Volume 15 2007

1

Formalized Mathematics

(a computer assisted approach)

Edited by
Roman Matuszewski

University of Białystok

ISSN 1426 - 2630

Contents

Formaliz. Math. 15 (**1**)

FORMALIZED MATHEMATICS
2007, Vol. **15**, No. **1**, Pages 1–9

Several Classes of BCI-algebras and their Properties

Yuzhong Ding
Qingdao University of Science
and Technology
China

Summary. I have formalized the BCI-algebras closely following the book [6], sections 1.1 to 1.3, 1.6, 2.1 to 2.3, and 2.7. In this article the general theory of BCI-algebras and several classes of BCI-algebras are given.

MML identifier: `BCIALG_1`, version: `7.8.04 4.81.962`

The articles [10], [4], [13], [9], [3], [12], [2], [11], [5], [7], [8], [1], and [14] provide the notation and terminology for this paper.

1. The Basics of General Theory of BCI-algebras

We introduce BCI structures which are extensions of 1-sorted structure and are systems

⟨ a carrier, an internal complement ⟩,

where the carrier is a set and the internal complement is a binary operation on the carrier.

Let us note that there exists a BCI structure which is non empty and strict.

Let A be a BCI structure and let x, y be elements of A. The functor $x \setminus y$ yielding an element of A is defined by:

(Def. 1) $x \setminus y =$ (the internal complement of A)(x, y).

We introduce BCI structures with 0 which are extensions of BCI structure and zero structure and are systems

⟨ a carrier, an internal complement, a zero ⟩,

ISSN 1426–2630

where the carrier is a set, the internal complement is a binary operation on the carrier, and the zero is an element of the carrier.

Let us note that there exists a BCI structure with 0 which is non empty and strict.

Let I_1 be a non empty BCI structure with 0 and let x be an element of I_1. The functor $x^{\rm c}$ yields an element of I_1 and is defined by:

(Def. 2) $x^{\rm c} = 0_{(I_1)} \setminus x$.

Let I_1 be a non empty BCI structure with 0. We say that I_1 is B if and only if:

(Def. 3) For all elements x, y, z of I_1 holds $x \setminus y \setminus (z \setminus y) \setminus (x \setminus z) = 0_{(I_1)}$.

We say that I_1 is C if and only if:

(Def. 4) For all elements x, y, z of I_1 holds $x \setminus y \setminus z \setminus (x \setminus z \setminus y) = 0_{(I_1)}$.

We say that I_1 is I if and only if:

(Def. 5) For every element x of I_1 holds $x \setminus x = 0_{(I_1)}$.

We say that I_1 is K if and only if:

(Def. 6) For all elements x, y of I_1 holds $x \setminus y \setminus x = 0_{(I_1)}$.

We say that I_1 is BCI-4 if and only if:

(Def. 7) For all elements x, y of I_1 such that $x \setminus y = 0_{(I_1)}$ and $y \setminus x = 0_{(I_1)}$ holds $x = y$.

We say that I_1 is BCK-5 if and only if:

(Def. 8) For every element x of I_1 holds $x^{\rm c} = 0_{(I_1)}$.

The BCI structure BCI-EXAMPLE with 0 is defined as follows:

(Def. 9) BCI-EXAMPLE $= \langle \{\emptyset\}, {\rm op}_2, {\rm op}_0 \rangle$.

Let us note that BCI-EXAMPLE is strict and non empty.

One can verify that there exists a non empty BCI structure with 0 which is strict, B, C, I, and BCI-4.

A BCI-algebra is B C I BCI-4 non empty BCI structure with 0.

Let X be a BCI-algebra. A BCI-algebra is called a subalgebra of X if it satisfies the conditions (Def. 10).

(Def. 10)(i) $0_{\rm it} = 0_X$,

(ii) the carrier of it $\subseteq$ the carrier of X, and

(iii) the internal complement of it $=$ (the internal complement of X) $\restriction$ (the carrier of it).

The following proposition is true

(1) Let X be a non empty BCI structure with 0. Then X is a BCI-algebra if and only if the following conditions are satisfied:

(i) X is I and BCI-4, and

(ii) for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) \setminus (z \setminus y) = 0_X$ and $x \setminus (x \setminus y) \setminus y = 0_X$.

One can check that there exists a BCI-algebra which is strict and BCK-5.

A BCK-algebra is BCK-5 BCI-algebra.

Let I_1 be a non empty BCI structure with 0 and let x, y be elements of I_1. The predicate $x \leq y$ is defined as follows:

(Def. 11) $x \setminus y = 0_{(I_1)}$.

We use the following convention: X denotes a BCI-algebra, x, y, z, u, a, b denote elements of X, and I_1 denotes a non empty subset of X.

We now state a number of propositions:

(2) $x \setminus 0_X = x$.

(3) If $x \setminus y = 0_X$ and $y \setminus z = 0_X$, then $x \setminus z = 0_X$.

(4) If $x \setminus y = 0_X$, then $x \setminus z \setminus (y \setminus z) = 0_X$ and $z \setminus y \setminus (z \setminus x) = 0_X$.

(5) If $x \leq y$, then $x \setminus z \leq y \setminus z$ and $z \setminus y \leq z \setminus x$.

(6) If $x \setminus y = 0_X$, then $(y \setminus x)^{\rm c} = 0_X$.

(7) $x \setminus y \setminus z = x \setminus z \setminus y$.

(8) $x \setminus (x \setminus (x \setminus y)) = x \setminus y$.

(9) $(x \setminus y)^{\rm c} = x^{\rm c} \setminus y^{\rm c}$.

(10) $x \setminus (x \setminus y) \setminus (y \setminus x) \setminus (x \setminus (x \setminus (y \setminus (y \setminus x)))) = 0_X$.

(11) Let X be a non empty BCI structure with 0. Then X is a BCI-algebra if and only if the following conditions are satisfied:

(i) X is BCI-4, and

(ii) for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) \setminus (z \setminus y) = 0_X$ and $x \setminus 0_X = x$.

(12) If for every BCI-algebra X and for all elements x, y of X holds $x \setminus (x \setminus y) = y \setminus (y \setminus x)$, then X is a BCK-algebra.

(13) If for every BCI-algebra X and for all elements x, y of X holds $x \setminus y \setminus y = x \setminus y$, then X is a BCK-algebra.

(14) If for every BCI-algebra X and for all elements x, y of X holds $x \setminus (y \setminus x) = x$, then X is a BCK-algebra.

(15) If for every BCI-algebra X and for all elements x, y, z of X holds $(x \setminus y) \setminus y = x \setminus z \setminus (y \setminus z)$, then X is a BCK-algebra.

(16) If for every BCI-algebra X and for all elements x, y of X holds $x \setminus y \setminus (y \setminus x) = x \setminus y$, then X is a BCK-algebra.

(17) If for every BCI-algebra X and for all elements x, y of X holds $x \setminus y \setminus (x \setminus y \setminus (y \setminus x)) = 0_X$, then X is a BCK-algebra.

(18) For every BCI-algebra X holds X is K iff X is a BCK-algebra.

Let X be a BCI-algebra. The functor BCK-part X yielding a non empty subset of X is defined by:

(Def. 12) BCK-part $X = \{x; x$ ranges over elements of X: $0_X \leq x\}$.

Next we state the proposition

(19) $0_X \in$ BCK-part X.

Let us consider X. Note that 0_X

Next we state three propositions:

(20) For all elements x, y of BCK-part X holds $x \setminus y \in$ BCK-part X.

