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Alexandroff One Point Compactification

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Summary. In the article, I introduce the notions of the compactification of topological spaces and the Alexandroff one point compactification. Some properties of the locally compact spaces and one point compactification are proved.

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The articles [15], [5], [16], [17], [4], [18], [1], [8], [14], [13], [19], [7], [9], [10], [6], [12], [2], [3], and [11] provide the notation and terminology for this paper.

Let X be a topological space and let P be a family of subsets of X. We say that P is compact if and only if:

(Def. 1) For every subset U of X such that $U \in P$ holds U is compact.

Let X be a topological space and let U be a subset of X. We say that U is relatively-compact if and only if:

(Def. 2) $\overline{U}$ is compact.

Let X be a topological space. Note that $\emptyset_X$ is relatively-compact.

Let X be a topological space. Observe that there exists a subset of X which is relatively-compact.

Let X be a topological space and let U be a relatively-compact subset of X. Observe that $\overline{U}$ is compact.

Let X be a topological space and let U be a subset of X. We introduce U is pre-compact as a synonym of U is relatively-compact.

Let X be a non empty topological space. We introduce X is liminally-compact as a synonym of X is locally-compact.

Let X be a non empty topological space. Let us observe that X is liminally-compact if and only if:

(Def. 3) For every point x of X holds there exists a generalized basis of x which is compact.

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Let X be a non empty topological space. We say that X is locally-relatively-compact if and only if:

(Def. 4) For every point x of X holds there exists a neighbourhood of x which is relatively-compact.

Let X be a non empty topological space. We say that X is locally-closed/compact if and only if:

(Def. 5) For every point x of X holds there exists a neighbourhood of x which is closed and compact.

Let X be a non empty topological space. We say that X is locally-compact if and only if:

(Def. 6) For every point x of X holds there exists a neighbourhood of x which is compact.

Let us observe that every non empty topological space which is liminally-compact is also locally-compact.

Let us note that every non empty T_3 topological space which is locally-compact is also liminally-compact.

One can verify that every non empty topological space which is locally-relatively-compact is also locally-closed/compact.

Let us observe that every non empty topological space which is locally-closed/compact is also locally-relatively-compact.

Let us observe that every non empty topological space which is locally-relatively-compact is also locally-compact.

One can verify that every non empty Hausdorff topological space which is locally-compact is also locally-relatively-compact.

One can check that every non empty topological space which is compact is also locally-compact.

Let us observe that every non empty topological space which is discrete is also locally-compact.

Let us mention that there exists a topological space which is discrete and non empty.

Let X be a locally-compact non empty topological space and let C be a closed non empty subset of X. Note that $X \restriction C$ is locally-compact.

Let X be a locally-compact non empty T_3 topological space and let P be an open non empty subset of X. Note that $X \restriction P$ is locally-compact.

One can prove the following two propositions:

(1) Let X be a Hausdorff non empty topological space and E be a non empty subset of X. If $X \restriction E$ is dense and locally-compact, then $X \restriction E$ is open.

(2) For all topological spaces X, Y and for every subset A of X such that $\Omega_X \subseteq \Omega_Y$ holds $(\mathrm{incl}(X, Y))^{\circ} A = A$.

Let X, Y be topological spaces and let f be a function from X into Y. We say that f is embedding if and only if:

(Def. 7) There exists a function h from X into $Y \restriction \operatorname{rng} f$ such that $h = f$ and h is a homeomorphism.

The following proposition is true

(3) Let X, Y be topological spaces. Suppose $\Omega_X \subseteq \Omega_Y$ and there exists a subset X_1 of Y such that $X_1 = \Omega_X$ and the topology of $Y \restriction X_1$ = the topology of X. Then $\operatorname{incl}(X, Y)$ is embedding.

Let X be a topological space, let Y be a topological space, and let h be a function from X into Y. We say that h is compactification if and only if:

(Def. 8) h is embedding and Y is compact and $h^{\circ}(\Omega_X)$ is dense.

Let X be a topological space and let Y be a topological space. Note that every function from X into Y which is compactification is also embedding.

Let X be a topological structure. The one-point compactification of X yields a strict topological structure and is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of the one-point compactification of $X = \operatorname{succ}(\Omega_X)$, and

(ii) the topology of the one-point compactification of X = (the topology of $X) \cup \{U \cup \{\Omega_X\}; U$ ranges over subsets of X: U is open $\wedge$ U^{c} is compact$\}$.

Let X be a topological structure. Note that the one-point compactification of X is non empty.

We now state the proposition

(4) For every topological structure X holds $\Omega_X \subseteq \Omega_{\text{the one-point compactification of } X}$.

Let X be a topological space. Note that the one-point compactification of X is topological space-like.

Next we state the proposition

(5) Every topological structure X is a subspace of the one-point compactification of X.

Let X be a topological space. One can verify that the one-point compactification of X is compact.

One can prove the following propositions:

(6) Let X be a non empty topological space. Then X is Hausdorff and locally-compact if and only if the one-point compactification of X is Hausdorff.

(7) Let X be a non empty topological space. Then X is non compact if and only if there exists a subset X' of the one-point compactification of X such that $X' = \Omega_X$ and X' is dense.

(8) Let X be a non empty topological space. Suppose X is non compact. Then $\operatorname{incl}(X, \text{the one-point compactification of } X)$ is compactification.

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Arrow's Impossibility Theorem

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Summary. A formalization of the first proof from [6].

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The terminology and notation used here are introduced in the following articles: [11], [13], [12], [10], [9], [5], [2], [3], [1], [8], [4], and [7].

1. Preliminaries

Let A, B' be non empty sets, let B be a non empty subset of B', let f be a function from A into B, and let x be an element of A. Then $f(x)$ is an element of B.

Next we state two propositions:

(1) For every finite set A such that $\operatorname{card} A \geq 2$ and for every element a of A there exists an element b of A such that $b \neq a$.

(2) Let A be a finite set. Suppose $\operatorname{card} A \geq 3$. Let a, b be elements of A. Then there exists an element c of A such that $c \neq a$ and $c \neq b$.

2. Linear Preorders and Linear Orders

In the sequel A denotes a non empty set and a, b, c denote elements of A.

Let us consider A. The functor $\operatorname{LinPreorders} A$ is defined by the condition (Def. 1).

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(Def. 1) Let R be a set. Then $R \in \mathrm{LinPreorders}\, A$ if and only if the following conditions are satisfied:

(i) R is a binary relation on A,

(ii) for all a, b holds $\langle a, b\rangle \in R$ or $\langle b, a\rangle \in R$, and

(iii) for all a, b, c such that $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$ holds $\langle a, c\rangle \in R$.

Let us consider A. Note that $\mathrm{LinPreorders}\, A$ is non empty.

Let us consider A. The functor $\mathrm{LinOrders}\, A$ yielding a subset of $\mathrm{LinPreorders}\, A$ is defined by:

(Def. 2) For every element R of $\mathrm{LinPreorders}\, A$ holds $R \in \mathrm{LinOrders}\, A$ iff for all a, b such that $\langle a, b\rangle \in R$ and $\langle b, a\rangle \in R$ holds $a = b$.

Let A be a set. One can verify that there exists an order in A which is connected.

Let us consider A. Then $\mathrm{LinOrders}\, A$ can be characterized by the condition:

(Def. 3) For every set R holds $R \in \mathrm{LinOrders}\, A$ iff R is a connected order in A.

Let us consider A. One can verify that $\mathrm{LinOrders}\, A$ is non empty.

In the sequel o, o' are elements of $\mathrm{LinPreorders}\, A$ and o'' is an element of $\mathrm{LinOrders}\, A$.

Let us consider A, o, a, b. The predicate $a \leq_o b$ is defined by:

(Def. 4) $\langle a, b\rangle \in o$.

Let us consider A, o, a, b. We introduce $b \geq_o a$ as a synonym of $a \leq_o b$. We introduce $b <_o a$ as an antonym of $a \leq_o b$. We introduce $a >_o b$ as an antonym of $a \leq_o b$.

We now state a number of propositions:

(3) $a \leq_o a$.

(4) $a \leq_o b$ or $b \leq_o a$.

(5) If $a \leq_o b$ or $a <_o b$ and if $b \leq_o c$ or $b <_o c$, then $a \leq_o c$.

(6) If $a \leq_{o''} b$ and $b \leq_{o''} a$, then $a = b$.

(7) If $a \neq b$ and $b \neq c$ and $a \neq c$, then there exists o such that $a <_o b$ and $b <_o c$.

(8) There exists o such that for every a such that $a \neq b$ holds $b <_o a$.

(9) There exists o such that for every a such that $a \neq b$ holds $a <_o b$.

(10) If $a \neq b$ and $a \neq c$, then there exists o such that $a <_o b$ and $a <_o c$ and $b <_o c$ iff $b <_{o'} c$ and $c <_o b$ iff $c <_{o'} b$.

(11) If $a \neq b$ and $a \neq c$, then there exists o such that $b <_o a$ and $c <_o a$ and $b <_o c$ iff $b <_{o'} c$ and $c <_o b$ iff $c <_{o'} b$.

(12) Let o, o' be elements of $\mathrm{LinOrders}\, A$. Then $a <_o b$ iff $a <_{o'} b$ and $b <_o a$ iff $b <_{o'} a$ if and only if $a <_o b$ iff $a <_{o'} b$.

(13) Let o be an element of $\mathrm{LinOrders}\, A$ and o' be an element of $\mathrm{LinPreorders}\, A$. Then for all a, b such that $a <_o b$ holds $a <_{o'} b$ if and only

if for all a, b holds $a <_o b$ iff $a <_{o'} b$.

3. Arrow's Theorem

For simplicity, we follow the rules: A, N are finite non empty sets, a, b are elements of A, i, n are elements of N, p, p' are elements of $(\mathrm{LinPreorders}\,A)^N$, and f is a function from $(\mathrm{LinPreorders}\,A)^N$ into $\mathrm{LinPreorders}\,A$.

We now state the proposition

(14) Suppose that

(i) for all p, a, b such that for every i holds $a <_{p(i)} b$ holds $a <_{f(p)} b$,

(ii) for all p, p', a, b such that for every i holds $a <_{p(i)} b$ iff $a <_{p'(i)} b$ and $b <_{p(i)} a$ iff $b <_{p'(i)} a$ holds $a <_{f(p)} b$ iff $a <_{f(p')} b$, and

(iii) $\mathrm{card}\,A \geq 3$.

Then there exists n such that for all p, a, b such that $a <_{p(n)} b$ holds $a <_{f(p)} b$.

In the sequel p, p' denote elements of $(\mathrm{LinOrders}\,A)^N$ and f denotes a function from $(\mathrm{LinOrders}\,A)^N$ into $\mathrm{LinPreorders}\,A$.

One can prove the following proposition

(15) Suppose that

(i) for all p, a, b such that for every i holds $a <_{p(i)} b$ holds $a <_{f(p)} b$,

(ii) for all p, p', a, b such that for every i holds $a <_{p(i)} b$ iff $a <_{p'(i)} b$ holds $a <_{f(p)} b$ iff $a <_{f(p')} b$, and

(iii) $\mathrm{card}\,A \geq 3$.

Then there exists n such that for all p, a, b holds $a <_{p(n)} b$ iff $a <_{f(p)} b$.

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Congruences and Quotient Algebras of BCI-algebras

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Summary. We have formalized the BCI-algebras closely following the book [7] pp.16-19 and pp.58-65. Firstly, the article focuses on the properties of the element and then the definition and properties of congruences and quotient algebras are given. Quotient algebras are the basic tools for exploring the structures of BCI-algebras.

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The articles [11], [5], [12], [10], [13], [2], [3], [1], [8], [14], [6], [15], [4], and [9] provide the terminology and notation for this paper.

1. Basic Properties of the Element

For simplicity, we adopt the following convention: X is a BCI-algebra, I is an ideal of X, a, x, y, z, u are elements of X, f is a function from $\mathbb{N}$ into the carrier of X, and j, i, k, n, m are elements of $\mathbb{N}$.

Let us consider X, x, y and let n be an element of $\mathbb{N}$. The functor $(x \setminus y)^n$ yielding an element of X is defined by:

(Def. 1) There exists f such that $(x \setminus y)^n = f(n)$ and $f(0) = x$ and for every element j of $\mathbb{N}$ such that $j < n$ holds $f(j+1) = f(j) \setminus y$.

One can prove the following propositions:

(1) $(x \setminus y)^0 = x$.

(2) $(x \setminus y)^1 = x \setminus y$.

(3) $(x \setminus y)^2 = x \setminus y \setminus y$.

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(4) $(x \setminus y)^{n+1} = ((x \setminus y)^n) \setminus y$.
(5) $(x \setminus 0_X)^{n+1} = x$.
(6) $(0_X \setminus 0_X)^n = 0_X$.
(7) $((x \setminus y)^n) \setminus z = ((x \setminus z) \setminus y)^n$.
(8) $(x \setminus (x \setminus (x \setminus y)))^n = (x \setminus y)^n$.
(9) $((0_X \setminus x)^n)^{\rm c} = (0_X \setminus x^{\rm c})^n$.
(10) $((x \setminus y)^n \setminus y)^m = (x \setminus y)^{n+m}$.
(11) $((x \setminus y)^n \setminus z)^m = ((x \setminus z)^m \setminus y)^n$.
(12) $(((0_X \setminus x)^n)^{\rm c})^{\rm c} = (0_X \setminus x)^n$.
(13) $(0_X \setminus x)^{n+m} = ((0_X \setminus x)^n) \setminus ((0_X \setminus x)^m)^{\rm c}$.
(14) $((0_X \setminus x)^{m+n})^{\rm c} = ((0_X \setminus x)^m)^{\rm c} \setminus ((0_X \setminus x)^n)$.
(15) $((0_X \setminus ((0_X \setminus x)^m))^n)^{\rm c} = (0_X \setminus x)^{m \cdot n}$.
(16) If $(0_X \setminus x)^m = 0_X$, then $(0_X \setminus x)^{m \cdot n} = 0_X$.
(17) If $x \setminus y = x$, then $(x \setminus y)^n = x$.
(18) $(0_X \setminus (x \setminus y))^n = ((0_X \setminus x)^n) \setminus ((0_X \setminus y)^n)$.
(19) If $x \leq y$, then $(x \setminus z)^n \leq (y \setminus z)^n$.
(20) If $x \leq y$, then $(z \setminus y)^n \leq (z \setminus x)^n$.
(21) $((x \setminus z)^n) \setminus ((y \setminus z)^n) \leq x \setminus y$.
(22) $((x \setminus (x \setminus y))^n \setminus (y \setminus x))^n \leq x$.

