FORMALIZED MATHEMATICS
Volume 14, Number 4, Pages 207–212
University of Białystok, 2006

Integrability and the Integral of Partial Functions from $\mathbb{R}$ into $\mathbb{R}$[1]

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Summary. In this paper, we showed the linearity of the indefinite integral $\int_a^b f dx$, the form of which was introduced in [11]. In addition, we proved some theorems about the integral calculus on the subinterval of $[a, b]$. As a result, we described the fundamental theorem of calculus, that we developed in [11], by a more general expression.

MML identifier: `INTEGRA6`, version: `7.8.03 4.75.958`

The articles [23], [25], [26], [2], [22], [4], [14], [1], [24], [5], [27], [7], [6], [21], [9], [3], [17], [16], [15], [18], [20], [8], [10], [13], [19], [12], and [11] provide the notation and terminology for this paper.

1. Preliminaries

We use the following convention: a, b, c, d, e, x are real numbers, A is a closed-interval subset of $\mathbb{R}$, and f, g are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

We now state several propositions:

(1) If $a \leq b$ and $c \leq d$ and $a + c = b + d$, then $a = b$ and $c = d$.

(2) If $a \leq b$, then $]x - a, x + a[\subseteq]x - b, x + b[$.

[1]This work has been partially supported by the MEXT grant Grant-in-Aid for Young Scientists (B)16700156.

ISSN 1426–2630

(3) For every binary relation R and for all sets A, B, C such that $A \subseteq B$ and $A \subseteq C$ holds $R{\upharpoonright}B{\upharpoonright}A = R{\upharpoonright}C{\upharpoonright}A$.

(4) For all sets A, B, C such that $A \subseteq B$ and $A \subseteq C$ holds $\chi_{B,B}{\upharpoonright}A = \chi_{C,C}{\upharpoonright}A$.

(5) If $a \leq b$, then $\mathrm{vol}([\prime a, b\prime]) = b - a$.

(6) $\mathrm{vol}([\prime \min(a, b), \max(a, b)\prime]) = |b - a|$.

2. Integrability and the Integral of Partial Functions

The following propositions are true:

(7) If $A \subseteq \mathrm{dom}\, f$ and f is integrable on A and f is bounded on A, then $|f|$ is integrable on A and $|\int\limits_A f(x)dx| \leq \int\limits_A |f|(x)dx$.

(8) If $a \leq b$ and $[\prime a, b\prime] \subseteq \mathrm{dom}\, f$ and f is integrable on $[\prime a, b\prime]$ and f is bounded on $[\prime a, b\prime]$, then $|\int\limits_a^b f(x)dx| \leq \int\limits_a^b |f|(x)dx$.

(9) Let r be a real number. Suppose $A \subseteq \mathrm{dom}\, f$ and f is integrable on A and f is bounded on A. Then $r\,f$ is integrable on A and $\int\limits_A (r\,f)(x)dx = r \cdot \int\limits_A f(x)dx$.

(10) If $a \leq b$ and $[\prime a, b\prime] \subseteq \mathrm{dom}\, f$ and f is integrable on $[\prime a, b\prime]$ and f is bounded on $[\prime a, b\prime]$, then $\int\limits_a^b (c\,f)(x)dx = c \cdot \int\limits_a^b f(x)dx$.

(11) Suppose $A \subseteq \mathrm{dom}\, f$ and $A \subseteq \mathrm{dom}\, g$ and f is integrable on A and f is bounded on A and g is integrable on A and g is bounded on A. Then $f+g$ is integrable on A and $f-g$ is integrable on A and $\int\limits_A (f+g)(x)dx = \int\limits_A f(x)dx + \int\limits_A g(x)dx$ and $\int\limits_A (f-g)(x)dx = \int\limits_A f(x)dx - \int\limits_A g(x)dx$.

(12) Suppose that $a \leq b$ and $[\prime a, b\prime] \subseteq \mathrm{dom}\, f$ and $[\prime a, b\prime] \subseteq \mathrm{dom}\, g$ and f is integrable on $[\prime a, b\prime]$ and g is integrable on $[\prime a, b\prime]$ and f is bounded on $[\prime a, b\prime]$ and g is bounded on $[\prime a, b\prime]$. Then $\int\limits_a^b (f+g)(x)dx = \int\limits_a^b f(x)dx + \int\limits_a^b g(x)dx$

and $\int_{a}^{b}(f-g)(x)dx = \int_{a}^{b} f(x)dx - \int_{a}^{b} g(x)dx$.

(13) If f is bounded on A and g is bounded on A, then $f\,g$ is bounded on A.

(14) Suppose $A \subseteq \operatorname{dom} f$ and $A \subseteq \operatorname{dom} g$ and f is integrable on A and f is bounded on A and g is integrable on A and g is bounded on A. Then $f\,g$ is integrable on A.

(15) Let n be an element of $\mathbb{N}$. Suppose $n > 0$ and $\operatorname{vol}(A) > 0$. Then there exists an element D of $\operatorname{divs} A$ such that $\operatorname{len} D = n$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} D$ holds $D(i) = \inf A + \frac{\operatorname{vol}(A)}{n} \cdot i$.

3. Integrability on a Subinterval

The following propositions are true:

(16) Suppose $\operatorname{vol}(A) > 0$. Then there exists a DivSequence T of A such that

(i) δ_T is convergent,

(ii) $\lim(\delta_T) = 0$, and

(iii) for every element n of $\mathbb{N}$ there exists an element T_1 of $\operatorname{divs} A$ such that T_1 divides into equal $n+1$ and $T(n) = T_1$.