(21) For every element x of X and for every element y of BCK-part X holds $x \setminus y \leq x$.

(22) X is a subalgebra of X.

Let X be a BCI-algebra and let I_1 be a subalgebra of X. We say that I_1 is proper if and only if:

(Def. 13) $I_1 \neq X$.

Let us consider X. Note that there exists a subalgebra of X which is non proper.

Let X be a BCI-algebra and let I_1 be an element of X. We say that I_1 is atom if and only if:

(Def. 14) For every element z of X such that $z \setminus I_1 = 0_X$ holds $z = I_1$.

Let X be a BCI-algebra. The functor AtomSet X yields a non empty subset of X and is defined by:

(Def. 15) AtomSet $X = \{x; x$ ranges over elements of X: x is atom$\}$.

One can prove the following propositions:

(23) $0_X \in$ AtomSet X.

(24) For every element x of X holds $x \in$ AtomSet X iff for every element z of X holds $z \setminus (z \setminus x) = x$.

(25) For every element x of X holds $x \in$ AtomSet X iff for all elements u, z of X holds $z \setminus u \setminus (z \setminus x) = x \setminus u$.

(26) For every element x of X holds $x \in$ AtomSet X iff for all elements y, z of X holds $x \setminus (z \setminus y) \leq y \setminus (z \setminus x)$.

(27) For every element x of X holds $x \in$ AtomSet X iff for all elements y, z, u of X holds $(x \setminus u) \setminus (z \setminus y) \leq y \setminus u \setminus (z \setminus x)$.

(28) For every element x of X holds $x \in$ AtomSet X iff for every element z of X holds $z^{\rm c} \setminus x^{\rm c} = x \setminus z$.

(29) For every element x of X holds $x \in$ AtomSet X iff $(x^{\rm c})^{\rm c} = x$.

(30) For every element x of X holds $x \in$ AtomSet X iff for every element z of X holds $(z \setminus x)^{\rm c} = x \setminus z$.

(31) For every element x of X holds $x \in$ AtomSet X iff for every element z of X holds $((x \setminus z)^{\rm c})^{\rm c} = x \setminus z$.

(32) For every element x of X holds $x \in$ AtomSet X iff for all elements z, u of X holds $z \setminus (z \setminus (x \setminus u)) = x \setminus u$.

(33) For every element a of AtomSet X and for every element x of X holds $a \setminus x \in$ AtomSet X.

Let X be a BCI-algebra and let a, b be elements of AtomSet X. Then $a \setminus b$ is an element of AtomSet X.

One can prove the following propositions:

(34) For every element x of X holds $x^{c} \in$ AtomSet X.

(35) For every element x of X there exists an element a of AtomSet X such that $a \leq x$.

Let X be a BCI-algebra. We say that X is generated by atom if and only if:

(Def. 16) For every element x of X there exists an element a of AtomSet X such that $a \leq x$.

Let X be a BCI-algebra and let a be an element of AtomSet X. The functor BranchV a yields a non empty subset of X and is defined as follows:

(Def. 17) BranchV $a = \{x; x$ ranges over elements of X: $a \leq x\}$.

We now state several propositions:

(36) Every BCI-algebra is generated by atom.

(37) For all elements a, b of AtomSet X and for every element x of BranchV b holds $a \setminus x = a \setminus b$.

(38) For every element a of AtomSet X and for every element x of BCK-part X holds $a \setminus x = a$.

(39) For all elements a, b of AtomSet X and for every element x of BranchV a and for every element y of BranchV b holds $x \setminus y \in$ BranchV$(a \setminus b)$.

(40) For every element a of AtomSet X and for all elements x, y of BranchV a holds $x \setminus y \in$ BCK-part X.

(41) For all elements a, b of AtomSet X and for every element x of BranchV a and for every element y of BranchV b such that $a \neq b$ holds $x \setminus y \notin$ BCK-part X.

(42) For all elements a, b of AtomSet X such that $a \neq b$ holds BranchV $a \cap$ BranchV $b = \emptyset$.

Let X be a BCI-algebra. A non empty subset of X is said to be an ideal of X if:

(Def. 18) $0_X \in$ it and for all elements x, y of X such that $x \setminus y \in$ it and $y \in$ it holds $x \in$ it.

Let X be a BCI-algebra and let I_1 be an ideal of X. We say that I_1 is closed if and only if:

(Def. 19) For every element x of I_1 holds $x^{c} \in I_1$.

Let us consider X. Note that there exists an ideal of X which is closed.

Next we state four propositions:

(43) $\{0_X\}$ is a closed ideal of X.

(44) The carrier of X is a closed ideal of X.

(45) BCK-part X is a closed ideal of X.

(46) If I_1 is an ideal of X, then for all elements x, y of X such that $x \in I_1$ and $y \leq x$ holds $y \in I_1$.

2. Associative BCI-algebras

Let I_1 be a BCI-algebra. We say that I_1 is associative if and only if:

(Def. 20) For all elements x, y, z of I_1 holds $(x \setminus y) \setminus z = x \setminus (y \setminus z)$.

We say that I_1 is quasi-associative if and only if:

(Def. 21) For every element x of I_1 holds $(x^{\rm c})^{\rm c} = x^{\rm c}$.

We say that I_1 is positive-implicative if and only if:

(Def. 22) For all elements x, y of I_1 holds $(x \setminus (x \setminus y)) \setminus (y \setminus x) = x \setminus (x \setminus (y \setminus (y \setminus x)))$.

We say that I_1 is weakly-positive-implicative if and only if:

(Def. 23) For all elements x, y, z of I_1 holds $(x \setminus y) \setminus z = x \setminus z \setminus z \setminus (y \setminus z)$.

We say that I_1 is implicative if and only if:

(Def. 24) For all elements x, y of I_1 holds $(x \setminus (x \setminus y)) \setminus (y \setminus x) = y \setminus (y \setminus x)$.

We say that I_1 is weakly-implicative if and only if:

(Def. 25) For all elements x, y of I_1 holds $x \setminus (y \setminus x) \setminus (y \setminus x)^{\rm c} = x$.

We say that I_1 is p-semisimple if and only if:

(Def. 26) For all elements x, y of I_1 holds $x \setminus (x \setminus y) = y$.

We say that I_1 is alternative if and only if:

(Def. 27) For all elements x, y of I_1 holds $x \setminus (x \setminus y) = (x \setminus x) \setminus y$ and $(x \setminus y) \setminus y = x \setminus (y \setminus y)$.

One can check that there exists a BCI-algebra which is implicative, positive-implicative, p-semisimple, associative, weakly-implicative, and weakly-positive-implicative.

Next we state several propositions:

(47) X is associative iff for every element x of X holds $x^{\rm c} = x$.

(48) For all elements x, y of X holds $y \setminus x = x \setminus y$ iff X is associative.

(49) Let X be a non empty BCI structure with 0. Then X is an associative BCI-algebra if and only if for all elements x, y, z of X holds $y \setminus x \setminus (z \setminus x) = z \setminus y$ and $x \setminus 0_X = x$.

(50) Let X be a non empty BCI structure with 0. Then X is an associative BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) = z \setminus y$ and $x^{\rm c} = x$.

(51) Let X be a non empty BCI structure with 0. Then X is an associative BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) = y \setminus z$ and $x \setminus 0_X = x$.

3. p-semisimple BCI-algebras

One can prove the following propositions:

(52) X is p-semisimple iff every element of X is atom.

(53) If X is p-semisimple, then BCK-part $X = \{0_X\}$.

(54) X is p-semisimple iff for every element x of X holds $(x^{\rm c})^{\rm c} = x$.