Let us consider X, a. We introduce a is minimal as a synonym of a is atom.

Let us consider X, a. We say that a is positive if and only if:

(Def. 2) $0_X \leq a$.

We say that a is least if and only if:

(Def. 3) For every x holds $a \leq x$.

We say that a is maximal if and only if:

(Def. 4) For every x such that $a \leq x$ holds $x = a$.

We say that a is greatest if and only if:

(Def. 5) For every x holds $x \leq a$.

Let us consider X. Observe that there exists an element of X which is positive.

Let us consider X. Note that 0_X is positive and minimal.

Next we state several propositions:

(23) a is minimal iff for every x holds $a \setminus x = x^{\rm c} \setminus a^{\rm c}$.
(24) $x^{\rm c}$ is minimal iff for every y such that $y \leq x$ holds $x^{\rm c} = y^{\rm c}$.
(25) $x^{\rm c}$ is minimal iff for all y, z holds $((x \setminus z \setminus (y \setminus z))^{\rm c})^{\rm c} = y^{\rm c} \setminus x^{\rm c}$.
(26) If 0_X is maximal, then every a is minimal.
(27) If there exists x which is greatest, then every a is positive.

(28) $x \setminus (x^{c})^{c}$ is a positive element of X.

(29) a is minimal iff $(a^{c})^{c} = a$.

(30) a is minimal iff there exists x such that $a = x^{c}$.

Let us consider X, x. We say that x is nilpotent if and only if:

(Def. 6) There exists a non empty element k of $\mathbb{N}$ such that $(0_X \setminus x)^k = 0_X$.

Let us consider X. We say that X is nilpotent if and only if:

(Def. 7) Every element of X is nilpotent.

Let us consider X, x. Let us assume that x is nilpotent. The functor $\mathrm{ord}(x)$ yielding a non empty element of $\mathbb{N}$ is defined by:

(Def. 8) $(0_X \setminus x)^{\mathrm{ord}(x)} = 0_X$ and for every element m of $\mathbb{N}$ such that $(0_X \setminus x)^m = 0_X$ and $m \neq 0$ holds $\mathrm{ord}(x) \leq m$.

Let us consider X. One can verify that 0_X is nilpotent.

We now state four propositions:

(31) x is a positive element of X iff x is nilpotent and $\mathrm{ord}(x) = 1$.

(32) X is a BCK-algebra iff for every x holds $\mathrm{ord}(x) = 1$ and x is nilpotent.

(33) $(0_X \setminus x^{c})^n$ is minimal.

(34) If x is nilpotent, then $\mathrm{ord}(x) = \mathrm{ord}(x^{c})$.

2. Congruences and Quotient Algebras

Let X be a BCI-algebra. An equivalence relation of X is said to be a congruence of X if:

(Def. 9) For all elements x, y, u, v of X such that $\langle x, y\rangle \in$ it and $\langle u, v\rangle \in$ it holds $\langle x \setminus u, y \setminus v\rangle \in$ it.

Let X be a BCI-algebra. An equivalence relation of X is said to be an L-congruence of X if:

(Def. 10) For all elements x, y of X such that $\langle x, y\rangle \in$ it and for every element u of X holds $\langle u \setminus x, u \setminus y\rangle \in$ it.

Let X be a BCI-algebra. An equivalence relation of X is said to be an R-congruence of X if:

(Def. 11) For all elements x, y of X such that $\langle x, y\rangle \in$ it and for every element u of X holds $\langle x \setminus u, y \setminus u\rangle \in$ it.

Let X be a BCI-algebra and let A be an ideal of X. A binary relation on X is said to be an I-congruence of X by A if:

(Def. 12) For all elements x, y of X holds $\langle x, y\rangle \in$ it iff $x \setminus y \in A$ and $y \setminus x \in A$.

Let X be a BCI-algebra and let A be an ideal of X. Note that every I-congruence of X by A is total, symmetric, and transitive.

Let X be a BCI-algebra. The functor $\mathrm{IConSet}\, X$ is defined as follows:

(Def. 13) For every set A_1 holds $A_1 \in \mathrm{IConSet}\, X$ iff there exists an ideal I of X such that A_1 is an I-congruence of X by I.

Let X be a BCI-algebra. The functor $\mathrm{ConSet}\, X$ is defined as follows:

(Def. 14) $\mathrm{ConSet}\, X = \{R : R$ ranges over congruences of $X\}$.

The functor $\mathrm{LConSet}\, X$ is defined by:

(Def. 15) $\mathrm{LConSet}\, X = \{R : R$ ranges over L-congruences of $X\}$.

The functor $\mathrm{RConSet}\, X$ is defined as follows:

(Def. 16) $\mathrm{RConSet}\, X = \{R : R$ ranges over R-congruences of $X\}$.

For simplicity, we adopt the following rules: R is an equivalence relation of X, R_1 is an I-congruence of X by I, E is a congruence of X, R_2 is an R-congruence of X, and L_1 is an L-congruence of X.

We now state three propositions:

(35) For all X, E holds $[0_X]_E$ is a closed ideal of X.

(36) R is a congruence of X iff R is an R-congruence of X and an L-congruence of X.

(37) R_1 is a congruence of X.

Let X be a BCI-algebra and let I be an ideal of X. We see that the I-congruence of X by I is a congruence of X.

One can prove the following propositions:

(38) $[0_X]_{(R_1)} \subseteq I$.

(39) I is closed iff $I = [0_X]_{(R_1)}$.

(40) If $\langle x, y\rangle \in E$, then $x \setminus y \in [0_X]_E$ and $y \setminus x \in [0_X]_E$.

(41) Let A, I be ideals of X, I_1 be an I-congruence of X by A, and I_2 be an I-congruence of X by I. If $[0_X]_{(I_1)} = [0_X]_{(I_2)}$, then $I_1 = I_2$.

(42) If $\langle x, y\rangle \in E$ and $u \in [0_X]_E$, then $\langle x, (y \setminus u)^k\rangle \in E$.

(43) Suppose that for all X, x, y there exist i, j, m, n such that $((x \setminus (x \setminus y))^i \setminus (y \setminus x))^j = ((y \setminus (y \setminus x))^m \setminus (x \setminus y))^n$. Let given E, I. If $I = [0_X]_E$, then E is an I-congruence of X by I.

(44) $\mathrm{IConSet}\, X \subseteq \mathrm{ConSet}\, X$.

(45) $\mathrm{ConSet}\, X \subseteq \mathrm{LConSet}\, X$.

(46) $\mathrm{ConSet}\, X \subseteq \mathrm{RConSet}\, X$.

(47) $\mathrm{ConSet}\, X = \mathrm{LConSet}\, X \cap \mathrm{RConSet}\, X$.

(48) If every L_1 is an I-congruence of X by I, then $E = R_1$.

(49) If every R_2 is an I-congruence of X by I, then $E = R_1$.

(50) $[0_X]_{(L_1)}$ is a closed ideal of X.

In the sequel E denotes a congruence of X and R_1 denotes an I-congruence of X by I.

Let us consider X, E. Note that $\mathrm{Classes}\, E$ is non empty.

Let us consider X, E. The functor $\mathrm{EqClaOp}\, E$ yielding a binary operation on $\mathrm{Classes}\, E$ is defined by:

(Def. 17) For all elements W_1, W_2 of $\mathrm{Classes}\, E$ and for all x, y such that $W_1 = [x]_E$ and $W_2 = [y]_E$ holds $(\mathrm{EqClaOp}\, E)(W_1, W_2) = [x \setminus y]_E$.

Let us consider X, E. The functor $\mathrm{zeroEqC}\, E$ yields an element of $\mathrm{Classes}\, E$ and is defined as follows:

(Def. 18) $\mathrm{zeroEqC}\, E = [0_X]_E$.

Let us consider X, E. The functor ${}^{X}/_{E}$ yielding a BCI structure with 0 is defined by:

(Def. 19) ${}^{X}/_{E} = \langle \mathrm{Classes}\, E, \mathrm{EqClaOp}\, E, \mathrm{zeroEqC}\, E \rangle$.

Let us consider X and let E be a congruence of X. One can check that ${}^{X}/_{E}$ is non empty.

In the sequel W_1, W_2 denote elements of $\mathrm{Classes}\, E$.

Let us consider X, E, W_1, W_2. The functor $W_1 \setminus W_2$ yielding an element of $\mathrm{Classes}\, E$ is defined by:

(Def. 20) $W_1 \setminus W_2 = (\mathrm{EqClaOp}\, E)(W_1, W_2)$.

Next we state the proposition

(51) ${}^{X}/_{R_1}$ is a BCI-algebra.

Let us consider X, I, R_1. Note that ${}^{X}/_{R_1}$ is strict, B, C, I, and BCI-4.

Next we state the proposition

(52) For all X, I such that $I = \text{BCK-part}\, X$ and for every I-congruence R_1 of X by I holds ${}^{X}/_{R_1}$ is a p-semisimple BCI-algebra.

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Linear Congruence Relation and Complete Residue Systems

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Summary. In this paper, we defined the congruence relation and proved its fundamental properties on the base of some useful theorems. Then we proved the existence of solution and the number of incongruent solution to a linear congruence and the linear congruent equation class, in particular, we proved the Chinese Remainder Theorem. Finally, we defined the complete residue system and proved its fundamental properties.

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The papers [21], [25], [2], [20], [1], [22], [3], [24], [14], [17], [16], [23], [26], [7], [5], [27], [9], [18], [13], [19], [28], [8], [10], [4], [15], [11], [6], and [12] provide the terminology and notation for this paper.

For simplicity, we adopt the following convention: a, b, m, x, y, i_1, i_2, i_3, i denote integers, k, p, q, n denote natural numbers, c, c_1, c_2 denote elements of $\mathbb{N}$, and X denotes a set.

Next we state the proposition

(1) For every real-membered set X and for every real number a holds $X \approx a \oplus X$.

Let X be a finite real-membered set and let a be a real number. One can verify that $a \oplus X$ is finite.

One can prove the following propositions:

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(2) For every finite real-membered set X and for every real number a holds $\operatorname{card} X = \operatorname{card}(a \oplus X)$.

(3) For every real-membered set X and for every real number a such that $a \neq 0$ holds $X \approx a \circ X$.

(4) Let X be a real-membered set and a be a real number. Then

(i) if $a = 0$ and X is non empty, then $a \circ X = \{0\}$, and

(ii) if $a \circ X = \{0\}$, then $a = 0$ or $X = \{0\}$.

Let X be a finite real-membered set and let a be a real number. One can verify that $a \circ X$ is finite.

The following propositions are true:

(5) For every finite real-membered set X and for every real number a such that $a \neq 0$ holds $\operatorname{card} X = \operatorname{card}(a \circ X)$.

(6) If $i \mid i_1$ and $i_1 \neq 0$, then $|i| \leq |i_1|$.

(7) For every i_3 such that $i_3 \neq 0$ holds $i_1 \mid i_2$ iff $i_1 \cdot i_3 \mid i_2 \cdot i_3$.

(8) For all natural numbers a, b, m and for every element n of $\mathbb{N}$ such that $a \bmod m = b \bmod m$ holds $a^n \bmod m = b^n \bmod m$.

(9) If $i_1 \cdot i \equiv i_2 \cdot i (\operatorname{mod} i_3)$ and i and i_3 are relative prime, then $i_1 \equiv i_2 (\operatorname{mod} i_3)$.

(10) If $i_1 \equiv i_2 (\operatorname{mod} i_3)$, then $i_1 \cdot k \equiv i_2 \cdot k (\operatorname{mod} i_3 \cdot k)$.

(11) If $i_1 \equiv i_2 (\operatorname{mod} i)$, then $i_1 \cdot i_3 \equiv i_2 \cdot i_3 (\operatorname{mod} i)$.

(12) For every integer i holds $0 = 0 \bmod i$.

(13) For every b such that $b > 0$ and for every a there exist integers q, r such that $a = b \cdot q + r$ and $r \geq 0$ and $r < b$.

(14) If $i_1 \equiv i_2 (\operatorname{mod} i_3)$, then $i_1 \operatorname{gcd} i_3 = i_2 \operatorname{gcd} i_3$.

(15) If a and m are relative prime, then there exists an integer x such that $(a \cdot x - b) \bmod m = 0$.

(16) If $m > 0$ and a and m are relative prime, then there exists a natural number n such that $(a \cdot n - b) \bmod m = 0$.

(17) If $m \neq 0$ and $a \operatorname{gcd} m \nmid b$, then it is not true that there exists an integer x such that $(a \cdot x - b) \bmod m = 0$.

(18) If $m \neq 0$ and $a \operatorname{gcd} m \mid b$, then there exists an integer x such that $(a \cdot x - b) \bmod m = 0$.

Let x be an integer. Observe that $x^{\mathbf{2}}$ is natural.

