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Difference and Difference Quotient

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Summary. In this article, we give the definitions of forward difference, backward difference, central difference and difference quotient, and some of their important properties.

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The articles [2], [6], [1], [13], [16], [17], [14], [4], [5], [9], [8], [12], [18], [7], [15], [11], [10], [3], and [19] provide the terminology and notation for this paper.

For simplicity, we follow the rules: n, m, i are elements of $\mathbb{N}$, h, r, r_1, r_2, x_0, x_1, x_2, x are real numbers, f is a partial function from $\mathbb{R}$ to $\mathbb{R}$, and S is a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let h be a real number. The functor $\mathrm{Shift}(f,h)$ yields a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:

(Def. 1) $\mathrm{dom}\,\mathrm{Shift}(f,h) = -h + \mathrm{dom}\,f$ and for every x such that $x \in -h + \mathrm{dom}\,f$ holds $(\mathrm{Shift}(f,h))(x) = f(x+h)$.

Let f be a function from $\mathbb{R}$ into $\mathbb{R}$ and let h be a real number. Then $\mathrm{Shift}(f,h)$ is a function from $\mathbb{R}$ into $\mathbb{R}$ and it can be characterized by the condition:

(Def. 2) For every x holds $(\mathrm{Shift}(f,h))(x) = f(x+h)$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let h be a real number. The functor $\mathrm{fD}(f,h)$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

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(Def. 3) $\mathrm{fD}(f, h) = \mathrm{Shift}(f, h) - f$.

Let f be a function from $\mathbb{R}$ into $\mathbb{R}$ and let h be a real number. Then $\mathrm{fD}(f, h)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let h be a real number. The functor $\mathrm{bD}(f, h)$ yields a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:

(Def. 4) $\mathrm{bD}(f, h) = f - \mathrm{Shift}(f, -h)$.

Let f be a function from $\mathbb{R}$ into $\mathbb{R}$ and let h be a real number. Then $\mathrm{bD}(f, h)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

We now state the proposition

(1) $\mathrm{bD}(f, h) = -\mathrm{fD}(f, -h)$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let h be a real number. The functor $\mathrm{cD}(f, h)$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by:

(Def. 5) $\mathrm{cD}(f, h) = \mathrm{Shift}(f, \frac{h}{2}) - \mathrm{Shift}(f, -\frac{h}{2})$.

Let f be a function from $\mathbb{R}$ into $\mathbb{R}$ and let h be a real number. Then $\mathrm{cD}(f, h)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let h be a real number. The forward difference of f and h yields a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ and is defined by the conditions (Def. 6).

(Def. 6)(i) (The forward difference of f and h)$(0) = f$, and

(ii) for every n holds (the forward difference of f and h)$(n + 1) = \mathrm{fD}(($the forward difference of f and $h)(n), h)$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let h be a real number. We introduce $\mathrm{fdif}(f, h)$ as a synonym of the forward difference of f and h.

In the sequel f, f_1, f_2 denote functions from $\mathbb{R}$ into $\mathbb{R}$.

The following propositions are true:

(2) For every n holds $(\mathrm{fdif}(f, h))(n)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

(3) For every x holds $(\mathrm{fD}(f, h))(x) = f(x + h) - f(x)$.

(4) For every x holds $(\mathrm{bD}(f, h))(x) = f(x) - f(x - h)$.

(5) For every x holds $(\mathrm{cD}(f, h))(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2})$.

(6) If f is constant, then for every x holds $(\mathrm{fdif}(f, h))(n + 1)(x) = 0$.

(7) $(\mathrm{fdif}(r\, f, h))(n + 1)(x) = r \cdot (\mathrm{fdif}(f, h))(n + 1)(x)$.

(8) $(\mathrm{fdif}(f_1 + f_2, h))(n+1)(x) = (\mathrm{fdif}(f_1, h))(n+1)(x) + (\mathrm{fdif}(f_2, h))(n+1)(x)$.

(9) $(\mathrm{fdif}(f_1 - f_2, h))(n+1)(x) = (\mathrm{fdif}(f_1, h))(n+1)(x) - (\mathrm{fdif}(f_2, h))(n+1)(x)$.