(17) Suppose $a \leq b$ and f is integrable on $['a,b']$ and f is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$ and $c \in ['a,b']$. Then f is integrable on $['a,c']$ and f is integrable on $['c,b']$ and $\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$.

(18) Suppose $a \leq c$ and $c \leq d$ and $d \leq b$ and f is integrable on $['a,b']$ and f is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$. Then f is integrable on $['c,d']$ and f is bounded on $['c,d']$ and $['c,d'] \subseteq \operatorname{dom} f$.

(19) Suppose that $a \leq c$ and $c \leq d$ and $d \leq b$ and f is integrable on $['a,b']$ and g is integrable on $['a,b']$ and f is bounded on $['a,b']$ and g is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$ and $['a,b'] \subseteq \operatorname{dom} g$. Then $f+g$ is integrable on $['c,d']$ and $f+g$ is bounded on $['c,d']$.

(20) Suppose $a \leq b$ and f is integrable on $['a,b']$ and f is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$ and $c \in ['a,b']$ and $d \in ['a,b']$. Then $\int_{a}^{d} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{d} f(x)dx$.

(21) Suppose $a \leq b$ and f is integrable on $['a,b']$ and f is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$ and $c \in ['a,b']$ and $d \in ['a,b']$. Then $['\min(c,d), \max(c,d)'] \subseteq \operatorname{dom}|f|$ and $|f|$ is integrable on

$['\min(c,d),\max(c,d)']$ and $|f|$ is bounded on $['\min(c,d),\max(c,d)']$ and $|\int\limits_{c}^{d} f(x)dx| \leq \int\limits_{\min(c,d)}^{\max(c,d)} |f|(x)dx$.

(22) Suppose $a \leq b$ and $c \leq d$ and f is integrable on $['a,b']$ and f is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$ and $c \in ['a,b']$ and $d \in ['a,b']$. Then $['c,d'] \subseteq \operatorname{dom}|f|$ and $|f|$ is integrable on $['c,d']$ and $|f|$ is bounded on $['c,d']$ and $|\int\limits_{c}^{d} f(x)dx| \leq \int\limits_{c}^{d} |f|(x)dx$ and $|\int\limits_{d}^{c} f(x)dx| \leq \int\limits_{c}^{d} |f|(x)dx$.

(23) Suppose that $a \leq b$ and $c \leq d$ and f is integrable on $['a,b']$ and f is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$ and $c \in ['a,b']$ and $d \in ['a,b']$ and for every real number x such that $x \in ['c,d']$ holds $|f(x)| \leq e$. Then $|\int\limits_{c}^{d} f(x)dx| \leq e \cdot (d-c)$ and $|\int\limits_{d}^{c} f(x)dx| \leq e \cdot (d-c)$.

(24) Suppose that $a \leq b$ and f is integrable on $['a,b']$ and g is integrable on $['a,b']$ and f is bounded on $['a,b']$ and g is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$ and $['a,b'] \subseteq \operatorname{dom} g$ and $c \in ['a,b']$ and $d \in ['a,b']$. Then $\int\limits_{c}^{d} (f+g)(x)dx = \int\limits_{c}^{d} f(x)dx + \int\limits_{c}^{d} g(x)dx$ and $\int\limits_{c}^{d} (f-g)(x)dx = \int\limits_{c}^{d} f(x)dx - \int\limits_{c}^{d} g(x)dx$.

(25) Suppose $a \leq b$ and f is integrable on $['a,b']$ and f is bounded on $['a,b']$ and $['a,b'] \subseteq \operatorname{dom} f$ and $c \in ['a,b']$ and $d \in ['a,b']$. Then $\int\limits_{c}^{d} (e\, f)(x)dx = e \cdot \int\limits_{c}^{d} f(x)dx$.

(26) Suppose $a \leq b$ and $['a,b'] \subseteq \operatorname{dom} f$ and for every real number x such that $x \in ['a,b']$ holds $f(x) = e$. Then f is integrable on $['a,b']$ and f is bounded on $['a,b']$ and $\int\limits_{a}^{b} f(x)dx = e \cdot (b-a)$.

(27) Suppose $a \leq b$ and for every real number x such that $x \in ['a,b']$ holds $f(x) = e$ and $['a,b'] \subseteq \operatorname{dom} f$ and $c \in ['a,b']$ and $d \in ['a,b']$. Then $\int\limits_{c}^{d} f(x)dx = e \cdot (d-c)$.

4. Fundamental Theorem of Calculus

Next we state two propositions:

(28) Let x_0 be a real number and F be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $a \leq b$ and f is integrable on $['a, b']$ and f is bounded on $['a, b']$ and $['a, b'] \subseteq \operatorname{dom} f$ and $]a, b[\subseteq \operatorname{dom} F$ and for every real number x such that $x \in]a, b[$ holds $F(x) = \int\limits_a^x f(x)dx$ and $x_0 \in]a, b[$ and f is continuous in x_0. Then F is differentiable in x_0 and $F'(x_0) = f(x_0)$.

(29) Let x_0 be a real number. Suppose $a \leq b$ and f is integrable on $['a, b']$ and f is bounded on $['a, b']$ and $['a, b'] \subseteq \operatorname{dom} f$ and $x_0 \in]a, b[$ and f is continuous in x_0. Then there exists a partial function F from $\mathbb{R}$ to $\mathbb{R}$ such that $]a, b[\subseteq \operatorname{dom} F$ and for every real number x such that $x \in]a, b[$ holds $F(x) = \int\limits_a^x f(x)dx$ and F is differentiable in x_0 and $F'(x_0) = f(x_0)$.

Acknowledgments

We would like to express our thanks to Prof. Yatsuka Nakamura.

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Received November 17, 2006