We now state several propositions:

(19) If $n > 0$ and $p > 0$, then $p \cdot q \bmod p^n = p \cdot (q \bmod p^{n -' 1})$.

(20) $p \cdot q \bmod p \cdot n = p \cdot (q \bmod n)$.

(21) If $n > 0$ and p is prime and p and q are relative prime, then $p \nmid q \bmod p^n$.

(22) For all natural numbers p, q, n such that $n > 0$ holds $(p - q) \bmod n = 0$ iff $p \bmod n = q \bmod n$.

(23) For all natural numbers a, b, m such that $m > 0$ holds $a \bmod m = b \bmod m$ iff $m \mid a - b$.

(24) If $n > 0$ and p is prime and p and q are relative prime, then it is not true that there exists an integer x such that $(p \cdot x^{\mathbf{2}} - q) \bmod p^n = 0$.

(25) If $n > 0$ and p is prime and p and q are relative prime, then it is not true that there exists an integer x such that $(p \cdot x - q) \bmod p^n = 0$.

Let m be an integer. The functor $\mathrm{Cong}\, m$ yielding a binary relation on $\mathbb{Z}$ is defined as follows:

(Def. 1) $\langle x, y\rangle \in \mathrm{Cong}\, m$ iff $x \equiv y (\bmod m)$.

Let m be an integer. One can check that $\mathrm{Cong}\, m$ is total.

Let m be an integer. One can check that $\mathrm{Cong}\, m$ is reflexive, symmetric, and transitive.

Next we state four propositions:

(26) Suppose $m \neq 0$ and $(a \cdot x - b) \bmod m = 0$. Let y be an integer. Then

(i) if a and m are relative prime and $(a \cdot y - b) \bmod m = 0$, then $y \in [x]_{\mathrm{Cong}\, m}$, and

(ii) if $y \in [x]_{\mathrm{Cong}\, m}$, then $(a \cdot y - b) \bmod m = 0$.

(27) Let a, b, m, x be integers. Suppose $m \neq 0$ and a and m are relative prime and $(a \cdot x - b) \bmod m = 0$. Then there exists an integer s such that $\langle x, b \cdot s\rangle \in \mathrm{Cong}\, m$.

(28) Let a, m be elements of $\mathbb{N}$. Suppose $a \neq 0$ and $m > 1$ and a and m are relative prime. Let b, x be integers. If $(a \cdot x - b) \bmod m = 0$, then $\langle x, b \cdot a^{\mathrm{Euler}\, m -' 1}\rangle \in \mathrm{Cong}\, m$.

(29) Suppose $m \neq 0$ and $a \gcd m \mid b$. Then there exists a finite sequence f_1 of elements of $\mathbb{Z}$ such that $\mathrm{len}\, f_1 = a \gcd m$ and for every c such that $c \in \mathrm{dom}\, f_1$ holds $(a \cdot f_1(c) - b) \bmod m = 0$ and for all c_1, c_2 such that $c_1 \in \mathrm{dom}\, f_1$ and $c_2 \in \mathrm{dom}\, f_1$ and $c_1 \neq c_2$ holds $f_1(c_1) \not\equiv f_1(c_2) (\bmod m)$.

We use the following convention: f_1, f_2 denote finite sequences of elements of $\mathbb{N}$ and a, b, c, d, n denote elements of $\mathbb{N}$.

Next we state a number of propositions:

(30) For all b, n such that $b \in \mathrm{dom}\, f_1$ and $\mathrm{len}\, f_1 = n + 1$ holds $(f_1 \frown \langle d\rangle)_{\restriction b} = ((f_1)_{\restriction b}) \frown \langle d\rangle$.

(31) Suppose $\mathrm{len}\, f_1 \geq 2$ and for all b, c such that $b \in \mathrm{dom}\, f_1$ and $c \in \mathrm{dom}\, f_1$ and $b \neq c$ holds $\gcd(f_1(b), f_1(c)) = 1$. Let given b. If $b \in \mathrm{dom}\, f_1$, then $\gcd(\prod((f_1)_{\restriction b}), f_1(b)) = 1$.

(32) For every a such that $a \in \mathrm{dom}\, f_1$ holds $f_1(a) \mid \prod f_1$.

(33) If $a \in \mathrm{dom}\, f_1$ and $p \mid f_1(a)$, then $p \mid \prod f_1$.

(34) If $\mathrm{len}\, f_1 = n + 1$ and $a \geq 1$ and $a \leq n$, then $(f_1)_{\restriction a}(n) = f_1(\mathrm{len}\, f_1)$.

(35) For all a, b such that $a \in \operatorname{dom} f_1$ and $b \in \operatorname{dom} f_1$ and $a \neq b$ and $\operatorname{len} f_1 \geq 1$ holds $f_1(b) \mid \prod((f_1)_{\restriction a})$.

(36) If for every b such that $b \in \operatorname{dom} f_1$ holds $a \mid f_1(b)$, then $a \mid \sum f_1$.

(37) Suppose $\operatorname{len} f_1 \geq 2$ and for all b, c such that $b \in \operatorname{dom} f_1$ and $c \in \operatorname{dom} f_1$ and $b \neq c$ holds $\gcd(f_1(b), f_1(c)) = 1$ and for every b such that $b \in \operatorname{dom} f_1$ holds $f_1(b) \neq 0$. Let given f_2. Suppose $\operatorname{len} f_2 = \operatorname{len} f_1$. Then there exists an integer x such that for every b such that $b \in \operatorname{dom} f_1$ holds $(x - f_2(b)) \bmod f_1(b) = 0$.

(38) If for all b, c such that $b \in \operatorname{dom} f_1$ and $c \in \operatorname{dom} f_1$ and $b \neq c$ holds $\gcd(f_1(b), f_1(c)) = 1$ and for every b such that $b \in \operatorname{dom} f_1$ holds $f_1(b) \mid a$, then $\prod f_1 \mid a$.

(39) Suppose $\operatorname{len} f_1 \geq 2$ and for all b, c such that $b \in \operatorname{dom} f_1$ and $c \in \operatorname{dom} f_1$ and $b \neq c$ holds $\gcd(f_1(b), f_1(c)) = 1$ and for every b such that $b \in \operatorname{dom} f_1$ holds $f_1(b) > 0$. Let given f_2. Suppose $\operatorname{len} f_2 = \operatorname{len} f_1$ and for every b such that $b \in \operatorname{dom} f_1$ holds $(x - f_2(b)) \bmod f_1(b) = 0$ and $(y - f_2(b)) \bmod f_1(b) = 0$. Then $x \equiv y (\operatorname{mod} \prod f_1)$.

We follow the rules: m_1, m_2, m_3, r, s, a, b, c, c_1, c_2, x denote integers and n_1, n_2, n_3 denote natural numbers.

The following propositions are true:

(40) Suppose $m_1 \neq 0$ and $m_2 \neq 0$ and m_1 and m_2 are relative prime. Then there exists an integer r such that for every x such that $(x - c_1) \bmod m_1 = 0$ and $(x - c_2) \bmod m_2 = 0$ holds $x \equiv c_1 + m_1 \cdot r (\operatorname{mod} m_1 \cdot m_2)$ and $(m_1 \cdot r - (c_2 - c_1)) \bmod m_2 = 0$.

(41) If $m_1 \neq 0$ and $m_2 \neq 0$ and $m_1 \gcd m_2 \nmid c_1 - c_2$, then it is not true that there exists x such that $(x - c_1) \bmod m_1 = 0$ and $(x - c_2) \bmod m_2 = 0$.

(42) Suppose $m_1 \neq 0$ and $m_2 \neq 0$ and $m_1 \gcd m_2 \mid c_2 - c_1$. Then there exists r such that for every x such that $(x - c_1) \bmod m_1 = 0$ and $(x - c_2) \bmod m_2 = 0$ holds $x \equiv c_1 + m_1 \cdot r (\operatorname{mod} \operatorname{lcm}(m_1, m_2))$ and $((m_1 \div (m_1 \gcd m_2)) \cdot r - ((c_2 - c_1) \div (m_1 \gcd m_2))) \bmod (m_2 \div (m_1 \gcd m_2)) = 0$.

(43) Suppose $m_1 \neq 0$ and $m_2 \neq 0$ and $a \gcd m_1 \mid c_1$ and $b \gcd m_2 \mid c_2$ and m_1 and m_2 are relative prime. Then there exist integers w, r, s such that
   (i) for every x such that $(a \cdot x - c_1) \bmod m_1 = 0$ and $(b \cdot x - c_2) \bmod m_2 = 0$ holds $x \equiv r + (m_1 \div (a \gcd m_1)) \cdot w (\operatorname{mod} (m_1 \div (a \gcd m_1)) \cdot (m_2 \div (b \gcd m_2)))$,
   (ii) $((a \div (a \gcd m_1)) \cdot r - (c_1 \div (a \gcd m_1))) \bmod (m_1 \div (a \gcd m_1)) = 0$,
   (iii) $((b \div (b \gcd m_2)) \cdot s - (c_2 \div (b \gcd m_2))) \bmod (m_2 \div (b \gcd m_2)) = 0$, and
   (iv) $((m_1 \div (a \gcd m_1)) \cdot w - (s - r)) \bmod (m_2 \div (b \gcd m_2)) = 0$.

(44) Suppose that
   (i) $m_1 \neq 0$,
   (ii) $m_2 \neq 0$,
   (iii) $m_3 \neq 0$,

(iv) m_1 and m_2 are relative prime,

(v) m_1 and m_3 are relative prime, and

(vi) m_2 and m_3 are relative prime.

Then there exist r, s such that for every x if $(x-a) \bmod m_1 = 0$ and $(x-b) \bmod m_2 = 0$ and $(x-c) \bmod m_3 = 0$, then $x \equiv a + m_1 \cdot r + m_1 \cdot m_2 \cdot s (\operatorname{mod} m_1 \cdot m_2 \cdot m_3)$ and $(m_1 \cdot r - (b-a)) \bmod m_2 = 0$ and $(m_1 \cdot m_2 \cdot s - (c - a - m_1 \cdot r)) \bmod m_3 = 0$.

(45) Suppose $m_1 \neq 0$ and $m_2 \neq 0$ and $m_3 \neq 0$ and $m_1 \operatorname{gcd} m_2 \nmid a - b$ or $m_1 \operatorname{gcd} m_3 \nmid a - c$ or $m_2 \operatorname{gcd} m_3 \nmid b - c$. Then it is not true that there exists x such that $(x-a) \bmod m_1 = 0$ and $(x-b) \bmod m_2 = 0$ and $(x-c) \bmod m_3 = 0$.

(46) For all non zero natural numbers n_1, n_2, n_3 holds $\operatorname{lcm}(\gcd(n_1,n_3), \gcd(n_2,n_3)) = \gcd(\operatorname{lcm}(n_1,n_2), n_3)$.

(47) Let n_1, n_2, n_3 be non zero natural numbers. Suppose $\gcd(n_1,n_2) \mid a-b$ and $\gcd(n_1,n_3) \mid a-c$ and $\gcd(n_2,n_3) \mid b-c$. Then there exist r, s such that for every x if $(x-a) \bmod n_1 = 0$ and $(x-b) \bmod n_2 = 0$ and $(x-c) \bmod n_3 = 0$, then $x \equiv a + n_1 \cdot r + \operatorname{lcm}(n_1,n_2) \cdot s (\operatorname{mod} \operatorname{lcm}(\operatorname{lcm}(n_1,n_2), n_3))$ and $((n_1 \div \gcd(n_1,n_2)) \cdot r - ((b-a) \div \gcd(n_1,n_2))) \bmod (n_2 \div \gcd(n_1,n_2)) = 0$ and $((\operatorname{lcm}(n_1,n_2) \div \gcd(\operatorname{lcm}(n_1,n_2),n_3)) \cdot s - ((c - (a + n_1 \cdot r)) \div \gcd(\operatorname{lcm}(n_1,n_2),n_3))) \bmod (n_3 \div \gcd(\operatorname{lcm}(n_1,n_2),n_3)) = 0$.

In the sequel f_1 denotes a finite sequence of elements of $\mathbb{N}$ and a, b, m denote elements of $\mathbb{N}$.

Let m be an element of $\mathbb{N}$ and let X be a set. We say that X is a complete residue system modulo m if and only if:

(Def. 2) There exists a finite sequence f_1 of elements of $\mathbb{Z}$ such that $X = \operatorname{rng} f_1$ and $\operatorname{len} f_1 = m$ and for every b such that $b \in \operatorname{dom} f_1$ holds $f_1(b) \in [b -' 1]_{\operatorname{Cong} m}$.

One can prove the following propositions:

(48) $\{a : a < m\}$ is a complete residue system modulo m.

(49) Let X be a finite set. Suppose X is a complete residue system modulo m. Then $\operatorname{card} X = m$ and for all integers x, y such that $x \in X$ and $y \in X$ and $x \neq y$ holds $\langle x, y \rangle \notin \operatorname{Cong} m$.

(50) $\emptyset$ is a complete residue system modulo m iff $m = 0$.

(51) Let X be a finite set. Suppose $\operatorname{card} X = m$. Then there exists a finite sequence f_1 such that $\operatorname{len} f_1 = m$ and for every a such that $a \in \operatorname{dom} f_1$ holds $f_1(a) \in X$ and f_1 is one-to-one.

(52) Let X be a finite subset of $\mathbb{Z}$. Suppose $\operatorname{card} X = m$ and for all integers x, y such that $x \in X$ and $y \in X$ and $x \neq y$ holds $\langle x, y \rangle \notin \operatorname{Cong} m$. Then X is a complete residue system modulo m.

In the sequel a is an integer.

The following two propositions are true:

(53) Let X be a finite subset of $\mathbb{Z}$. Suppose X is a complete residue system modulo m. Then $a \oplus X$ is a complete residue system modulo m.