(10) If $f = r_1\, f_1 + r_2\, f_2$, then for every x holds $(\mathrm{fdif}(f, h))(n + 1)(x) = r_1 \cdot (\mathrm{fdif}(f_1, h))(n + 1)(x) + r_2 \cdot (\mathrm{fdif}(f_2, h))(n + 1)(x)$.

(11) For every x holds $(\mathrm{fdif}(f, h))(1)(x) = (\mathrm{Shift}(f, h))(x) - f(x)$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let h be a real number. The backward difference of f and h yielding a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ is defined by the conditions (Def. 7).

(Def. 7)(i) (The backward difference of f and h)(0) $= f$, and

(ii) for every n holds (the backward difference of f and h)($n+1$) $=$ bD((the backward difference of f and h)(n), h).

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let h be a real number. We introduce bdif(f, h) as a synonym of the backward difference of f and h.

We now state several propositions:

(12) For every n holds $(\mathrm{bdif}(f, h))(n)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

(13) If f is constant, then for every x holds $(\mathrm{bdif}(f, h))(n+1)(x) = 0$.

(14) $(\mathrm{bdif}(r\, f, h))(n+1)(x) = r \cdot (\mathrm{bdif}(f, h))(n+1)(x)$.

(15) $(\mathrm{bdif}(f_1 + f_2, h))(n+1)(x) = (\mathrm{bdif}(f_1, h))(n+1)(x) + (\mathrm{bdif}(f_2, h))(n+1)(x)$.

(16) $(\mathrm{bdif}(f_1 - f_2, h))(n+1)(x) = (\mathrm{bdif}(f_1, h))(n+1)(x) - (\mathrm{bdif}(f_2, h))(n+1)(x)$.

(17) If $f = r_1\, f_1 + r_2\, f_2$, then for every x holds $(\mathrm{bdif}(f, h))(n+1)(x) = r_1 \cdot (\mathrm{bdif}(f_1, h))(n+1)(x) + r_2 \cdot (\mathrm{bdif}(f_2, h))(n+1)(x)$.

(18) $(\mathrm{bdif}(f, h))(1)(x) = f(x) - (\mathrm{Shift}(f, -h))(x)$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let h be a real number. The central difference of f and h yielding a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ is defined by the conditions (Def. 8).

(Def. 8)(i) (The central difference of f and h)(0) $= f$, and

(ii) for every n holds (the central difference of f and h)($n+1$) $=$ cD((the central difference of f and h)(n), h).

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let h be a real number. We introduce cdif(f, h) as a synonym of the central difference of f and h.

One can prove the following propositions:

(19) For every n holds $(\mathrm{cdif}(f, h))(n)$ is a function from $\mathbb{R}$ into $\mathbb{R}$.

(20) If f is constant, then for every x holds $(\mathrm{cdif}(f, h))(n+1)(x) = 0$.

(21) $(\mathrm{cdif}(r\, f, h))(n+1)(x) = r \cdot (\mathrm{cdif}(f, h))(n+1)(x)$.

(22) $(\mathrm{cdif}(f_1 + f_2, h))(n+1)(x) = (\mathrm{cdif}(f_1, h))(n+1)(x) + (\mathrm{cdif}(f_2, h))(n+1)(x)$.

(23) $(\mathrm{cdif}(f_1 - f_2, h))(n+1)(x) = (\mathrm{cdif}(f_1, h))(n+1)(x) - (\mathrm{cdif}(f_2, h))(n+1)(x)$.

(24) If $f = r_1\, f_1 + r_2\, f_2$, then for every x holds $(\mathrm{cdif}(f, h))(n+1)(x) = r_1 \cdot (\mathrm{cdif}(f_1, h))(n+1)(x) + r_2 \cdot (\mathrm{cdif}(f_2, h))(n+1)(x)$.

(25) $(\mathrm{cdif}(f, h))(1)(x) = (\mathrm{Shift}(f, \frac{h}{2}))(x) - (\mathrm{Shift}(f, -\frac{h}{2}))(x)$.

(26) $(\mathrm{fdif}(f, h))(n)(x) = (\mathrm{bdif}(f, h))(n)(x + n \cdot h)$.

(27) $(\mathrm{fdif}(f, h))(2 \cdot n)(x) = (\mathrm{cdif}(f, h))(2 \cdot n)(x + n \cdot h)$.