(55) X is p-semisimple iff for all x, y holds $y \setminus (y \setminus x) = x$.

(56) X is p-semisimple iff for all x, y, z holds $z \setminus y \setminus (z \setminus x) = x \setminus y$.

(57) X is p-semisimple iff for all x, y, z holds $x \setminus (z \setminus y) = y \setminus (z \setminus x)$.

(58) X is p-semisimple iff for all x, y, z, u holds $(x \setminus u) \setminus (z \setminus y) = y \setminus u \setminus (z \setminus x)$.

(59) X is p-semisimple iff for all x, z holds $z^{\rm c} \setminus x^{\rm c} = x \setminus z$.

(60) X is p-semisimple iff for all x, z holds $((x \setminus z)^{\rm c})^{\rm c} = x \setminus z$.

(61) X is p-semisimple iff for all x, u, z holds $z \setminus (z \setminus (x \setminus u)) = x \setminus u$.

(62) X is p-semisimple iff for every x such that $x^{\rm c} = 0_X$ holds $x = 0_X$.

(63) X is p-semisimple iff for all x, y holds $x \setminus y^{\rm c} = y \setminus x^{\rm c}$.

(64) X is p-semisimple iff for all x, y, z, u holds $(x \setminus y) \setminus (z \setminus u) = x \setminus z \setminus (y \setminus u)$.

(65) X is p-semisimple iff for all x, y, z holds $x \setminus y \setminus (z \setminus y) = x \setminus z$.

(66) X is p-semisimple iff for all x, y, z holds $x \setminus (y \setminus z) = (z \setminus y) \setminus x^{\rm c}$.

(67) X is p-semisimple iff for all x, y, z such that $y \setminus x = z \setminus x$ holds $y = z$.

(68) X is p-semisimple iff for all x, y, z such that $x \setminus y = x \setminus z$ holds $y = z$.

(69) Let X be a non empty BCI structure with 0. Then X is a p-semisimple BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) = z \setminus y$ and $x \setminus 0_X = x$.

(70) Let X be a non empty BCI structure with 0. Then X is a p-semisimple BCI-algebra if and only if X is I and for all elements x, y, z of X holds $x \setminus (y \setminus z) = z \setminus (y \setminus x)$ and $x \setminus 0_X = x$.

4. Quasi-associative BCI-algebras

Next we state several propositions:

(71) X is quasi-associative iff for every element x of X holds $x^{\rm c} \leq x$.

(72) X is quasi-associative iff for all elements x, y of X holds $(x \setminus y)^{\rm c} = (y \setminus x)^{\rm c}$.

(73) X is quasi-associative iff for all elements x, y of X holds $x^{\rm c} \setminus y = (x \setminus y)^{\rm c}$.

(74) X is quasi-associative iff for all elements x, y of X holds $x \setminus y \setminus (y \setminus x) \in$ BCK-part X.

(75) X is quasi-associative iff for all elements x, y, z of X holds $(x \setminus y) \setminus z \leq x \setminus (y \setminus z)$.

5. Alternative BCI-algebras

We now state several propositions:

(76) If X is alternative, then $x^{c} = x$ and $x \setminus (x \setminus y) = y$ and $x \setminus y \setminus y = x$.

(77) If X is alternative and $x \setminus a = x \setminus b$, then $a = b$.

(78) If X is alternative and $a \setminus x = b \setminus x$, then $a = b$.

(79) If X is alternative and $x \setminus y = 0_X$, then $x = y$.

(80) If X is alternative and $x \setminus a \setminus b = 0_X$, then $a = x \setminus b$ and $b = x \setminus a$.

One can check the following observations:

* every BCI-algebra which is alternative is also associative,
* every BCI-algebra which is associative is also alternative, and
* every BCI-algebra which is alternative is also implicative.

The following two propositions are true:

(81) If X is alternative, then $x \setminus (x \setminus y) \setminus (y \setminus x) = x$.

(82) If X is alternative, then $y \setminus (y \setminus (x \setminus (x \setminus y))) = y$.

6. Implicative, Positive-Implicative, and Weakly-Positive-Implicative BCI-algebras

Let us observe that every BCI-algebra which is associative is also weakly-positive-implicative and every BCI-algebra which is p-semisimple is also weakly-positive-implicative.

We now state two propositions:

(83) Let X be a non empty BCI structure with 0. Then X is an implicative BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) \setminus (z \setminus y) = 0_X$ and $x \setminus 0_X = x$ and $(x \setminus (x \setminus y)) \setminus (y \setminus x) = y \setminus (y \setminus x)$.

(84) X is weakly-positive-implicative iff for all elements x, y of X holds $x \setminus y = x \setminus y \setminus y \setminus y^{c}$.

One can verify that every BCI-algebra which is positive-implicative is also weakly-positive-implicative and every BCI-algebra which is alternative is also weakly-positive-implicative.

One can prove the following two propositions:

(85) Suppose X is a weakly-positive-implicative BCI-algebra. Let x, y be elements of X. Then $(x \setminus (x \setminus y)) \setminus (y \setminus x) = y \setminus (y \setminus x) \setminus (y \setminus x) \setminus (x \setminus y)$.

(86) Let X be a non empty BCI structure with 0. Then X is a positive-implicative BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) \setminus (z \setminus y) = 0_X$ and $x \setminus 0_X = x$ and $x \setminus y = x \setminus y \setminus y \setminus y^{\mathrm{c}}$ and $(x \setminus (x \setminus y)) \setminus (y \setminus x) = y \setminus (y \setminus x) \setminus (y \setminus x) \setminus (x \setminus y)$.

References

[1] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.
[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[5] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[6] Jie Meng and YoungLin Liu. *An Introduction to BCI-algebras.* Shaanxi Scientific and Technological Press, 2001.
[7] Michał Muzalewski. Midpoint algebras. *Formalized Mathematics*, 1(**3**):483–488, 1990.
[8] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.
[9] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[11] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[12] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[13] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[14] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received February 23, 2007

FORMALIZED MATHEMATICS
2007, Vol. **15**, No. **1**, Pages 11–15

Formal Languages – Concatenation and Closure

Michał Trybulec
Motorola Software Group
Cracow, Poland

Summary. Formal languages are introduced as subsets of the set of all 0-based finite sequences over a given set (the alphabet). Concatenation, the n-th power and closure are defined and their properties are shown. Finally, it is shown that the closure of the alphabet (understood here as the language of words of length 1) equals to the set of all words over that alphabet, and that the alphabet is the minimal set with this property. Notation and terminology were taken from [5] and [13].

MML identifier: `FLANG_1`, version: `7.8.04 4.81.962`

The terminology and notation used here are introduced in the following articles: [10], [4], [11], [8], [9], [2], [14], [3], [1], [6], [12], and [7].

1. Preliminaries

For simplicity, we follow the rules: E is a set, x is a set, A, B, C, D are subsets of E^ω, a, b, c are elements of E^ω, e is an element of E, i, n, n_1, n_2, m are natural numbers, and p, q, r_1, r_2 are real numbers.

Let us consider E, a, b. Then $a \frown b$ is an element of E^ω.

Let us consider E. Then $\langle\rangle_E$ is an element of E^ω.

Let E be a non empty set and let e be an element of E. Then $\langle e \rangle$ is an element of E^ω.

Let us consider E, a. Then $\{a\}$ is a subset of E^ω.

Let us consider E, let f be a function from $\mathbb{N}$ into 2^{E^ω}, and let us consider n. Then $f(n)$ is a subset of E^ω.