(54) Let X be a finite subset of $\mathbb{Z}$. Suppose a and m are relative prime and X is a complete residue system modulo m. Then $a \circ X$ is a complete residue system modulo m.

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Several Integrability Formulas of Special Functions

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Summary. In this article, we give several integrability formulas of special and composite functions including trigonometric function, inverse trigonometric function, hyperbolic function and logarithmic function.

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The notation and terminology used here are introduced in the following papers: [21], [20], [7], [12], [6], [25], [3], [8], [26], [24], [5], [22], [18], [19], [17], [9], [16], [11], [14], [1], [15], [23], [13], [10], [2], and [4].

1. Preliminaries

For simplicity, we adopt the following convention: f, f_1, f_2, g denote partial functions from $\mathbb{R}$ to $\mathbb{R}$, A denotes a closed-interval subset of $\mathbb{R}$, r, x, x_0 denote real numbers, n denotes an element of $\mathbb{N}$, and Z denotes an open subset of $\mathbb{R}$.

The following propositions are true:

(1) $\sin(x + 2 \cdot n \cdot \pi) = \sin x$.

(2) $\sin(x + (2 \cdot n + 1) \cdot \pi) = -\sin x$.

(3) $\cos(x + 2 \cdot n \cdot \pi) = \cos x$.

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(4) $\cos(x + (2 \cdot n + 1) \cdot \pi) = -\cos x$.

(5) If $\sin(\frac{x}{2}) \geq 0$, then $\sin(\frac{x}{2}) = \sqrt{\frac{1-\cos x}{2}}$.

(6) If $\sin(\frac{x}{2}) < 0$, then $\sin(\frac{x}{2}) = -\sqrt{\frac{1-\cos x}{2}}$.

(7) $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.

(8) $\sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$.

(9) $[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}] \subseteq]-1, 1[$.

(10) $\arcsin(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$.

(11) $\arcsin(-\frac{\sqrt{2}}{2}) = -\frac{\pi}{4}$.

(12) If $\cos(\frac{x}{2}) \geq 0$, then $\cos(\frac{x}{2}) = \sqrt{\frac{1+\cos x}{2}}$.

(13) $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.

(14) $\cos(\frac{3 \cdot \pi}{4}) = -\frac{\sqrt{2}}{2}$.

(15) $\arccos(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$.

(16) $\arccos(-\frac{\sqrt{2}}{2}) = \frac{3 \cdot \pi}{4}$.

(17) (The function sinh)$(1) = \frac{e^{\mathbf{2}}-1}{2 \cdot e}$.

(18) (The function cosh)$(0) = 1$.

(19) (The function cosh)$(1) = \frac{e^{\mathbf{2}}+1}{2 \cdot e}$.

(20) For every linear function L_1 holds $-L_1$ is a linear function.

(21) For every rest R_1 holds $-R_1$ is a rest.

(22) For all f_1, x_0 such that f_1 is differentiable in x_0 holds $-f_1$ is differentiable in x_0 and $(-f_1)'(x_0) = -{f_1}'(x_0)$.

(23) Let given f_1, Z. Suppose $Z \subseteq \mathrm{dom}(-f_1)$ and f_1 is differentiable on Z. Then $-f_1$ is differentiable on Z and for every x such that $x \in Z$ holds $(-f_1)'_{\restriction Z}(x) = -{f_1}'(x)$.

(24) $-$the function sin is differentiable on $\mathbb{R}$.

(25) $-$the function cos is differentiable in x and ($-$the function cos)$'(x) =$ (the function sin)(x).

(26)(i) $-$the function cos is differentiable on $\mathbb{R}$, and

(ii) for every x such that $x \in \mathbb{R}$ holds ($-$the function cos)$'(x) =$ (the function sin)(x).

(27) (The function sin)$'_{\restriction \mathbb{R}} =$ the function cos.

(28) (The function cos)$'_{\restriction \mathbb{R}} = -$the function sin.

(29) ($-$the function cos)$'_{\restriction \mathbb{R}} =$ the function sin.

(30) (The function sinh)$'_{\restriction \mathbb{R}} =$ the function cosh.

(31) (The function cosh)$'_{\restriction \mathbb{R}} =$ the function sinh.

(32) (The function exp)$'_{\restriction \mathbb{R}} =$ the function exp.

(33) Suppose $Z \subseteq \operatorname{dom}$ (the function tan) and for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}$ and (the function cos)$(x) \neq 0$. Then

(i) the function tan is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds (the function tan)$'_{\restriction Z}(x) = \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(34) Suppose that

(i) $Z \subseteq \operatorname{dom}$ (the function cot), and

(ii) for every x such that $x \in Z$ holds $f(x) = -\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$ and (the function sin)$(x) \neq 0$.

Then

(iii) the function cot is differentiable on Z, and

(iv) for every x such that $x \in Z$ holds (the function cot)$'_{\restriction Z}(x) = -\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(35) For every real number r holds $\operatorname{dom}(\mathbb{R} \longmapsto r) = \mathbb{R}$ and $\operatorname{rng}(\mathbb{R} \longmapsto r) \subseteq \mathbb{R}$.

Let r be a real number. The functor $\operatorname{Cst} r$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined as follows:

(Def. 1) $\operatorname{Cst} r = \mathbb{R} \longmapsto r$.

We now state two propositions:

(36) For all real numbers a, b and for every closed-interval subset A of $\mathbb{R}$ holds $\chi_{A,A} = \operatorname{Cst} 1 \restriction A$.

(37) For all real numbers a, b and for every closed-interval subset A of $\mathbb{R}$ such that $A = [a, b]$ holds $\sup A = b$ and $\inf A = a$.

2. Several Integrability Formulas of Special Functions

The following propositions are true:

(38) For all real numbers a, b such that $a \leq b$ holds $\int\limits_a^b \operatorname{Cst} 1(x) dx = b - a$.

(39) $\int\limits_A$ (the function cos)$(x)dx$ = (the function sin)$(\sup A)$ − (the function sin)$(\inf A)$.

(40) If $A = [0, \frac{\pi}{2}]$, then $\int\limits_A$ (the function cos)$(x)dx = 1$.

(41) If $A = [0, \pi]$, then $\int\limits_A$ (the function cos)$(x)dx = 0$.

(42) If $A = [0, \frac{\pi \cdot 3}{2}]$, then $\int\limits_A$ (the function cos)$(x)dx = -1$.

(43) If $A = [0, \pi \cdot 2]$, then $\int\limits_A$ (the function cos)$(x)dx = 0$.

(44) If $A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$, then $\int\limits_A$ (the function cos)$(x)dx = 0$.

(45) If $A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$, then $\int\limits_A$ (the function cos)$(x)dx =$ $-2 \cdot \sin x$.

(46) $\int\limits_A$ ($-$the function sin)$(x)dx =$ (the function cos)$(\sup A) -$ (the function cos)$(\inf A)$.

(47) If $A = [0, \frac{\pi}{2}]$, then $\int\limits_A$ ($-$the function sin)$(x)dx = -1$.

(48) If $A = [0, \pi]$, then $\int\limits_A$ ($-$the function sin)$(x)dx = -2$.

(49) If $A = [0, \frac{\pi \cdot 3}{2}]$, then $\int\limits_A$ ($-$the function sin)$(x)dx = -1$.

(50) If $A = [0, \pi \cdot 2]$, then $\int\limits_A$ ($-$the function sin)$(x)dx = 0$.

(51) If $A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$, then $\int\limits_A$ ($-$the function sin)$(x)dx = -2$.

(52) If $A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$, then $\int\limits_A$ ($-$the function sin)$(x)dx =$ $-2 \cdot \cos x$.

(53) $\int\limits_A$ (the function exp)$(x)dx =$ (the function exp)$(\sup A) -$ (the function exp)$(\inf A)$.

(54) If $A = [0, 1]$, then $\int\limits_A$ (the function exp)$(x)dx = e - 1$.

(55) $\int\limits_A$ (the function sinh)$(x)dx =$ (the function cosh)$(\sup A) -$ (the function cosh)$(\inf A)$.

(56) If $A = [0, 1]$, then $\int\limits_A$ (the function sinh)$(x)dx = \frac{(e-1)^{\mathbf{2}}}{2 \cdot e}$.

(57) $\int\limits_A$ (the function cosh)$(x)dx =$ (the function sinh)$(\sup A) -$ (the function

sinh)(inf A).

(58) If $A = [0, 1]$, then $\int\limits_A$ (the function cosh)$(x)dx = \dfrac{e^{\mathbf{2}} - 1}{2 \cdot e}$.

(59) Suppose that

(i) $A \subseteq Z$,

(ii) dom (the function tan) $= Z$,

(iii) dom (the function tan) $= \operatorname{dom} f_2$,

(iv) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}$ and (the function cos)$(x) \neq 0$, and

(v) f_2 is continuous on A.

Then $\int\limits_A f_2(x)dx =$ (the function tan)(sup A) − (the function tan)(inf A).

(60) Suppose that

(i) $A \subseteq Z$,

(ii) dom (the function cot) $= Z$,

(iii) dom (the function cot) $= \operatorname{dom} f_2$,

(iv) for every x such that $x \in Z$ holds $f_2(x) = -\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$ and (the function sin)$(x) \neq 0$, and

(v) f_2 is continuous on A.

Then $\int\limits_A f_2(x)dx =$ (the function cot)(sup A) − (the function cot)(inf A).

(61) Suppose dom (the function tanh) $= \operatorname{dom} f_2$ and for every x such that $x \in \mathbb{R}$ holds $f_2(x) = \frac{1}{(\text{the function cosh})(x)^{\mathbf{2}}}$ and f_2 is continuous on A. Then $\int\limits_A f_2(x)dx =$ (the function tanh)(sup A) − (the function tanh)(inf A).

(62) Suppose $A \subseteq]-1, 1[$ and dom((the function arcsin)$'_{\restriction]-1,1[}$) $= \operatorname{dom} f_2$ and for every x holds $x \in]-1, 1[$ and $f_2(x) = \frac{1}{\sqrt{1-x^{\mathbf{2}}}}$ and f_2 is continuous on A. Then $\int\limits_A f_2(x)dx =$ (the function arcsin)(sup A) − (the function arcsin)(inf A).

(63) Suppose $A \subseteq]-1, 1[$ and dom((the function arccos)$'_{\restriction]-1,1[}$) $= \operatorname{dom} f_2$ and for every x holds $x \in]-1, 1[$ and $f_2(x) = -\frac{1}{\sqrt{1-x^{\mathbf{2}}}}$ and f_2 is continuous on A. Then $\int\limits_A f_2(x)dx =$ (the function arccos)(sup A) − (the function arccos)(inf A).

(64) Suppose that

(i) $A = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$,

(ii) dom((the function arcsin)$'_{\restriction]-1,1[}$) $= \operatorname{dom} f_2$,

(iii) for every x holds $x \in]-1,1[$ and $f_2(x) = \frac{1}{\sqrt{1-x^2}}$, and
(iv) f_2 is continuous on A.
Then $\int\limits_A f_2(x)dx = \frac{\pi}{2}$.

(65) Suppose that
(i) $A = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$,
(ii) $\mathrm{dom}((\text{the function arccos})'_{\restriction]-1,1[}) = \mathrm{dom}\, f_2$,
(iii) for every x holds $x \in]-1,1[$ and $f_2(x) = -\frac{1}{\sqrt{1-x^2}}$, and
(iv) f_2 is continuous on A.
Then $\int\limits_A f_2(x)dx = -\frac{\pi}{2}$.

(66) Suppose that f is differentiable on Z and g is differentiable on Z and $A \subseteq Z$ and $f'_{\restriction Z}$ is integrable on A and $f'_{\restriction Z}$ is bounded on A and $g'_{\restriction Z}$ is integrable on A and $g'_{\restriction Z}$ is bounded on A. Then $\int\limits_A (f'_{\restriction Z} + g'_{\restriction Z})(x)dx = ((f(\sup A) - f(\inf A)) + g(\sup A)) - g(\inf A)$.

(67) Suppose that f is differentiable on Z and g is differentiable on Z and $A \subseteq Z$ and $f'_{\restriction Z}$ is integrable on A and $f'_{\restriction Z}$ is bounded on A and $g'_{\restriction Z}$ is integrable on A and $g'_{\restriction Z}$ is bounded on A. Then $\int\limits_A (f'_{\restriction Z} - g'_{\restriction Z})(x)dx = f(\sup A) - f(\inf A) - (g(\sup A) - g(\inf A))$.

(68) Suppose f is differentiable on Z and $A \subseteq Z$ and $f'_{\restriction Z}$ is integrable on A and $f'_{\restriction Z}$ is bounded on A. Then $\int\limits_A (r\, f'_{\restriction Z})(x)dx = r \cdot f(\sup A) - r \cdot f(\inf A)$.

(69) $\int\limits_A ((\text{the function sin}) + (\text{the function cos}))(x)dx = (((-\text{the function cos})(\sup A) - (-\text{the function cos})(\inf A)) + (\text{the function sin})(\sup A)) - (\text{the function sin})(\inf A)$.

(70) If $A = [0, \frac{\pi}{2}]$, then $\int\limits_A ((\text{the function sin}) + (\text{the function cos}))(x)dx = 2$.

(71) If $A = [0, \pi]$, then $\int\limits_A ((\text{the function sin}) + (\text{the function cos}))(x)dx = 2$.

(72) If $A = [0, \frac{\pi \cdot 3}{2}]$, then $\int\limits_A ((\text{the function sin}) + (\text{the function cos}))(x)dx = 0$.

(73) If $A = [0, \pi \cdot 2]$, then $\int\limits_A ((\text{the function sin}) + (\text{the function cos}))(x)dx =$

0.