(28) $(\mathrm{fdif}(f, h))(2 \cdot n + 1)(x) = (\mathrm{cdif}(f, h))(2 \cdot n + 1)(x + n \cdot h + \frac{h}{2})$.

Let f be a function from $\mathbb{R}$ into $\mathbb{R}$ and let us consider x_0, x_1. The functor $\Delta(f, x_0, x_1)$ yielding a real number is defined as follows:

(Def. 9)(i) $\Delta(f, x_0, x_1) = \frac{f(x_0)-f(x_1)}{x_0-x_1}$ if $x_0 \neq x_1$,

(ii) $x_0 \neq x_1$, otherwise.

Let x_0, x_1, x_2 be real numbers and let f be a function from $\mathbb{R}$ into $\mathbb{R}$. The functor $[!f, x_0, x_1, x_2!]$ yielding a real number is defined as follows:

(Def. 10)(i) $[!f, x_0, x_1, x_2!] = \frac{\Delta(f,x_0,x_1)-\Delta(f,x_1,x_2)}{x_0-x_2}$ if $x_0 \neq x_2$,

(ii) $x_0 \neq x_2$, otherwise.

Let x_0, x_1, x_2, x_3 be real numbers and let f be a function from $\mathbb{R}$ into $\mathbb{R}$. The functor $[!f, x_0, x_1, x_2, x_3!]$ yielding a real number is defined by:

(Def. 11)(i) $[!f, x_0, x_1, x_2, x_3!] = \frac{[!f,x_0,x_1,x_2!]-[!f,x_1,x_2,x_3!]}{x_0-x_3}$ if $x_0 \neq x_3$,

(ii) $x_0 \neq x_3$, otherwise.

We now state several propositions:

(29) If $x_0 \neq x_1$, then $\Delta(f, x_0, x_1) = \Delta(f, x_1, x_0)$.

(30) If f is constant and $x_0 \neq x_1$, then $\Delta(f, x_0, x_1) = 0$.

(31) If $x_0 \neq x_1$, then $\Delta(r\,f, x_0, x_1) = r \cdot \Delta(f, x_0, x_1)$.

(32) If $x_0 \neq x_1$, then $\Delta(f_1 + f_2, x_0, x_1) = \Delta(f_1, x_0, x_1) + \Delta(f_2, x_0, x_1)$.

(33) If $x_0 \neq x_1$, then $\Delta(r_1\,f_1 + r_2\,f_2, x_0, x_1) = r_1 \cdot \Delta(f_1, x_0, x_1) + r_2 \cdot \Delta(f_2, x_0, x_1)$.

(34) If $x_0 \neq x_1$ and $x_0 \neq x_2$ and $x_1 \neq x_2$, then $[!f, x_0, x_1, x_2!] = [!f, x_1, x_2, x_0!]$ and $[!f, x_0, x_1, x_2!] = [!f, x_2, x_1, x_0!]$.

(35) If $x_0 \neq x_1$ and $x_0 \neq x_2$ and $x_1 \neq x_2$, then $[!f, x_0, x_1, x_2!] = [!f, x_2, x_0, x_1!]$ and $[!f, x_0, x_1, x_2!] = [!f, x_1, x_0, x_2!]$.

(36) $(\mathrm{fdif}((\mathrm{fdif}(f, h))(m), h))(n)(x) = (\mathrm{fdif}(f, h))(m + n)(x)$.

Let us consider S. We say that S is sequence-yielding if and only if:

(Def. 12) For every n holds $S(n)$ is a sequence of real numbers.

Let us note that there exists a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ which is sequence-yielding.

A seq sequence is a sequence-yielding sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$.

Let S be a seq sequence and let us consider n. Then $S(n)$ is a sequence of real numbers.

In the sequel S denotes a seq sequence.

Next we state the proposition

(37) Suppose that for every n and for every i such that $i \leq n$ holds $S(n)(i) = \binom{n}{i} \cdot (\mathrm{fdif}(f_1, h))(i)(x) \cdot (\mathrm{fdif}(f_2, h))(n -' i)(x + i \cdot h)$. Then $(\mathrm{fdif}(f_1\,f_2, h))(1)(x) = \sum_{\kappa=0}^{1} S(1)(\kappa)$ and $(\mathrm{fdif}(f_1\,f_2, h))(2)(x) = \sum_{\kappa=0}^{2} S(2)(\kappa)$.

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