One can prove the following propositions:

ISSN 1426–2630

(1) If $\{x\} \not\subseteq X$, then $\{x\}$ misses X.
(2) If $n_1 > 1$ or $n_2 > 1$, then $n_1 + n_2 > 1$.
(3) $n > 0$ iff $n \geq 1$.
(4) If $r_1 + p \leq r_2 + q$ and $p \geq q$, then $r_1 \leq r_2$.
(5) If $n_1 + n \leq n_2 + 1$ and $n > 0$, then $n_1 \leq n_2$.
(6) $n_1 + n_2 = 1$ iff $n_1 = 1$ and $n_2 = 0$ or $n_1 = 0$ and $n_2 = 1$.
(7) $a \frown b = \langle x \rangle$ iff $a = \langle\rangle_E$ and $b = \langle x \rangle$ or $b = \langle\rangle_E$ and $a = \langle x \rangle$.
(8) For all finite 0-sequences p, q such that $a = p \frown q$ holds p is an element of E^ω and q is an element of E^ω.
(9) If $\langle x \rangle$ is an element of E^ω, then $x \in E$.
(10) If $\operatorname{len} b = n + 1$, then there exist c, e such that $\operatorname{len} c = n$ and $b = c \frown \langle e \rangle$.
(11) If $a \frown a = a$, then $a = \emptyset$.

2. Concatenation of Languages

Let us consider E, A, B. The functor $A \frown B$ yields a subset of E^ω and is defined by:

(Def. 1) $x \in A \frown B$ iff there exist a, b such that $a \in A$ and $b \in B$ and $x = a \frown b$.

The following propositions are true:

(12) $A \frown B = \emptyset$ iff $A = \emptyset$ or $B = \emptyset$.
(13) $A \frown \{\langle\rangle_E\} = A$ and $\{\langle\rangle_E\} \frown A = A$.
(14) $A \frown B = \{\langle\rangle_E\}$ iff $A = \{\langle\rangle_E\}$ and $B = \{\langle\rangle_E\}$.
(15) $\langle\rangle_E \in A \frown B$ iff $\langle\rangle_E \in A$ and $\langle\rangle_E \in B$.
(16) If $\langle\rangle_E \in B$, then $A \subseteq A \frown B$ and $A \subseteq B \frown A$.
(17) If $A \subseteq C$ and $B \subseteq D$, then $A \frown B \subseteq C \frown D$.
(18) $(A \frown B) \frown C = A \frown (B \frown C)$.
(19) $A \frown (B \cap C) \subseteq (A \frown B) \cap (A \frown C)$ and $(B \cap C) \frown A \subseteq (B \frown A) \cap (C \frown A)$.
(20) $A \frown B \cup A \frown C = A \frown (B \cup C)$ and $B \frown A \cup C \frown A = (B \cup C) \frown A$.
(21) $A \frown B \setminus A \frown C \subseteq A \frown (B \setminus C)$ and $B \frown A \setminus C \frown A \subseteq (B \setminus C) \frown A$.
(22) $A \frown B \dot{-} A \frown C \subseteq A \frown (B \dot{-} C)$ and $B \frown A \dot{-} C \frown A \subseteq (B \dot{-} C) \frown A$.

3. n-th Power of a Language

Let us consider E, A, n. The functor A^n yields a subset of E^ω and is defined by:

(Def. 2) There exists a function c_1 from $\mathbb{N}$ into 2^{E^ω} such that $A^n = c_1(n)$ and $c_1(0) = \{\langle\rangle_E\}$ and for every i holds $c_1(i + 1) = c_1(i) \frown A$.

Next we state a number of propositions:

(23) $A^{n+1} = (A^n) \frown A$.

(24) $A^0 = \{\langle\rangle_E\}$.

(25) $A^1 = A$.

(26) $A^2 = A \frown A$.

(27) If $n \geq 1$, then $(\emptyset_{E^\omega})^n = \emptyset$.

(28) $\{\langle\rangle_E\}^n = \{\langle\rangle_E\}$.

(29) $A^n = \{\langle\rangle_E\}$ iff $n = 0$ or $A = \{\langle\rangle_E\}$.

(30) If $\langle\rangle_E \in A$, then $\langle\rangle_E \in A^n$.

(31) $(A^n) \frown A = A \frown A^n$.

(32) $A^{m+n} = (A^m) \frown A^n$.

(33) $(A^m)^n = A^{m \cdot n}$.

(34) If $\langle\rangle_E \in A$ and $n > 0$, then $A \subseteq A^n$.

(35) If $\langle\rangle_E \in A$ and $n > 0$ and $m > n$, then $A^n \subseteq A^m$.

(36) If $A \subseteq B$, then $A^n \subseteq B^n$.

(37) $A^n \cup B^n \subseteq (A \cup B)^n$.

(38) $(A \cap B)^n \subseteq A^n \cap B^n$.

(39) If $a \in C^m$ and $b \in C^n$, then $a \frown b \in C^{m+n}$.

4. Closure of a Language

Let us consider E, A. The functor A^* yielding a subset of E^ω is defined as follows:

(Def. 3) $A^* = \bigcup\{B : \bigvee_n \ B = A^n\}$.

The following propositions are true:

(40) $x \in A^*$ iff there exists n such that $x \in A^n$.

(41) $A^n \subseteq A^*$.

(42) If $x \in A$, then $x \in A^*$.

(43) $A \subseteq A^*$.

(44) $A \frown A \subseteq A^*$.

(45) If $a \in C^*$ and $b \in C^*$, then $a \frown b \in C^*$.

(46) If $A \subseteq C^*$ and $B \subseteq C^*$, then $A \frown B \subseteq C^*$.

(47) $A^* = \{\langle\rangle_E\}$ iff $A = \emptyset$ or $A = \{\langle\rangle_E\}$.

(48) $\langle\rangle_E \in A^*$.

(49) If $A^* = \{x\}$, then $x = \langle\rangle_E$.

(50) If $x \in A^{m+1}$, then $x \in (A^*) \frown A$ and $x \in A \frown A^*$.

(51) If $x \in (A^*) \frown A$ or $x \in A \frown A^*$, then $x \in A^*$.

(52) If $\langle\rangle_E \in A$, then $A^* = (A^*) \frown A$ and $A^* = A \frown A^*$.

(53) If $\langle\rangle_E \in A$, then $A^* = (A^*) \frown A^n$ and $A^* = (A^n) \frown A^*$.

(54) $A^* = \{\langle\rangle_E\} \cup A \frown A^*$ and $A^* = \{\langle\rangle_E\} \cup (A^*) \frown A$.

(55) $A \frown A^* = (A^*) \frown A$.

(56) $(A^n) \frown A^* = (A^*) \frown A^n$.

(57) If $A \subseteq B^*$, then $A^n \subseteq B^*$.

(58) If $A \subseteq B^*$, then $A^* \subseteq B^*$.

(59) If $A \subseteq B$, then $A^* \subseteq B^*$.

(60) $(A^*)^* = A^*$.

(61) $(A^*) \frown A^* = A^*$.

(62) $(A^n)^* \subseteq A^*$.

(63) $(A^*)^n \subseteq A^*$.

(64) If $n > 0$, then $(A^*)^n = A^*$.

(65) If $A \subseteq B^*$, then $B^* = (B \cup A)^*$.

(66) If $a \in A^*$, then $A^* = (A \cup \{a\})^*$.

(67) $A^* = (A \setminus \{\langle\rangle_E\})^*$.

(68) $A^* \cup B^* \subseteq (A \cup B)^*$.