(74) If $A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$, then $\int\limits_A ((\text{the function sin}) + (\text{the function cos}))(x)dx = 2.$

(75) If $A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$, then $\int\limits_A ((\text{the function sin}) + (\text{the function cos}))(x)dx = 2 \cdot \cos x - 2 \cdot \sin x.$

(76) $\int\limits_A ((\text{the function sinh}) + (\text{the function cosh}))(x)dx = (((\text{the function cosh})(\sup A) - (\text{the function cosh})(\inf A)) + (\text{the function sinh})(\sup A)) - (\text{the function sinh})(\inf A).$

(77) If $A = [0, 1]$, then $\int\limits_A ((\text{the function sinh}) + (\text{the function cosh}))(x)dx = e - 1.$

(78) $\int\limits_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = (-\text{the function cos})(\sup A) - (-\text{the function cos})(\inf A) - ((\text{the function sin})(\sup A) - (\text{the function sin})(\inf A)).$

(79) If $A = [0, \frac{\pi}{2}]$, then $\int\limits_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = 0.$

(80) If $A = [0, \pi]$, then $\int\limits_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = 2.$

(81) If $A = [0, \frac{\pi \cdot 3}{2}]$, then $\int\limits_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = 2.$

(82) If $A = [0, \pi \cdot 2]$, then $\int\limits_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = 0.$

(83) If $A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$, then $\int\limits_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = 2.$

(84) If $A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$, then $\int\limits_A ((\text{the function sin}) - (\text{the function cos}))(x)dx = 2 \cdot \cos x + 2 \cdot \sin x.$

(85) $\int\limits_A (r\,(\text{the function sin}))(x)dx = r \cdot (-\text{the function cos})(\sup A) - r \cdot (-\text{the function cos})(\inf A).$

(86) $\int_A (r\,(\text{the function cos}))(x)dx = r \cdot (\text{the function sin})(\sup A) - r \cdot (\text{the function sin})(\inf A)$.

(87) $\int_A (r\,(\text{the function sinh}))(x)dx = r \cdot (\text{the function cosh})(\sup A) - r \cdot (\text{the function cosh})(\inf A)$.

(88) $\int_A (r\,(\text{the function cosh}))(x)dx = r \cdot (\text{the function sinh})(\sup A) - r \cdot (\text{the function sinh})(\inf A)$.

(89) $\int_A (r\,(\text{the function exp}))(x)dx = r \cdot (\text{the function exp})(\sup A) - r \cdot (\text{the function exp})(\inf A)$.

(90) $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = \frac{1}{2} \cdot ((\text{the function cos})(\inf A) \cdot (\text{the function cos})(\inf A) - (\text{the function cos})(\sup A) \cdot (\text{the function cos})(\sup A))$.

(91) If $A = [0, \frac{\pi}{2}]$, then $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = \frac{1}{2}$.

(92) If $A = [0, \pi]$, then $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = 0$.

(93) If $A = [0, \pi \cdot \frac{3}{2}]$, then $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = \frac{1}{2}$.

(94) If $A = [0, \pi \cdot 2]$, then $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = 0$.

(95) If $A = [2 \cdot n \cdot \pi, (2 \cdot n + 1) \cdot \pi]$, then $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = 0$.

(96) If $A = [x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$, then $\int_A ((\text{the function sin})\,(\text{the function cos}))(x)dx = 0$.

(97) $\int_A ((\text{the function sin})\,(\text{the function sin}))(x)dx = ((\text{the function cos})(\inf A) \cdot (\text{the function sin})(\inf A) - (\text{the function cos})(\sup A) \cdot (\text{the function sin})(\sup A)) + \int_A ((\text{the function cos})\,(\text{the function cos}))(x)dx$.

(98) $\int\limits_A ((\text{the function sinh})\,(\text{the function sinh}))(x)dx = (\text{the function cosh})$ $(\sup A)\cdot(\text{the function sinh})(\sup A) - (\text{the function cosh})(\inf A)\cdot(\text{the function sinh})(\inf A) - \int\limits_A ((\text{the function cosh})\,(\text{the function cosh}))(x)dx.$

(99) $\int\limits_A ((\text{the function sinh})\,(\text{the function cosh}))(x)dx = \frac{1}{2}\cdot((\text{the function cosh})(\sup A)\cdot(\text{the function cosh})(\sup A) - (\text{the function cosh})(\inf A)\cdot(\text{the function cosh})(\inf A)).$

(100) $\int\limits_A ((\text{the function exp})\,(\text{the function exp}))(x)dx = \frac{1}{2}\cdot((\text{the function exp})(\sup A)^{\mathbf{2}} - (\text{the function exp})(\inf A)^{\mathbf{2}}).$

(101) $\int\limits_A ((\text{the function exp})\,((\text{the function sin}) + (\text{the function cos})))(x)dx = ((\text{the function exp})\ (\text{the function sin}))(\sup A) - ((\text{the function exp})\ (\text{the function sin}))(\inf A).$

(102) $\int\limits_A ((\text{the function exp})\,((\text{the function cos}) - (\text{the function sin})))(x)dx = ((\text{the function exp})\ (\text{the function cos}))(\sup A) - ((\text{the function exp})\ (\text{the function cos}))(\inf A).$

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Basic Properties of the Rank of Matrices over a Field

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Summary. In this paper I present selected properties of triangular matrices and basic properties of the rank of matrices over a field.

I define a submatrix as a matrix formed by selecting certain rows and columns from a bigger matrix. That is in my considerations, as an array, it is cut down to those entries constrained by row and column. Then I introduce the concept of the rank of a $m \times n$ matrix A by the condition: A has the rank r if and only if, there is a $r \times r$ submatrix of A with a non-zero determinant, and for every $k \times k$ submatrix of A with a non-zero determinant we have $k \leq r$.

At the end, I prove that the rank defined by the size of the biggest submatrix with a non-zero determinant of a matrix A, is the same as the maximal number of linearly independent rows of A.

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The articles [27], [10], [37], [23], [1], [2], [12], [38], [39], [7], [8], [3], [4], [24], [36], [31], [15], [6], [13], [28], [14], [41], [30], [19], [34], [42], [9], [22], [16], [11], [25], [40], [18], [20], [26], [33], [21], [17], [35], [32], [29], [43], and [5] provide the terminology and notation for this paper.

1. Triangular Matrices

For simplicity, we use the following convention: x, X, Y are sets, D is a non empty set, i, j, k, m, n, m', n' are elements of $\mathbb{N}$, i_0, j_0, n_0, m_0 are non zero elements of $\mathbb{N}$, K is a field, a, b are elements of K, p is a finite sequence of elements of K, and M is a matrix over K of dimension n.

Next we state a number of propositions:

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(1) For every matrix A over D of dimension $n \times m$ holds if $n = 0$, then $m = 0$ iff $\operatorname{len} A = n$ and $\operatorname{width} A = m$.

(2) The following statements are equivalent

(i) M is a lower triangular matrix over K of dimension n,

(ii) M^{T} is an upper triangular matrix over K of dimension n.

(3) The diagonal of $M =$ the diagonal of M^{T}.

(4) Let p_1 be an element of the permutations of n-element set. Suppose $p_1 \neq \operatorname{idseq}(n)$. Then there exists i such that $i \in \operatorname{Seg} n$ and $p_1(i) > i$ and there exists j such that $j \in \operatorname{Seg} n$ and $p_1(j) < j$.

(5) Let M be a matrix over K of dimension n and p_1 be an element of the permutations of n-element set. Suppose that

(i) $p_1 \neq \operatorname{idseq}(n)$, and

(ii) M is a lower triangular matrix over K of dimension n or an upper triangular matrix over K of dimension n.

Then (the product on paths of $M)(p_1) = 0_K$.

(6) Let M be a matrix over K of dimension n and I be an element of the permutations of n-element set. If $I = \operatorname{idseq}(n)$, then the diagonal of $M = I$-Path M.

(7) Let M be an upper triangular matrix over K of dimension n. Then $\operatorname{Det} M =$ (the multiplication of $K) \circledast$ (the diagonal of M).

(8) Let M be a lower triangular matrix over K of dimension n. Then $\operatorname{Det} M =$ (the multiplication of $K) \circledast$ (the diagonal of M).

(9) For every finite set X and for every n holds $\overline{\overline{\{Y; Y \text{ ranges over subsets of } X\colon \operatorname{card} Y = n\}}} = \binom{\operatorname{card} X}{n}$.

(10) $\overline{\overline{2\mathrm{Set}\,\operatorname{Seg} n}} = \binom{n}{2}$.

(11) Let R be an element of the permutations of n-element set. If $R = \operatorname{Rev}(\operatorname{idseq}(n))$, then R is even iff $\binom{n}{2} \bmod 2 = 0$.

(12) Let M be a matrix over K of dimension n and R be a permutation of $\operatorname{Seg} n$. Suppose $R = \operatorname{Rev}(\operatorname{idseq}(n))$ and for all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i + j \leq n$ holds $M_{i,j} = 0_K$. Then $M \cdot R$ is an upper triangular matrix over K of dimension n.

(13) Let M be a matrix over K of dimension n and R be a permutation of $\operatorname{Seg} n$. Suppose $R = \operatorname{Rev}(\operatorname{idseq}(n))$ and for all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i + j > n + 1$ holds $M_{i,j} = 0_K$. Then $M \cdot R$ is a lower triangular matrix over K of dimension n.

(14) Let M be a matrix over K of dimension n and R be an element of the permutations of n-element set. Suppose that

(i) $R = \operatorname{Rev}(\operatorname{idseq}(n))$, and

(ii) for all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i + j \leq n$ holds $M_{i,j} = 0_K$ or for all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i+j > n+1$

holds $M_{i,j} = 0_K$.
Then $\mathrm{Det}\, M = (-1)^{\mathrm{sgn}(R)}$(the multiplication of $K \odot (R\text{-Path}\, M)$).

(15) Let M be a matrix over K of dimension n and M_1, M_2 be upper triangular matrices over K of dimension n. Suppose $M = M_1 \cdot M_2$. Then
 (i) M is an upper triangular matrix over K of dimension n, and
 (ii) the diagonal of M = (the diagonal of M_1) $\bullet$ (the diagonal of M_2).

(16) Let M be a matrix over K of dimension n and M_1, M_2 be lower triangular matrices over K of dimension n. Suppose $M = M_1 \cdot M_2$. Then
 (i) M is a lower triangular matrix over K of dimension n, and
 (ii) the diagonal of M = (the diagonal of M_1) $\bullet$ (the diagonal of M_2).

2. The Rank of Matrices

Let D be a non empty set, let M be a matrix over D, let n, m be natural numbers, let n_1 be an element of $\mathbb{N}^n$, and let m_1 be an element of $\mathbb{N}^m$. The functor $\mathrm{Segm}(M, n_1, m_1)$ yielding a matrix over D of dimension $n \times m$ is defined as follows:

(Def. 1) For all natural numbers i, j such that $\langle i, j \rangle \in$ the indices of $\mathrm{Segm}(M, n_1, m_1)$ holds $(\mathrm{Segm}(M, n_1, m_1))_{i,j} = M_{n_1(i), m_1(j)}$.

For simplicity, we follow the rules: A denotes a matrix over D, A' denotes a matrix over D of dimension $n' \times m'$, M' denotes a matrix over K of dimension $n' \times m'$, n_1, n_2, n_3 denote elements of $\mathbb{N}^n$, m_1, m_2 denote elements of $\mathbb{N}^m$, and M denotes a matrix over K.

Next we state a number of propositions:

(17) If $[\!:\mathrm{rng}\, n_1, \mathrm{rng}\, m_1 :\!] \subseteq$ the indices of A, then $\langle i, j \rangle \in$ the indices of $\mathrm{Segm}(A, n_1, m_1)$ iff $\langle n_1(i), m_1(j) \rangle \in$ the indices of A.

(18) If $[\!:\mathrm{rng}\, n_1, \mathrm{rng}\, m_1 :\!] \subseteq$ the indices of A and $n = 0$ iff $m = 0$, then $(\mathrm{Segm}(A, n_1, m_1))^{\mathrm{T}} = \mathrm{Segm}(A^{\mathrm{T}}, m_1, n_1)$.

(19) If $[\!:\mathrm{rng}\, n_1, \mathrm{rng}\, m_1 :\!] \subseteq$ the indices of A and if $m = 0$, then $n = 0$, then $\mathrm{Segm}(A, n_1, m_1) = (\mathrm{Segm}(A^{\mathrm{T}}, m_1, n_1))^{\mathrm{T}}$.

(20) For every matrix A over D of dimension 1 holds $A = \langle\langle A_{1,1} \rangle\rangle$.

(21) If $n = 1$ and $m = 1$, then $\mathrm{Segm}(A, n_1, m_1) = \langle\langle A_{n_1(1), m_1(1)} \rangle\rangle$.

(22) For every matrix A over D of dimension 2 holds $A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$.

(23) If $n = 2$ and $m = 2$, then $\mathrm{Segm}(A, n_1, m_1) =$
$\begin{pmatrix} A_{n_1(1), m_1(1)} & A_{n_1(1), m_1(2)} \\ A_{n_1(2), m_1(1)} & A_{n_1(2), m_1(2)} \end{pmatrix}$.

(24) If $i \in \mathrm{Seg}\, n$ and $\mathrm{rng}\, m_1 \subseteq \mathrm{Seg}\, \mathrm{width}\, A$, then $\mathrm{Line}(\mathrm{Segm}(A, n_1, m_1), i) = \mathrm{Line}(A, n_1(i)) \cdot m_1$.

