrm{d}M + \int f{\restriction}B \, \mathrm{d}M$.

(106) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Given an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A. Then f is integrable on M if and only if $|f|$ is integrable on M.

(107) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. If f is integrable on M, then $|\int f \, \mathrm{d}M| \leq \int |f| \, \mathrm{d}M$.

(108) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose that

(i) there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A,

(ii) $\operatorname{dom} f = \operatorname{dom} g$,

(iii) g is integrable on M, and

(iv) for every element x of X such that $x \in \operatorname{dom} f$ holds $|f(x)| \leq g(x)$.

Then f is integrable on M and $\int |f| \, \mathrm{d}M \leq \int g \, \mathrm{d}M$.

(109) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and r be a real number. Suppose $\operatorname{dom} f \in S$ and $0 \leq r$ and $\operatorname{dom} f \neq \emptyset$ and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) = r$. Then $\int\limits_X f \, \mathrm{d}M = \overline{\mathbb{R}}(r) \cdot M(\operatorname{dom} f)$.

(110) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and r be a real number. Suppose $\operatorname{dom} f \in S$ and $0 \leq r$ and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) = r$. Then $\int' f \, \mathrm{d}M = \overline{\mathbb{R}}(r) \cdot M(\operatorname{dom} f)$.

(111) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M. Then $f^{-1}(\{+\infty\}) \in S$ and $f^{-1}(\{-\infty\}) \in S$ and $M(f^{-1}(\{+\infty\})) = 0$ and $M(f^{-1}(\{-\infty\})) = 0$ and $f^{-1}(\{+\infty\}) \cup f^{-1}(\{-\infty\}) \in S$ and $M(f^{-1}(\{+\infty\}) \cup f^{-1}(\{-\infty\})) = 0$.

(112) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M and g is integrable on M and f is non-negative and g is non-negative. Then $f + g$ is integrable on M.

(113) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. If f is integrable on M and g is integrable on M, then $\operatorname{dom}(f + g) \in S$.

(114) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M and g is integrable on M. Then $f + g$ is integrable on M.

(115) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\overline{\mathbb{R}}$. Suppose f is

integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and $\int f+g \,\mathrm{d}M = \int f \restriction E \,\mathrm{d}M + \int g \restriction E \,\mathrm{d}M$.

(116) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and c be a real number. Suppose f is integrable on M. Then $c\, f$ is integrable on M and $\int c\, f \,\mathrm{d}M = \overline{\mathbb{R}}(c) \cdot \int f \,\mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, let f be a partial function from X to $\overline{\mathbb{R}}$, and let B be an element of S. The functor $\int\limits_B f \,\mathrm{d}M$ yielding an element of $\overline{\mathbb{R}}$ is defined as follows:

(Def. 18) $\int\limits_B f \,\mathrm{d}M = \int f \restriction B \,\mathrm{d}M$.

The following propositions are true:

(117) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f, g be partial functions from X to $\overline{\mathbb{R}}$, and B be an element of S. Suppose f is integrable on M and g is integrable on M and $B \subseteq \mathrm{dom}(f+g)$. Then $f+g$ is integrable on M and $\int\limits_B f+g \,\mathrm{d}M = \int\limits_B f \,\mathrm{d}M + \int\limits_B g \,\mathrm{d}M$.

(118) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, c be a real number, and B be an element of S. Suppose f is integrable on M and f is measurable on B. Then $f \restriction B$ is integrable on M and $\int\limits_B c\, f \,\mathrm{d}M = \overline{\mathbb{R}}(c) \cdot \int\limits_B f \,\mathrm{d}M$.

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