(69) $(A \cap B)^* \subseteq A^* \cap B^*$.

(70) $\langle x\rangle \in A^*$ iff $\langle x\rangle \in A$.

5. Alphabet as a Language

Let us consider E. The functor $\mathrm{Lex}\, E$ yielding a subset of E^ω is defined by:

(Def. 4) $x \in \mathrm{Lex}\, E$ iff there exists e such that $e \in E$ and $x = \langle e\rangle$.

Next we state three propositions:

(71) $a \in (\mathrm{Lex}\, E)^{\mathrm{len}\, a}$.

(72) $(\mathrm{Lex}\, E)^* = E^\omega$.

(73) If $A^* = E^\omega$, then $\mathrm{Lex}\, E \subseteq A$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[5] John E. Hopcroft and Jeffrey D. Ullman. *Introduction to Automata Theory, Languages and Computation*. Addison-Wesley Publishing Company, 1979.
[6] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[7] Karol Pąk. The Catalan numbers. Part II. *Formalized Mathematics*, 14(**4**):153–159, 2006.

[8] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[9] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[11] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[12] Tetsuya Tsunetou, Grzegorz Bancerek, and Yatsuka Nakamura. Zero-based finite sequences. *Formalized Mathematics*, 9(**4**):825–829, 2001.
[13] William M. Waite and Gerhard Goos. *Compiler Construction.* Springer-Verlag New York Inc., 1984.
[14] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received March 9, 2007

FORMALIZED MATHEMATICS
2007, Vol. 15, No. 1, Pages 17–25

Basic Properties of Determinants of Square Matrices over a Field[1]

Karol Pąk
Institute of Computer Science
University of Białystok
Poland

Summary. In this paper I present basic properties of the determinant of square matrices over a field and selected properties of the sign of a permutation. First, I define the sign of a permutation by the requirement

$$\mathrm{sgn}(p) = \prod_{1 \leq i < j \leq n} \mathrm{sgn}(p(j) - p(i)),$$

where p is any fixed permutation of a set with n elements. I prove that the sign of a product of two permutations is the same as the product of their signs and show the relation between signs and parity of permutations. Then I consider the determinant of a linear combination of lines, the determinant of a matrix with permutated lines and the determinant of a matrix with a repeated line. Finally, at the end I prove that the determinant of a product of two square matrices is equal to the product of their determinants.

MML identifier: `MATRIX11`, version: `7.8.04 4.81.962`

The articles [21], [12], [27], [18], [13], [28], [7], [10], [8], [3], [4], [19], [25], [24], [16], [20], [11], [6], [5], [14], [22], [15], [31], [23], [26], [32], [1], [29], [9], [2], [17], and [30] provide the terminology and notation for this paper.

[1]This work has been partially supported by the KBN grant 4 T11C 039 24 and the FP6 IST grant TYPES No. 510996.

ISSN 1426–2630

1. The Sign of a Permutation

For simplicity, we use the following convention: x, X denote sets, i, j, k, l, n, m denote natural numbers, D denotes a non empty set, K denotes a field, a, b denote elements of K, p_1, p, q denote elements of the permutations of n-element set, P_1, P denote permutations of $\mathrm{Seg}\,n$, F denotes a function from $\mathrm{Seg}\,n$ into $\mathrm{Seg}\,n$, p_2, p_3, q_2, p_4 denote elements of the permutations of $(n+2)$-element set, and P_2 denotes a permutation of $\mathrm{Seg}(n+2)$.

Let X be a set. We introduce $\mathrm{2Set}\,X$ as a synonym of $\mathrm{TwoElementSets}(X)$.

The following three propositions are true:

(1) $X \in \mathrm{2Set}\,\mathrm{Seg}\,n$ iff there exist i, j such that $i \in \mathrm{Seg}\,n$ and $j \in \mathrm{Seg}\,n$ and $i < j$ and $X = \{i, j\}$.

(2) $\mathrm{2Set}\,\mathrm{Seg}\,0 = \emptyset$ and $\mathrm{2Set}\,\mathrm{Seg}\,1 = \emptyset$.

(3) For every n such that $n \geq 2$ holds $\{1, 2\} \in \mathrm{2Set}\,\mathrm{Seg}\,n$.

Let us consider n. Observe that $\mathrm{2Set}\,\mathrm{Seg}(n+2)$ is non empty and finite.

Let us consider n, x and let p_1 be an element of the permutations of n-element set. Note that $p_1(x)$ is natural.

Let us consider K. One can verify that the multiplication of K is unital and the multiplication of K is associative.

Let us consider n, K and let p_2 be an element of the permutations of $(n+2)$-element set. The functor $\text{Part-sgn}(p_2, K)$ yielding a function from $\mathrm{2Set}\,\mathrm{Seg}(n+2)$ into the carrier of K is defined by the condition (Def. 1).

(Def. 1) Let i, j be elements of $\mathbb{N}$ such that $i \in \mathrm{Seg}(n+2)$ and $j \in \mathrm{Seg}(n+2)$ and $i < j$. Then

(i) if $p_2(i) < p_2(j)$, then $(\text{Part-sgn}(p_2, K))(\{i, j\}) = \mathbf{1}_K$, and

(ii) if $p_2(i) > p_2(j)$, then $(\text{Part-sgn}(p_2, K))(\{i, j\}) = -\mathbf{1}_K$.

One can prove the following proposition

(4) Let X be an element of $\mathrm{Fin}\,\mathrm{2Set}\,\mathrm{Seg}(n+2)$. Suppose that for every x such that $x \in X$ holds $(\text{Part-sgn}(p_3, K))(x) = \mathbf{1}_K$. Then (the multiplication of K)-$\sum_X \text{Part-sgn}(p_3, K) = \mathbf{1}_K$.

In the sequel s denotes an element of $\mathrm{2Set}\,\mathrm{Seg}(n+2)$.

The following propositions are true:

(5) $(\text{Part-sgn}(p_3, K))(s) = \mathbf{1}_K$ or $(\text{Part-sgn}(p_3, K))(s) = -\mathbf{1}_K$.

(6) For all i, j such that $i \in \mathrm{Seg}(n+2)$ and $j \in \mathrm{Seg}(n+2)$ and $i < j$ and $p_3(i) = q_2(i)$ and $p_3(j) = q_2(j)$ holds $(\text{Part-sgn}(p_3, K))(\{i, j\}) = (\text{Part-sgn}(q_2, K))(\{i, j\})$.

(7) Let X be an element of $\mathrm{Fin}\,\mathrm{2Set}\,\mathrm{Seg}(n+2)$, given p_3, q_2, and F be a finite set such that $F = \{s : s \in X \wedge (\text{Part-sgn}(p_3, K))(s) \neq (\text{Part-sgn}(q_2, K))(s)\}$. Then

(i) if card $F \bmod 2 = 0$, then (the multiplication of K)- $\sum_X$ Part-sgn(p_3, K) = (the multiplication of K)- $\sum_X$ Part-sgn(q_2, K), and

(ii) if card $F \bmod 2 = 1$, then (the multiplication of K)- $\sum_X$ Part-sgn(p_3, K) = $-$((the multiplication of K)- $\sum_X$ Part-sgn(q_2, K)).

(8) Let P be a permutation of Seg n. Suppose P is a transposition. Let given i, j. Suppose $i < j$. Then $P(i) = j$ if and only if the following conditions are satisfied:

(i) $i \in \operatorname{dom} P$,

(ii) $j \in \operatorname{dom} P$,

(iii) $P(i) = j$,

(iv) $P(j) = i$, and

(v) for every k such that $k \neq i$ and $k \neq j$ and $k \in \operatorname{dom} P$ holds $P(k) = k$.