(25) If $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $n_1(i) = n_1(j)$, then $\operatorname{Line}(\operatorname{Segm}(A, n_1, m_1), i) = \operatorname{Line}(\operatorname{Segm}(A, n_1, m_1), j)$.

(26) If $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $n_1(i) = n_1(j)$ and $i \neq j$, then $\operatorname{Det} \operatorname{Segm}(M, n_1, n_2) = 0_K$.

(27) If n_1 is not one-to-one, then $\operatorname{Det} \operatorname{Segm}(M, n_1, n_2) = 0_K$.

(28) If $j \in \operatorname{Seg} m$ and $\operatorname{rng} n_1 \subseteq \operatorname{Seg} \operatorname{len} A$, then $(\operatorname{Segm}(A, n_1, m_1))_{\square, j} = A_{\square, m_1(j)} \cdot n_1$.

(29) If $i \in \operatorname{Seg} m$ and $j \in \operatorname{Seg} m$ and $m_1(i) = m_1(j)$, then $(\operatorname{Segm}(A, n_1, m_1))_{\square, i} = (\operatorname{Segm}(A, n_1, m_1))_{\square, j}$.

(30) If $i \in \operatorname{Seg} m$ and $j \in \operatorname{Seg} m$ and $m_1(i) = m_1(j)$ and $i \neq j$, then $\operatorname{Det} \operatorname{Segm}(M, m_2, m_1) = 0_K$.

(31) If m_1 is not one-to-one, then $\operatorname{Det} \operatorname{Segm}(M, m_2, m_1) = 0_K$.

(32) Let n_1, n_2 be elements of $\mathbb{N}^n$. Suppose n_1 is one-to-one and n_2 is one-to-one and $\operatorname{rng} n_1 = \operatorname{rng} n_2$. Then there exists a permutation p_1 of $\operatorname{Seg} n$ such that $n_2 = n_1 \cdot p_1$.

(33) For every function f from $\operatorname{Seg} n$ into $\operatorname{Seg} n$ such that $n_2 = n_1 \cdot f$ holds $\operatorname{Segm}(A, n_2, m_1) = \operatorname{Segm}(A, n_1, m_1) \cdot f$.

(34) For every function f from $\operatorname{Seg} m$ into $\operatorname{Seg} m$ such that $m_2 = m_1 \cdot f$ holds $(\operatorname{Segm}(A, n_1, m_2))^{\mathrm{T}} = (\operatorname{Segm}(A, n_1, m_1))^{\mathrm{T}} \cdot f$.

(35) Let p_1 be an element of the permutations of n-element set. If $n_2 = n_3 \cdot p_1$, then $\operatorname{Det} \operatorname{Segm}(M, n_2, n_1) = (-1)^{\operatorname{sgn}(p_1)} \operatorname{Det} \operatorname{Segm}(M, n_3, n_1)$ and $\operatorname{Det} \operatorname{Segm}(M, n_1, n_2) = (-1)^{\operatorname{sgn}(p_1)} \operatorname{Det} \operatorname{Segm}(M, n_1, n_3)$.

(36) For all elements n_1, n_2, n'_1, n'_2 of $\mathbb{N}^n$ such that $\operatorname{rng} n_1 = \operatorname{rng} n'_1$ and $\operatorname{rng} n_2 = \operatorname{rng} n'_2$ holds $\operatorname{Det} \operatorname{Segm}(M, n_1, n_2) = \operatorname{Det} \operatorname{Segm}(M, n'_1, n'_2)$ or $\operatorname{Det} \operatorname{Segm}(M, n_1, n_2) = -\operatorname{Det} \operatorname{Segm}(M, n'_1, n'_2)$.

(37) Let F, F_1 be finite sequences of elements of D and given n_1, m_1. Suppose $\operatorname{len} F = \operatorname{width} A'$ and $F_1 = F \cdot m_1$ and $[\!:\operatorname{rng} n_1, \operatorname{rng} m_1 :\!] \subseteq$ the indices of A'. Let given i, j. If ${n_1}^{-1}(\{j\}) = \{i\}$, then $\operatorname{RLine}(\operatorname{Segm}(A', n_1, m_1), i, F_1) = \operatorname{Segm}(\operatorname{RLine}(A', j, F), n_1, m_1)$.

(38) Let F be a finite sequence of elements of D and given i, n_1. If $i \notin \operatorname{rng} n_1$ and $[\!:\operatorname{rng} n_1, \operatorname{rng} m_1 :\!] \subseteq$ the indices of A', then $\operatorname{Segm}(A', n_1, m_1) = \operatorname{Segm}(\operatorname{RLine}(A', i, F), n_1, m_1)$.

(39) If $i \in \operatorname{Seg} n'$ and $i \in \operatorname{rng} n_1$ and $[\!:\operatorname{rng} n_1, \operatorname{rng} m_1 :\!] \subseteq$ the indices of A', then there exists n_2 such that $\operatorname{rng} n_2 = (\operatorname{rng} n_1 \setminus \{i\}) \cup \{j\}$ and $\operatorname{Segm}(\operatorname{RLine}(A', i, \operatorname{Line}(A', j)), n_1, m_1) = \operatorname{Segm}(A', n_2, m_1)$.

(40) For every finite sequence F of elements of D such that $i \notin \operatorname{Seg} \operatorname{len} A'$ holds $\operatorname{RLine}(A', i, F) = A'$.

Let n, m be natural numbers, let K be a field, let M be a matrix over K of dimension $n \times m$, and let a be an element of K. Then $a \cdot M$ is a matrix over

K of dimension $n \times m$.

We now state two propositions:

(41) If $[\!: \operatorname{rng} n_1, \operatorname{rng} m_1 :\!] \subseteq$ the indices of M, then $a \cdot \operatorname{Segm}(M, n_1, m_1) = \operatorname{Segm}(a \cdot M, n_1, m_1)$.

(42) If $n_1 = \operatorname{idseq}(\operatorname{len} A)$ and $m_1 = \operatorname{idseq}(\operatorname{width} A)$, then $\operatorname{Segm}(A, n_1, m_1) = A$.

Let us observe that there exists a subset of $\mathbb{N}$ which is empty, without zero, and finite and there exists a subset of $\mathbb{N}$ which is non empty, without zero, and finite.

Let us consider n. Observe that $\operatorname{Seg} n$ is without zero.

Let X be a without zero set and let Y be a set. One can verify that $X \setminus Y$ is without zero and $X \cap Y$ is without zero.

One can prove the following proposition

(43) For every finite without zero subset N of $\mathbb{N}$ there exists k such that $N \subseteq \operatorname{Seg} k$.

Let N be a finite without zero subset of $\mathbb{N}$. Then $\operatorname{Sgm} N$ is an element of $\mathbb{N}^{\operatorname{card} N}$.

Let D be a non empty set, let A be a matrix over D, and let P, Q be without zero finite subsets of $\mathbb{N}$. The functor $\operatorname{Segm}(A, P, Q)$ yields a matrix over D of dimension $\operatorname{card} P \times \operatorname{card} Q$ and is defined by:

(Def. 2) $\operatorname{Segm}(A, P, Q) = \operatorname{Segm}(A, \operatorname{Sgm} P, \operatorname{Sgm} Q)$.

Next we state two propositions:

(44) $\operatorname{Segm}(A, \{i_0\}, \{j_0\}) = \langle\langle A_{i_0, j_0} \rangle\rangle$.

(45) If $i_0 < j_0$ and $n_0 < m_0$, then $\operatorname{Segm}(A, \{i_0, j_0\}, \{n_0, m_0\}) = \begin{pmatrix} A_{i_0,n_0} & A_{i_0,m_0} \\ A_{j_0,n_0} & A_{j_0,m_0} \end{pmatrix}$.

In the sequel P, P_1, P_2, Q, Q_1, Q_2 are without zero finite subsets of $\mathbb{N}$.

The following propositions are true:

(46) $\operatorname{Segm}(A, \operatorname{Seg} \operatorname{len} A, \operatorname{Seg} \operatorname{width} A) = A$.

(47) If $i \in \operatorname{Seg} \operatorname{card} P$ and $Q \subseteq \operatorname{Seg} \operatorname{width} A$, then $\operatorname{Line}(\operatorname{Segm}(A, P, Q), i) = \operatorname{Line}(A, (\operatorname{Sgm} P)(i)) \cdot \operatorname{Sgm} Q$.

(48) If $i \in \operatorname{Seg} \operatorname{card} P$, then $\operatorname{Line}(\operatorname{Segm}(A, P, \operatorname{Seg} \operatorname{width} A), i) = \operatorname{Line}(A, (\operatorname{Sgm} P)(i))$.

(49) If $j \in \operatorname{Seg} \operatorname{card} Q$ and $P \subseteq \operatorname{Seg} \operatorname{len} A$, then $(\operatorname{Segm}(A, P, Q))_{\square, j} = A_{\square, (\operatorname{Sgm} Q)(j)} \cdot \operatorname{Sgm} P$.

(50) If $j \in \operatorname{Seg} \operatorname{card} Q$, then $(\operatorname{Segm}(A, \operatorname{Seg} \operatorname{len} A, Q))_{\square, j} = A_{\square, (\operatorname{Sgm} Q)(j)}$.

(51) $\operatorname{Segm}(A, \operatorname{Seg} \operatorname{len} A \setminus \{i\}, \operatorname{Seg} \operatorname{width} A) = A_{\restriction i}$.

(52) $\operatorname{Segm}(M, \operatorname{Seg} \operatorname{len} M, \operatorname{Seg} \operatorname{width} M \setminus \{i\}) =$ the deleting of i-column in M.

(53) $(\operatorname{Sgm} P)^{-1}(X)$ is a without zero finite subset of $\mathbb{N}$.

(54) If $X \subseteq P$, then $\mathrm{Sgm}\, X = \mathrm{Sgm}\, P \cdot \mathrm{Sgm}((\mathrm{Sgm}\, P)^{-1}(X))$.

(55) $[\!:(\mathrm{Sgm}\, P)^{-1}(X), (\mathrm{Sgm}\, Q)^{-1}(Y)\,:\!] \subseteq$ the indices of $\mathrm{Segm}(A, P, Q)$.

(56) If $P \subseteq P_1$ and $Q \subseteq Q_1$ and $P_2 = (\mathrm{Sgm}\, P_1)^{-1}(P)$ and $Q_2 = (\mathrm{Sgm}\, Q_1)^{-1}(Q)$, then $[\!:\mathrm{rng}\,\mathrm{Sgm}\, P_2, \mathrm{rng}\,\mathrm{Sgm}\, Q_2\,:\!] \subseteq$ the indices of $\mathrm{Segm}(A, P_1, Q_1)$ and $\mathrm{Segm}(\mathrm{Segm}(A, P_1, Q_1), P_2, Q_2) = \mathrm{Segm}(A, P, Q)$.

(57) Suppose $P = \emptyset$ iff $Q = \emptyset$ and $[\!:P, Q\,:\!] \subseteq$ the indices of $\mathrm{Segm}(A, P_1, Q_1)$. Then there exist P_2, Q_2 such that $P_2 \subseteq P_1$ and $Q_2 \subseteq Q_1$ and $P_2 = (\mathrm{Sgm}\, P_1)^{\circ} P$ and $Q_2 = (\mathrm{Sgm}\, Q_1)^{\circ} Q$ and $\mathrm{card}\, P_2 = \mathrm{card}\, P$ and $\mathrm{card}\, Q_2 = \mathrm{card}\, Q$ and $\mathrm{Segm}(\mathrm{Segm}(A, P_1, Q_1), P, Q) = \mathrm{Segm}(A, P_2, Q_2)$.

(58) For every matrix M over K of dimension n holds $\mathrm{Segm}(M, \mathrm{Seg}\, n \setminus \{i\}, \mathrm{Seg}\, n \setminus \{j\}) =$ the deleting of i-row and j-column in M.

(59) Let F, F_2 be finite sequences of elements of D. Suppose $\mathrm{len}\, F = \mathrm{width}\, A'$ and $F_2 = F \cdot \mathrm{Sgm}\, Q$ and $[\!:P, Q\,:\!] \subseteq$ the indices of A'. Then $\mathrm{RLine}(\mathrm{Segm}(A', P, Q), i, F_2) = \mathrm{Segm}(\mathrm{RLine}(A', (\mathrm{Sgm}\, P)(i), F), P, Q)$.

(60) Let F be a finite sequence of elements of D and given i, P. If $i \notin P$ and $[\!:P, Q\,:\!] \subseteq$ the indices of A', then $\mathrm{Segm}(A', P, Q) = \mathrm{Segm}(\mathrm{RLine}(A', i, F), P, Q)$.

(61) If $[\!:P, Q\,:\!] \subseteq$ the indices of A and $\mathrm{card}\, P = 0$ iff $\mathrm{card}\, Q = 0$, then $(\mathrm{Segm}(A, P, Q))^{\mathrm{T}} = \mathrm{Segm}(A^{\mathrm{T}}, Q, P)$.

(62) If $[\!:P, Q\,:\!] \subseteq$ the indices of A and if $\mathrm{card}\, Q = 0$, then $\mathrm{card}\, P = 0$, then $\mathrm{Segm}(A, P, Q) = (\mathrm{Segm}(A^{\mathrm{T}}, Q, P))^{\mathrm{T}}$.

(63) If $[\!:P, Q\,:\!] \subseteq$ the indices of M, then $a \cdot \mathrm{Segm}(M, P, Q) = \mathrm{Segm}(a \cdot M, P, Q)$.

Let D be a non empty set, let A be a matrix over D, and let P, Q be without zero finite subsets of $\mathbb{N}$. Let us assume that $\mathrm{card}\, P = \mathrm{card}\, Q$. The functor $\mathrm{EqSegm}(A, P, Q)$ yields a matrix over D of dimension $\mathrm{card}\, P$ and is defined by:

(Def. 3) $\mathrm{EqSegm}(A, P, Q) = \mathrm{Segm}(A, P, Q)$.

Next we state several propositions:

(64) For all P, Q, i, j such that $i \in \mathrm{Seg}\,\mathrm{card}\, P$ and $j \in \mathrm{Seg}\,\mathrm{card}\, P$ and $\mathrm{card}\, P = \mathrm{card}\, Q$ holds $\mathrm{Delete}(\mathrm{EqSegm}(M, P, Q), i, j) = \mathrm{EqSegm}(M, P \setminus \{(\mathrm{Sgm}\, P)(i)\}, Q \setminus \{(\mathrm{Sgm}\, Q)(j)\})$ and $\mathrm{card}(P \setminus \{(\mathrm{Sgm}\, P)(i)\}) = \mathrm{card}(Q \setminus \{(\mathrm{Sgm}\, Q)(j)\})$.

(65) For all M, P, P_1, Q_1 such that $\mathrm{card}\, P_1 = \mathrm{card}\, Q_1$ and $P \subseteq P_1$ and $\mathrm{Det}\,\mathrm{EqSegm}(M, P_1, Q_1) \neq 0_K$ there exists Q such that $Q \subseteq Q_1$ and $\mathrm{card}\, P = \mathrm{card}\, Q$ and $\mathrm{Det}\,\mathrm{EqSegm}(M, P, Q) \neq 0_K$.

(66) For all M, P_1, Q, Q_1 such that $\mathrm{card}\, P_1 = \mathrm{card}\, Q_1$ and $Q \subseteq Q_1$ and $\mathrm{Det}\,\mathrm{EqSegm}(M, P_1, Q_1) \neq 0_K$ there exists P such that $P \subseteq P_1$ and $\mathrm{card}\, P = \mathrm{card}\, Q$ and $\mathrm{Det}\,\mathrm{EqSegm}(M, P, Q) \neq 0_K$.

(67) If $\mathrm{card}\, P = \mathrm{card}\, Q$, then $[\!:P, Q\,:\!] \subseteq$ the indices of A iff $P \subseteq \mathrm{Seg}\,\mathrm{len}\, A$

and $Q \subseteq \operatorname{Seg} \operatorname{width} A$.

(68) Let given P, Q, i, j_0. Suppose $i \in \operatorname{Seg} n'$ and $j_0 \in \operatorname{Seg} n'$ and $i \in P$ and $j_0 \notin P$ and $\operatorname{card} P = \operatorname{card} Q$ and $[\!: P, Q :\!] \subseteq$ the indices of M'. Then $\operatorname{card} P = \operatorname{card}((P \setminus \{i\}) \cup \{j_0\})$ but $[\!: (P \setminus \{i\}) \cup \{j_0\}, Q :\!] \subseteq$ the indices of M' but $\operatorname{Det} \operatorname{EqSegm}(\operatorname{RLine}(M', i, \operatorname{Line}(M', j_0)), P, Q) = \operatorname{Det} \operatorname{EqSegm}(M', (P \setminus \{i\}) \cup \{j_0\}, Q)$ or $\operatorname{Det} \operatorname{EqSegm}(\operatorname{RLine}(M', i, \operatorname{Line}(M', j_0)), P, Q) = -\operatorname{Det} \operatorname{EqSegm}(M', (P \setminus \{i\}) \cup \{j_0\}, Q)$.

(69) If $\operatorname{card} P = \operatorname{card} Q$, then $[\!: P, Q :\!] \subseteq$ the indices of A iff $[\!: Q, P :\!] \subseteq$ the indices of A^{T}.

(70) If $[\!: P, Q :\!] \subseteq$ the indices of M and $\operatorname{card} P = \operatorname{card} Q$, then $\operatorname{Det} \operatorname{EqSegm}(M, P, Q) = \operatorname{Det} \operatorname{EqSegm}(M^{\mathrm{T}}, Q, P)$.

(71) For every matrix M over K of dimension n holds $\operatorname{Det}(a \cdot M) = \operatorname{power}_K(a, n) \cdot \operatorname{Det} M$.

(72) If $[\!: P, Q :\!] \subseteq$ the indices of M and $\operatorname{card} P = \operatorname{card} Q$, then $\operatorname{Det} \operatorname{EqSegm}(a \cdot M, P, Q) = \operatorname{power}_K(a, \operatorname{card} P) \cdot \operatorname{Det} \operatorname{EqSegm}(M, P, Q)$.

Let K be a field and let M be a matrix over K. The functor $\operatorname{rk}(M)$ yielding an element of $\mathbb{N}$ is defined by the conditions (Def. 4).

(Def. 4)(i) There exist P, Q such that $[\!: P, Q :\!] \subseteq$ the indices of M and $\operatorname{card} P = \operatorname{card} Q$ and $\operatorname{card} P = \operatorname{rk}(M)$ and $\operatorname{Det} \operatorname{EqSegm}(M, P, Q) \neq 0_K$, and

(ii) for all P_1, Q_1 such that $[\!: P_1, Q_1 :\!] \subseteq$ the indices of M and $\operatorname{card} P_1 = \operatorname{card} Q_1$ and $\operatorname{Det} \operatorname{EqSegm}(M, P_1, Q_1) \neq 0_K$ holds $\operatorname{card} P_1 \leq \operatorname{rk}(M)$.

The following propositions are true:

(73) For all P, Q such that $[\!: P, Q :\!] \subseteq$ the indices of M and $\operatorname{card} P = \operatorname{card} Q$ holds $\operatorname{card} P \leq \operatorname{len} M$ and $\operatorname{card} Q \leq \operatorname{width} M$.

(74) $\operatorname{rk}(M) \leq \operatorname{len} M$ and $\operatorname{rk}(M) \leq \operatorname{width} M$.

(75) If $[\!: \operatorname{rng} n_2, \operatorname{rng} n_3 :\!] \subseteq$ the indices of M and $\operatorname{Det} \operatorname{Segm}(M, n_2, n_3) \neq 0_K$, then there exist P_1, P_2 such that $P_1 = \operatorname{rng} n_2$ and $P_2 = \operatorname{rng} n_3$ and $\operatorname{card} P_1 = \operatorname{card} P_2$ and $\operatorname{card} P_1 = n$ and $\operatorname{Det} \operatorname{EqSegm}(M, P_1, P_2) \neq 0_K$.

(76) Let R_1 be an element of $\mathbb{N}$. Then $\operatorname{rk}(M) = R_1$ if and only if the following conditions are satisfied:

(i) there exist elements r_1, r_2 of $\mathbb{N}^{R_1}$ such that $[\!: \operatorname{rng} r_1, \operatorname{rng} r_2 :\!] \subseteq$ the indices of M and $\operatorname{Det} \operatorname{Segm}(M, r_1, r_2) \neq 0_K$, and

(ii) for all n, n_2, n_3 such that $[\!: \operatorname{rng} n_2, \operatorname{rng} n_3 :\!] \subseteq$ the indices of M and $\operatorname{Det} \operatorname{Segm}(M, n_2, n_3) \neq 0_K$ holds $n \leq R_1$.

(77) If $n = 0$ or $m = 0$, then $\operatorname{rk}(\operatorname{Segm}(M, n_1, m_1)) = 0$.

(78) If $[\!: \operatorname{rng} n_1, \operatorname{rng} m_1 :\!] \subseteq$ the indices of M, then $\operatorname{rk}(M) \geq \operatorname{rk}(\operatorname{Segm}(M, n_1, m_1))$.

(79) If $[\!: P, Q :\!] \subseteq$ the indices of M, then $\operatorname{rk}(M) \geq \operatorname{rk}(\operatorname{Segm}(M, P, Q))$.

(80) If $P \subseteq P_1$ and $Q \subseteq Q_1$, then $\operatorname{rk}(\operatorname{Segm}(M, P, Q)) \leq \operatorname{rk}(\operatorname{Segm}(M, P_1, Q_1))$.

(81) For all functions f, g such that $\operatorname{rng} f \subseteq \operatorname{rng} g$ there exists a function h such that $\operatorname{dom} h = \operatorname{dom} f$ and $\operatorname{rng} h \subseteq \operatorname{dom} g$ and $f = g \cdot h$.

(82) If $[\!:\operatorname{rng} n_1, \operatorname{rng} m_1 :\!]$ = the indices of M, then $\operatorname{rk}(M) = \operatorname{rk}(\operatorname{Segm}(M, n_1, m_1))$.

(83) For every matrix M over K of dimension n holds $\operatorname{rk}(M) = n$ iff $\operatorname{Det} M \neq 0_K$.

(84) $\operatorname{rk}(M) = \operatorname{rk}(M^{\mathrm{T}})$.

(85) For every matrix M over K of dimension $n \times m$ and for every permutation F of $\operatorname{Seg} n$ holds $\operatorname{rk}(M) = \operatorname{rk}(M \cdot F)$.

(86) If $a \neq 0_K$, then $\operatorname{rk}(M) = \operatorname{rk}(a \cdot M)$.

(87) Let p, p_2 be finite sequences of elements of K and f be a function. If $p_2 = p \cdot f$ and $\operatorname{rng} f \subseteq \operatorname{dom} p$, then $a \cdot p \cdot f = a \cdot p_2$.

(88) Let p, p_2, q, q_1 be finite sequences of elements of K and f be a function. If $p_2 = p \cdot f$ and $\operatorname{rng} f \subseteq \operatorname{dom} p$ and $q_1 = q \cdot f$ and $\operatorname{rng} f \subseteq \operatorname{dom} q$, then $(p + q) \cdot f = p_2 + q_1$.

(89) If $a \neq 0_K$, then $\operatorname{rk}(M') = \operatorname{rk}(\operatorname{RLine}(M', i, a \cdot \operatorname{Line}(M', i)))$.

(90) If $\operatorname{Line}(M, i) = \operatorname{width} M \mapsto 0_K$, then rk(the deleting of i-row in M) = $\operatorname{rk}(M)$.

(91) For every p such that $\operatorname{len} p = \operatorname{width} M'$ holds rk(the deleting of i-row in M') = $\operatorname{rk}(\operatorname{RLine}(M', i, 0_K \cdot p))$.

(92) If $j \in \operatorname{Seg} \operatorname{len} M'$ and if $i = j$, then $a \neq -\mathbf{1}_K$, then $\operatorname{rk}(M') = \operatorname{rk}(\operatorname{RLine}(M', i, \operatorname{Line}(M', i) + a \cdot \operatorname{Line}(M', j)))$.

(93) If $j \in \operatorname{Seg} \operatorname{len} M'$ and $j \neq i$, then rk(the deleting of i-row in M') = $\operatorname{rk}(\operatorname{RLine}(M', i, a \cdot \operatorname{Line}(M', j)))$.

(94) $\operatorname{rk}(M) > 0$ iff there exist i, j such that $\langle i, j \rangle \in$ the indices of M and $M_{i,j} \neq 0_K$.

(95) $\operatorname{rk}(M) = 0$ iff $M = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_K^{(\operatorname{len} M) \times (\operatorname{width} M)}$.

(96) $\operatorname{rk}(M) = 1$ if and only if the following conditions are satisfied:

(i) there exist i, j such that $\langle i, j \rangle \in$ the indices of M and $M_{i,j} \neq 0_K$, and

(ii) for all i_0, j_0, n_0, m_0 such that $i_0 \neq j_0$ and $n_0 \neq m_0$ and $[\!: \{i_0, j_0\}, \{n_0, m_0\} :\!] \subseteq$ the indices of M holds $\operatorname{Det} \operatorname{EqSegm}(M, \{i_0, j_0\}, \{n_0, m_0\}) = 0_K$.

(97) $\operatorname{rk}(M) = 1$ if and only if the following conditions are satisfied:

(i) there exist i, j such that $\langle i, j \rangle \in$ the indices of M and $M_{i,j} \neq 0_K$, and

(ii) for all i, j, n, m such that $[\!: \{i, j\}, \{n, m\} :\!] \subseteq$ the indices of M holds $M_{i,n} \cdot M_{j,m} = M_{i,m} \cdot M_{j,n}$.

(98) $\operatorname{rk}(M) = 1$ if and only if there exists i such that $i \in \operatorname{Seg} \operatorname{len} M$ and there exists j such that $j \in \operatorname{Seg} \operatorname{width} M$ and $M_{i,j} \neq 0_K$ and for every k such that $k \in \operatorname{Seg} \operatorname{len} M$ there exists a such that $\operatorname{Line}(M, k) = a \cdot \operatorname{Line}(M, i)$.

Let us consider K. Observe that there exists a matrix over K which is diagonal.

One can prove the following propositions:

(99) Let M be a diagonal matrix over K and N_1 be a set. Suppose $N_1 = \{i : \langle i, i \rangle \in$ the indices of $M \ \wedge \ M_{i,i} \neq 0_K\}$. Let given P, Q. If $[\!: P, Q :\!] \subseteq$ the indices of M and $\operatorname{card} P = \operatorname{card} Q$ and $\operatorname{Det} \operatorname{EqSegm}(M, P, Q) \neq 0_K$, then $P \subseteq N_1$ and $Q \subseteq N_1$.

(100) For every diagonal matrix M over K and for every P such that $[\!: P, P :\!] \subseteq$ the indices of M holds $\operatorname{Segm}(M, P, P)$ is diagonal.

(101) Let M be a diagonal matrix over K and N_1 be a set. If $N_1 = \{i : \langle i, i \rangle \in$ the indices of $M \ \wedge \ M_{i,i} \neq 0_K\}$, then $\operatorname{rk}(M) = \overline{\overline{N_1}}$.