(9) Let given p_3, q_2, p_4, i, j. Suppose $p_4 = p_3 \cdot q_2$ and q_2 is a transposition and $q_2(i) = j$ and $i < j$. Let given s. If (Part-sgn$(p_3, K))(s) \neq$ (Part-sgn$(p_4, K))(s)$, then $i \in s$ or $j \in s$.

(10) Let given p_3, q_2, p_4, i, j, K. Suppose $p_4 = p_3 \cdot q_2$ and q_2 is a transposition and $q_2(i) = j$ and $i < j$ and $\mathbf{1}_K \neq -\mathbf{1}_K$. Then

(i) (Part-sgn$(p_3, K))(\{i, j\}) \neq$ (Part-sgn$(p_4, K))(\{i, j\})$, and

(ii) for every k such that $k \in \operatorname{Seg}(n+2)$ and $i \neq k$ and $j \neq k$ holds (Part-sgn$(p_3, K))(\{i, k\}) \neq$ (Part-sgn$(p_4, K))(\{i, k\})$ iff (Part-sgn$(p_3, K))(\{j, k\}) \neq$ (Part-sgn$(p_4, K))(\{j, k\})$.

Let us consider n, K and let p_2 be an element of the permutations of $(n+2)$-element set. The functor sgn(p_2, K) yielding an element of K is defined by:

(Def. 2) sgn(p_2, K) = (the multiplication of K)- $\sum_{\Omega^{\rm f}_{2\mathrm{Set}\,\mathrm{Seg}(n+2)}}$ Part-sgn(p_2, K).

The following propositions are true:

(11) sgn$(p_3, K) = \mathbf{1}_K$ or sgn$(p_3, K) = -\mathbf{1}_K$.

(12) For every element I_1 of the permutations of $(n+2)$-element set such that $I_1 = \operatorname{idseq}(n+2)$ holds sgn$(I_1, K) = \mathbf{1}_K$.

(13) For all p_3, q_2, p_4 such that $p_4 = p_3 \cdot q_2$ and q_2 is a transposition holds sgn$(p_4, K) = -$sgn(p_3, K).

(14) For every element t_1 of the permutations of $(n+2)$-element set such that t_1 is a transposition holds sgn$(t_1, K) = -\mathbf{1}_K$.

(15) Let P be a finite sequence of elements of A_{n+2} and p_3 be an element of the permutations of $(n+2)$-element set such that $p_3 = \prod P$ and for every i such that $i \in \operatorname{dom} P$ there exists an element t_2 of the permutations of $(n+2)$-element set such that $P(i) = t_2$ and t_2 is a transposition. Then

(i) if len $P \bmod 2 = 0$, then sgn$(p_3, K) = \mathbf{1}_K$, and

(ii) if len $P \bmod 2 = 1$, then sgn$(p_3, K) = -\mathbf{1}_K$.

(16) Let given i, j, n. Suppose $i < j$ and $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$. Then there exists an element t_1 of the permutations of n-element set such that t_1 is a

transposition and $t_1(i) = j$.

(17) Let p be an element of the permutations of $(k+1)$-element set. Suppose $p(k+1) \neq k+1$. Then there exists an element t_1 of the permutations of $(k+1)$-element set such that t_1 is a transposition and $t_1(p(k+1)) = k+1$ and $(t_1 \cdot p)(k+1) = k+1$.

(18) Let given X, x. Suppose $x \notin X$. Let p_5 be a permutation of $X \cup \{x\}$. If $p_5(x) = x$, then there exists a permutation p of X such that $p_5 \restriction X = p$.

(19) Let p, q be permutations of X and p_5, q_1 be permutations of $X \cup \{x\}$. If $p_5 \restriction X = p$ and $q_1 \restriction X = q$ and $p_5(x) = x$ and $q_1(x) = x$, then $(p_5 \cdot q_1) \restriction X = p \cdot q$ and $(p_5 \cdot q_1)(x) = x$.

(20) For every element t_1 of the permutations of k-element set such that t_1 is a transposition holds $t_1 \cdot t_1 = \mathrm{idseq}(k)$ and $t_1 = {t_1}^{-1}$.

(21) Let given p_1. Then there exists a finite sequence P of elements of A_n such that

(i) $p_1 = \prod P$, and

(ii) for every i such that $i \in \mathrm{dom}\, P$ there exists an element t_2 of the permutations of n-element set such that $P(i) = t_2$ and t_2 is a transposition.

(22) K is Fanoian iff $\mathbf{1}_K \neq -\mathbf{1}_K$.

(23) For every Fanoian field K holds p_2 is even iff $\mathrm{sgn}(p_2, K) = \mathbf{1}_K$ and p_2 is odd iff $\mathrm{sgn}(p_2, K) = -\mathbf{1}_K$.

(24) For all p_3, q_2, p_4 such that $p_4 = p_3 \cdot q_2$ holds $\mathrm{sgn}(p_4, K) = \mathrm{sgn}(p_3, K) \cdot \mathrm{sgn}(q_2, K)$.

(25) p is even and q is even or p is odd and q is odd iff $p \cdot q$ is even.

(26) $(-1)^{\mathrm{sgn}(p_2)} a = \mathrm{sgn}(p_2, K) \cdot a$.

(27) For every element t_1 of the permutations of $(n+2)$-element set such that t_1 is a transposition holds t_1 is odd.

Let us consider n. Observe that there exists a permutation of $\mathrm{Seg}(n+2)$ which is odd.

2. The Determinant of a Linear Combination of Lines

For simplicity, we follow the rules: p_6 denotes a finite sequence of elements of D, M denotes a matrix over D of dimension $n \times m$, p_7, q_3 denote finite sequences of elements of K, and A, B denote matrices over K of dimension n.

Let us consider l, n, m, D, let M be a matrix over D of dimension $n \times m$, and let p_6 be a finite sequence of elements of D. The functor $\mathrm{ReplaceLine}(M, l, p_6)$ yields a matrix over D of dimension $n \times m$ and is defined as follows:

(Def. 3)(i) $\mathrm{len}\,\mathrm{ReplaceLine}(M, l, p_6) = \mathrm{len}\, M$ and $\mathrm{width}\,\mathrm{ReplaceLine}(M, l, p_6) = \mathrm{width}\, M$ and for all i, j such that $\langle i, j \rangle \in$ the indices of M holds if $i \neq l$, then $(\mathrm{ReplaceLine}(M, l, p_6))_{i,j} = M_{i,j}$ and if $i = l$, then $(\mathrm{ReplaceLine}(M, l, p_6))_{l,j} = p_6(j)$ if $\mathrm{len}\, p_6 = \mathrm{width}\, M$,

(ii) $\mathrm{ReplaceLine}(M, l, p_6) = M$, otherwise.

Let us consider l, n, m, D, let M be a matrix over D of dimension $n \times m$, and let p_6 be a finite sequence of elements of D. We introduce $\mathrm{RLine}(M, l, p_6)$ as a synonym of $\mathrm{ReplaceLine}(M, l, p_6)$.

The following propositions are true:

(28) For all l, M, p_6, i such that $i \in \mathrm{Seg}\, n$ holds if $i = l$ and $\mathrm{len}\, p_6 = \mathrm{width}\, M$, then $\mathrm{Line}(\mathrm{RLine}(M, l, p_6), i) = p_6$ and if $i \neq l$, then $\mathrm{Line}(\mathrm{RLine}(M, l, p_6), i) = \mathrm{Line}(M, i)$.

(29) For all M, p_6 such that $\mathrm{len}\, p_6 = \mathrm{width}\, M$ and for every element p' of D^* such that $p_6 = p'$ holds $\mathrm{RLine}(M, l, p_6) = \mathrm{Replace}(M, l, p')$.

(30) $M = \mathrm{RLine}(M, l, \mathrm{Line}(M, l))$.

(31) Let given l, p_7, q_3, p_1. Suppose $l \in \mathrm{Seg}\, n$ and $\mathrm{len}\, p_7 = n$ and $\mathrm{len}\, q_3 = n$. Let M be a matrix over K of dimension n. Then (the multiplication of K) $\circledast$ (p_1-Path $\mathrm{RLine}(M, l, a \cdot p_7 + b \cdot q_3)$) $= a \cdot$ ((the multiplication of K) $\circledast$ (p_1-Path $\mathrm{RLine}(M, l, p_7)$)) $+ b \cdot$ ((the multiplication of K) $\circledast$ (p_1-Path $\mathrm{RLine}(M, l, q_3)$)).

(32) Let given l, p_7, q_3, p_1. Suppose $l \in \mathrm{Seg}\, n$ and $\mathrm{len}\, p_7 = n$ and $\mathrm{len}\, q_3 = n$. Let M be a matrix over K of dimension n. Then (the product on paths of $\mathrm{RLine}(M, l, a \cdot p_7 + b \cdot q_3)$)$(p_1) = a \cdot$ (the product on paths of $\mathrm{RLine}(M, l, p_7)$)$(p_1) + b \cdot$ (the product on paths of $\mathrm{RLine}(M, l, q_3)$)$(p_1)$.

(33) Let given l, p_7, q_3. Suppose $l \in \mathrm{Seg}\, n$ and $\mathrm{len}\, p_7 = n$ and $\mathrm{len}\, q_3 = n$. Let M be a matrix over K of dimension n. Then $\mathrm{Det}\, \mathrm{RLine}(M, l, a \cdot p_7 + b \cdot q_3) = a \cdot \mathrm{Det}\, \mathrm{RLine}(M, l, p_7) + b \cdot \mathrm{Det}\, \mathrm{RLine}(M, l, q_3)$.

(34) If $l \in \mathrm{Seg}\, n$ and $\mathrm{len}\, p_7 = n$, then $\mathrm{Det}\, \mathrm{RLine}(A, l, a \cdot p_7) = a \cdot \mathrm{Det}\, \mathrm{RLine}(A, l, p_7)$.

(35) If $l \in \mathrm{Seg}\, n$, then $\mathrm{Det}\, \mathrm{RLine}(A, l, a \cdot \mathrm{Line}(A, l)) = a \cdot \mathrm{Det}\, A$.

(36) If $l \in \mathrm{Seg}\, n$ and $\mathrm{len}\, p_7 = n$ and $\mathrm{len}\, q_3 = n$, then $\mathrm{Det}\, \mathrm{RLine}(A, l, p_7 + q_3) = \mathrm{Det}\, \mathrm{RLine}(A, l, p_7) + \mathrm{Det}\, \mathrm{RLine}(A, l, q_3)$.

3. The Determinant of a Matrix with Permutated Lines and with a Repeated Line

Let us consider n, m, D, let F be a function from $\mathrm{Seg}\, n$ into $\mathrm{Seg}\, n$, and let M be a matrix over D of dimension $n \times m$. Then $M \cdot F$ is a matrix over D of dimension $n \times m$ and it can be characterized by the condition:

(Def. 4) $\mathrm{len}(M \cdot F) = \mathrm{len}\, M$ and $\mathrm{width}(M \cdot F) = \mathrm{width}\, M$ and for all i, j, k such that $\langle i, j \rangle \in$ the indices of M and $F(i) = k$ holds $(M \cdot F)_{i,j} = M_{k,j}$.

The following propositions are true:

(37)(i) The indices of M = the indices of $M \cdot F$, and

(ii) for all i, j such that $\langle i, j\rangle \in$ the indices of M there exists k such that $F(i) = k$ and $\langle k, j\rangle \in$ the indices of M and $(M \cdot F)_{i,j} = M_{k,j}$.

(38) For every matrix M over D of dimension $n \times m$ and for every F and for every k such that $k \in \mathrm{Seg}\, n$ holds $\mathrm{Line}(M \cdot F, k) = M(F(k))$.

(39) $M \cdot \mathrm{idseq}(n) = M$.

(40) For all p, P_1, q such that $q = p \cdot {P_1}^{-1}$ holds p-Path $A \cdot P_1 = (q$-Path $A) \cdot P_1$.

(41) For all p, P_1, q such that $q = p \cdot {P_1}^{-1}$ holds (the multiplication of K) $\circledast$ (p-Path $A \cdot P_1$) = (the multiplication of K) $\circledast$ (q-Path A).

(42) For all p_3, q_2 such that $q_2 = {p_3}^{-1}$ holds $\mathrm{sgn}(p_3, K) = \mathrm{sgn}(q_2, K)$.

(43) Let M be a matrix over K of dimension $n+2$ and given p_2, P_2. Suppose $p_2 = P_2$. Let given p_3, q_2. Suppose $q_2 = p_3 \cdot {P_2}^{-1}$. Then (the product on paths of M)$(q_2) = \mathrm{sgn}(p_2, K) \cdot$ (the product on paths of $M \cdot P_2$)(p_3).

(44) Let given p_1. Then there exists a permutation P of the permutations of n-element set such that for every element p of the permutations of n-element set holds $P(p) = p \cdot p_1$.

(45) For every matrix M over K of dimension $n+2 \times n+2$ and for all p_2, P_2 such that $p_2 = P_2$ holds $\mathrm{Det}(M \cdot P_2) = \mathrm{sgn}(p_2, K) \cdot \mathrm{Det}\, M$.

(46) For every matrix M over K of dimension n and for all p_1, P_1 such that $p_1 = P_1$ holds $\mathrm{Det}(M \cdot P_1) = (-1)^{\mathrm{sgn}(p_1)}\, \mathrm{Det}\, M$.

(47) Let P_3 be a permutation of the permutations of n-element set and given p_1. If p_1 is odd and for every p holds $P_3(p) = p \cdot p_1$, then $P_3{}^\circ\{p : p$ is even$\} = \{q : q$ is odd$\}$.

(48) Let given n. Suppose $n \geq 2$. Then there exist finite sets O_1, E_1 such that $E_1 = \{p : p$ is even$\}$ and $O_1 = \{q : q$ is odd$\}$ and $E_1 \cap O_1 = \emptyset$ and $E_1 \cup O_1$ = the permutations of n-element set and $\mathrm{card}\, E_1 = \mathrm{card}\, O_1$.

(49) Let given i, j. Suppose $i \in \mathrm{Seg}\, n$ and $j \in \mathrm{Seg}\, n$ and $i < j$. Let M be a matrix over K of dimension n. Suppose $\mathrm{Line}(M, i) = \mathrm{Line}(M, j)$. Let p, q, t_1 be elements of the permutations of n-element set. Suppose $q = p \cdot t_1$ and t_1 is a transposition and $t_1(i) = j$. Then (the product on paths of M)$(q) = -$(the product on paths of M)(p).

(50) Let given i, j. Suppose $i \in \mathrm{Seg}\, n$ and $j \in \mathrm{Seg}\, n$ and $i < j$. Let M be a matrix over K of dimension n. If $\mathrm{Line}(M, i) = \mathrm{Line}(M, j)$, then $\mathrm{Det}\, M = 0_K$.

(51) For all i, j such that $i \in \mathrm{Seg}\, n$ and $j \in \mathrm{Seg}\, n$ and $i \neq j$ holds $\mathrm{Det}\, \mathrm{RLine}(A, i, \mathrm{Line}(A, j)) = 0_K$.

(52) For all i, j such that $i \in \mathrm{Seg}\, n$ and $j \in \mathrm{Seg}\, n$ and $i \neq j$ holds $\mathrm{Det}\, \mathrm{RLine}(A, i, a \cdot \mathrm{Line}(A, j)) = 0_K$.

(53) For all i, j such that $i \in \mathrm{Seg}\, n$ and $j \in \mathrm{Seg}\, n$ and $i \neq j$ holds $\mathrm{Det}\, A =$

Det RLine$(A, i, \mathrm{Line}(A, i) + a \cdot \mathrm{Line}(A, j))$.

(54) If $F \notin$ the permutations of n-element set, then $\mathrm{Det}(A \cdot F) = 0_K$.

4. The Determinant of a Product of Two Square Matrices

Let K be a non empty loop structure. The functor addFinS K yielding a binary operation on (the carrier of K)* is defined as follows:

(Def. 5) For all elements p_5, p_3 of (the carrier of K)* holds (addFinS K)$(p_5, p_3) = p_5 + p_3$.

Let K be an Abelian non empty loop structure. One can verify that addFinS K is commutative.

Let K be an add-associative non empty loop structure. Note that addFinS K is associative.

The following propositions are true:

(55) Let A, B be matrices over K. Suppose width $A = \mathrm{len}\, B$ and $\mathrm{len}\, B > 0$. Let given i. Suppose $i \in \mathrm{Seg}\,\mathrm{len}\, A$. Then there exists a finite sequence P of elements of (the carrier of K)* such that $\mathrm{len}\, P = \mathrm{len}\, B$ and $\mathrm{Line}(A \cdot B, i) = \mathrm{addFinS}\, K \odot P$ and for every j such that $j \in \mathrm{Seg}\,\mathrm{len}\, B$ holds $P(j) = A_{i,j} \cdot \mathrm{Line}(B, j)$.

(56) Let A, B, C be matrices over K of dimension n and given i. Suppose $i \in \mathrm{Seg}\, n$. Then there exists a finite sequence P of elements of K such that $\mathrm{len}\, P = n$ and Det RLine$(C, i, \mathrm{Line}(A \cdot B, i))$ = the addition of $K \odot P$ and for every j such that $j \in \mathrm{Seg}\, n$ holds $P(j) = A_{i,j} \cdot$ Det RLine$(C, i, \mathrm{Line}(B, j))$.

(57) Let X be a set, Y be a non empty set, and given x. Suppose $x \notin X$. Then there exists a function B_1 from $[: Y^X, Y :]$ into $Y^{X \cup \{x\}}$ such that

(i) B_1 is bijective, and

(ii) for every function f from X into Y and for every function F from $X \cup \{x\}$ into Y such that $F \restriction X = f$ holds $B_1(\langle f, F(x) \rangle) = F$.

(58) Let X be a finite set, Y be a non empty finite set, and given x. Suppose $x \notin X$. Let F be a binary operation on D. Suppose F is commutative and associative and has a unity and an inverse operation. Let f be a function from Y^X into D and g be a function from $Y^{X \cup \{x\}}$ into D. Suppose that for every function H from X into Y and for every element S_1 of $\mathrm{Fin}(Y^{X \cup \{x\}})$ such that $S_1 = \{h; h$ ranges over functions from $X \cup \{x\}$ into Y: $h \restriction X = H\}$ holds $F\text{-}\sum_{S_1} g = f(H)$. Then $F\text{-}\sum_{\Omega^{\mathrm{f}}_{Y^X}} f = F\text{-}\sum_{\Omega^{\mathrm{f}}_{Y^{X \cup \{x\}}}} g$.

(59) Let A, B be matrices over D of dimension $n \times m$ and given i. Suppose $i \leq n$ and $0 < n$. Let F be a function from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, n$. Then there exists a matrix M over D of dimension $n \times m$ such that $M = A +\!\cdot (B \cdot$

(idseq(n)+·F))↾Seg i and for every j holds if $j \in$ Seg i, then $M(j) = B(F(j))$ and if $j \notin$ Seg i, then $M(j) = A(j)$.

(60) Let A, B be matrices over K of dimension n. Suppose $0 < n$. Then there exists a function P from $(\mathrm{Seg}\, n)^{\mathrm{Seg}\, n}$ into the carrier of K such that

(i) for every function F from Seg n into Seg n there exists a finite sequence P_4 of elements of K such that len $P_4 = n$ and for all natural numbers F_1, j such that $j \in$ Seg n and $F_1 = F(j)$ holds $P_4(j) = A_{j,F_1}$ and $P(F) =$ ((the multiplication of K) $\circledast$ (P_4)) $\cdot$ Det$(B \cdot F)$, and

(ii) Det$(A \cdot B) =$ (the addition of K)- $\sum_{\Omega^{\mathrm{f}}_{(\mathrm{Seg}\, n)^{\mathrm{Seg}\, n}}} P$.

(61) Let A, B be matrices over K of dimension n. Suppose $0 < n$. Then there exists a function P from the permutations of n-element set into the carrier of K such that

(i) Det$(A \cdot B) =$ (the addition of K)- $\sum_{\Omega^{\mathrm{f}}_{\text{the permutations of } n\text{-element set}}} P$, and

(ii) for every element p_1 of the permutations of n-element set holds $P(p_1) =$ ((the multiplication of K) $\circledast$ (p_1-Path A)) $\cdot (-1)^{\mathrm{sgn}(p_1)}$ Det B.

(62) For all matrices A, B over K of dimension n such that $0 < n$ holds Det$(A \cdot B) =$ Det $A \cdot$ Det B.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[5] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.
[6] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[10] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[11] Czesław Byliński. Semigroup operations on finite subsets. *Formalized Mathematics*, 1(**4**):651–656, 1990.
[12] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[13] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[14] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.
[15] Katarzyna Jankowska. Transpose matrices and groups of permutations. *Formalized Mathematics*, 2(**5**):711–717, 1991.
[16] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[17] Yozo Toda. The formalization of simple graphs. *Formalized Mathematics*, 5(**1**):137–144, 1996.
[18] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*

[19] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[20] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.
[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[22] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(**1**):187–190, 1990.
[23] Wojciech A. Trybulec. Binary operations on finite sequences. *Formalized Mathematics*, 1(**5**):979–981, 1990.
[24] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[25] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[26] Wojciech A. Trybulec. Lattice of subgroups of a group. Frattini subgroup. *Formalized Mathematics*, 2(**1**):41–47, 1991.
[27] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[28] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[29] Hiroshi Yamazaki, Yoshinori Fujisawa, and Yatsuka Nakamura. On replace function and swap function for finite sequences. *Formalized Mathematics*, 9(**3**):471–474, 2001.
[30] Xiaopeng Yue, Xiquan Liang, and Zhongpin Sun. Some properties of some special matrices. *Formalized Mathematics*, 13(**4**):541–547, 2005.
[31] Katarzyna Zawadzka. The sum and product of finite sequences of elements of a field. *Formalized Mathematics*, 3(**2**):205–211, 1992.
[32] Katarzyna Zawadzka. The product and the determinant of matrices with entries in a field. *Formalized Mathematics*, 4(**1**):1–8, 1993.

Received March 21, 2007