For simplicity, we adopt the following rules: v, v_1, v_2, u denote vectors of the n-dimension vector space over K, t, t_1, t_2 denote elements of (the carrier of $K)^n$, L denotes a linear combination of the n-dimension vector space over K, and M, M_1 denote matrices over K of dimension $m \times n$.

We now state the proposition

(102)(i) The carrier of the n-dimension vector space over $K =$ (the carrier of $K)^n$,

(ii) $0_{\text{the } n\text{-dimension vector space over } K} = n \mapsto 0_K$,

(iii) if $t_1 = v_1$ and $t_2 = v_2$, then $t_1 + t_2 = v_1 + v_2$, and

(iv) if $t = v$, then $a \cdot t = a \cdot v$.

Let us consider K, n. Then the n-dimension vector space over K is a strict vector space over K.

Let us consider K, n. One can verify that every vector of the n-dimension vector space over K is function-like and relation-like.

Let us consider K, m, n and let M be a matrix over K of dimension $m \times n$. We introduce $\operatorname{lines}(M)$ as a synonym of $\operatorname{rng} M$. We introduce M is without repeated line as a synonym of M is one-to-one.

Let K be a field, let us consider m, n, and let M be a matrix over K of dimension $m \times n$. Then $\operatorname{lines}(M)$ is a subset of the n-dimension vector space over K.

Next we state two propositions:

(103) $x \in \operatorname{lines}(M)$ iff there exists i such that $i \in \operatorname{Seg} m$ and $x = \operatorname{Line}(M, i)$.

(104) Let V be a finite subset of the n-dimension vector space over K. Then there exists a matrix M over K of dimension $\operatorname{card} V \times n$ such that M is without repeated line and $\operatorname{lines}(M) = V$.

Let us consider K, n and let F be a finite sequence of elements of the n-dimension vector space over K. The functor FinS2MX F yielding a matrix over K of dimension $\operatorname{len} F \times n$ is defined by:

(Def. 5) $\text{FinS2MX}\, F = F$.

Let us consider K, m, n and let M be a matrix over K of dimension $m \times n$. The functor MX2FinS M yielding a finite sequence of elements of the n-dimension vector space over K is defined as follows:

(Def. 6) $\text{MX2FinS}\, M = M$.

One can prove the following propositions:

(105) If $\operatorname{rk}(M) = m$, then M is without repeated line.

(106) If $i \in \operatorname{Seg} \operatorname{len} M$ and $a = L(M(i))$, then $\operatorname{Line}(\text{FinS2MX}(L\ \text{MX2FinS}\, M), i) = a \cdot \operatorname{Line}(M, i)$.

(107) If M is without repeated line and the support of $L \subseteq \operatorname{lines}(M)$ and $i \in \operatorname{Seg} n$, then $(\sum L)(i) = \sum((\text{FinS2MX}(L\ \text{MX2FinS}\, M))_{\square,i})$.

(108) Let given M, M_1. Suppose M is without repeated line and for every i such that $i \in \operatorname{Seg} m$ there exists a such that $\operatorname{Line}(M_1, i) = a \cdot \operatorname{Line}(M, i)$. Then there exists a linear combination L of $\operatorname{lines}(M)$ such that $L\ \text{MX2FinS}\, M = M_1$.

(109) Let given M. Suppose M is without repeated line. Then for every i such that $i \in \operatorname{Seg} m$ holds $\operatorname{Line}(M, i) \neq n \mapsto 0_K$ and for every M_1 such that for every i such that $i \in \operatorname{Seg} m$ there exists a such that $\operatorname{Line}(M_1, i) = a \cdot \operatorname{Line}(M, i)$ and for every j such that $j \in \operatorname{Seg} n$ holds $\sum((M_1)_{\square,j}) = 0_K$ holds $M_1 = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{m \times n}$ if and only if $\operatorname{lines}(M)$ is linearly independent.

(110) If $\operatorname{rk}(M) = m$, then $\operatorname{lines}(M)$ is linearly independent.

(111) Let M be a diagonal n-dimensional matrix over K. Suppose $\operatorname{rk}(M) = n$. Then $\operatorname{lines}(M)$ is a basis of the n-dimension vector space over K.

Let us consider K, n. Then the n-dimension vector space over K is a strict finite dimensional vector space over K.

The following propositions are true:

(112) $\dim(\text{the } n\text{-dimension vector space over } K) = n$.

(113) Let given M, i, a. Suppose that for every j such that $j \in \operatorname{Seg} m$ holds $M_{j,i} = a$. Then M is without repeated line if and only if $\operatorname{Segm}(M, \operatorname{Seg} \operatorname{len} M, \operatorname{Seg} \operatorname{width} M \setminus \{i\})$ is without repeated line.

(114) Let given M, i. Suppose M is without repeated line and $\operatorname{lines}(M)$ is linearly independent and for every j such that $j \in \operatorname{Seg} m$ holds $M_{j,i} = 0_K$. Then $\operatorname{lines}(\operatorname{Segm}(M, \operatorname{Seg} \operatorname{len} M, \operatorname{Seg} \operatorname{width} M \setminus \{i\}))$ is linearly independent.

(115) Let V be a vector space over K and U be a finite subset of V. Suppose U is linearly independent. Let u, v be vectors of V. If $u \in U$ and $v \in U$ and $u \neq v$, then $(U \setminus \{u\}) \cup \{u + a \cdot v\}$ is linearly independent.

(116) Let V be a vector space over K and u, v be vectors of V. Then $x \in \mathrm{Lin}(\{u, v\})$ if and only if there exist a, b such that $x = a \cdot u + b \cdot v$.

(117) Let given M. Suppose $\mathrm{lines}(M)$ is linearly independent and M is without repeated line. Let given i, j. Suppose $j \in \mathrm{Seg}\,\mathrm{len}\,M$ and $i \neq j$. Then $\mathrm{RLine}(M, i, \mathrm{Line}(M, i) + a \cdot \mathrm{Line}(M, j))$ is without repeated line and $\mathrm{lines}(\mathrm{RLine}(M, i, \mathrm{Line}(M, i) + a \cdot \mathrm{Line}(M, j)))$ is linearly independent.

(118) If $P \subseteq \mathrm{Seg}\,m$, then $\mathrm{lines}(\mathrm{Segm}(M, P, \mathrm{Seg}\,n)) \subseteq \mathrm{lines}(M)$.

(119) If $P \subseteq \mathrm{Seg}\,m$ and $\mathrm{lines}(M)$ is linearly independent, then $\mathrm{lines}(\mathrm{Segm}(M, P, \mathrm{Seg}\,n))$ is linearly independent.

(120) If $P \subseteq \mathrm{Seg}\,m$ and M is without repeated line, then $\mathrm{Segm}(M, P, \mathrm{Seg}\,n)$ is without repeated line.

(121) Let M be a matrix over K of dimension $m \times n$. Then $\mathrm{lines}(M)$ is linearly independent and M is without repeated line if and only if $\mathrm{rk}(M) = m$.

(122) Let U be a subset of the n-dimension vector space over K. Suppose $U \subseteq \mathrm{lines}(M)$. Then there exists P such that $P \subseteq \mathrm{Seg}\,m$ and $\mathrm{lines}(\mathrm{Segm}(M, P, \mathrm{Seg}\,n)) = U$ and $\mathrm{Segm}(M, P, \mathrm{Seg}\,n)$ is without repeated line.

(123) Let R_1 be an element of $\mathbb{N}$. Then $\mathrm{rk}(M) = R_1$ if and only if the following conditions are satisfied:
   (i) there exists a finite subset U of the n-dimension vector space over K such that U is linearly independent and $U \subseteq \mathrm{lines}(M)$ and $\mathrm{card}\,U = R_1$, and
   (ii) for every finite subset W of the n-dimension vector space over K such that W is linearly independent and $W \subseteq \mathrm{lines}(M)$ holds $\mathrm{card}\,W \leq R_1$.

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Basic Operations on Preordered Coherent Spaces

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Summary. This Mizar paper presents the definition of a "Preordered Coherent Space" (PCS). Furthermore, the paper defines a number of operations on PCS's and states and proves a number of elementary lemmas about these operations. PCS's have many useful properties which could qualify them for mathematical study in their own right. PCS's were invented, however, to construct Scott domains, to solve domain equations, and to construct models of various versions of lambda calculus.

For more on PCS's, see [11]. The present Mizar paper defines the operations on PCS's used in Chapter 8 of [3].

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The articles [16], [20], [7], [17], [15], [21], [4], [6], [22], [23], [14], [1], [13], [5], [18], [9], [19], [12], [8], [2], and [10] provide the notation and terminology for this paper.

1. Preliminaries

Let R_1, R_2 be sets and let R be a relation between R_1 and R_2. Then field R is a subset of $R_1 \cup R_2$.

[1]The author visited the Department of Computer Science, University of Copenhagen, while writing the article.

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Let R_1, R_2, S_1, S_2 be sets, let R be a relation between R_1 and R_2, and let S be a relation between S_1 and S_2. Then $R \cup S$ is a relation between $R_1 \cup S_1$ and $R_2 \cup S_2$.

Let R_1, S_1 be sets, let R be a total binary relation on R_1, and let S be a total binary relation on S_1. Note that $R \cup S$ is total.

Let R_1, S_1 be sets, let R be a reflexive binary relation on R_1, and let S be a reflexive binary relation on S_1. Observe that $R \cup S$ is reflexive.

Let R_1, S_1 be sets, let R be a symmetric binary relation on R_1, and let S be a symmetric binary relation on S_1. Observe that $R \cup S$ is symmetric.

One can prove the following proposition

(1) Let R_1, S_1 be sets, R be a transitive binary relation on R_1, and S be a transitive binary relation on S_1. If R_1 misses S_1, then $R \cup S$ is transitive.

Let A be an empty set and let B be a set. One can check that $\emptyset_{A,B}$ is total.

Let I be a non empty set and let C be a 1-sorted yielding many sorted set indexed by I. Then the support of C can be characterized by the condition:

(Def. 1) For every element i of I holds (the support of C)(i) = the carrier of $C(i)$.

Let R_1, R_2, S_1, S_2 be sets, let R be a relation between R_1 and R_2, and let S be a relation between S_1 and S_2. The functor $[\hat{}R, S\hat{}]$ yields a relation between $[\!: R_1, S_1 :\!]$ and $[\!: R_2, S_2 :\!]$ and is defined by the condition (Def. 2).

(Def. 2) Let x, y be sets. Then $\langle x, y \rangle \in [\hat{}R, S\hat{}]$ if and only if there exist sets r_1, s_1, r_2, s_2 such that $x = \langle r_1, s_1 \rangle$ and $y = \langle r_2, s_2 \rangle$ and $r_1 \in R_1$ and $s_1 \in S_1$ and $r_2 \in R_2$ and $s_2 \in S_2$ and $\langle r_1, r_2 \rangle \in R$ or $\langle s_1, s_2 \rangle \in S$.

Let R_1, R_2, S_1, S_2 be non empty sets, let R be a relation between R_1 and R_2, and let S be a relation between S_1 and S_2. Then $[\hat{}R, S\hat{}]$ can be characterized by the condition:

(Def. 3) Let r_1 be an element of R_1, r_2 be an element of R_2, s_1 be an element of S_1, and s_2 be an element of S_2. Then $\langle \langle r_1, s_1 \rangle, \langle r_2, s_2 \rangle \rangle \in [\hat{}R, S\hat{}]$ if and only if $\langle r_1, r_2 \rangle \in R$ or $\langle s_1, s_2 \rangle \in S$.

Let R_1, S_1 be sets, let R be a total binary relation on R_1, and let S be a total binary relation on S_1. Note that $[\hat{}R, S\hat{}]$ is total.

Let R_1, S_1 be sets, let R be a reflexive binary relation on R_1, and let S be a reflexive binary relation on S_1. One can check that $[\hat{}R, S\hat{}]$ is reflexive.

Let R_1, S_1 be sets, let R be a binary relation on R_1, and let S be a total reflexive binary relation on S_1. Observe that $[\hat{}R, S\hat{}]$ is reflexive.

Let R_1, S_1 be sets, let R be a total reflexive binary relation on R_1, and let S be a binary relation on S_1. Observe that $[\hat{}R, S\hat{}]$ is reflexive.

Let R_1, S_1 be sets, let R be a symmetric binary relation on R_1, and let S be a symmetric binary relation on S_1. Note that $[\hat{}R, S\hat{}]$ is symmetric.

2. Relational Structures

Let us observe that every relational structure which is empty is also total.

Let R be a binary relation. We say that R is transitive-yielding if and only if:

(Def. 4) For every relational structure S such that $S \in \operatorname{rng} R$ holds S is transitive.

Let us note that every binary relation which is poset-yielding is also transitive-yielding.

Let us mention that there exists a function which is poset-yielding.

Let I be a set. Observe that there exists a many sorted set indexed by I which is poset-yielding.

Let I be a set and let C be a relational structure yielding many sorted set indexed by I. The functor pcs-InternalRels C yields a many sorted set indexed by I and is defined by the condition (Def. 5).

(Def. 5) Let i be a set. Suppose $i \in I$. Then there exists a relational structure P such that $P = C(i)$ and $(\text{pcs-InternalRels}\, C)(i) =$ the internal relation of P.

Let I be a non empty set and let C be a relational structure yielding many sorted set indexed by I. Then pcs-InternalRels C can be characterized by the condition:

(Def. 6) For every element i of I holds $(\text{pcs-InternalRels}\, C)(i) =$ the internal relation of $C(i)$.

Let I be a set and let C be a relational structure yielding many sorted set indexed by I. One can check that pcs-InternalRels C is binary relation yielding.

Let I be a non empty set, let C be a transitive-yielding relational structure yielding many sorted set indexed by I, and let i be an element of I. Note that $C(i)$ is transitive.

3. Tolerance Structures

We introduce alternative relational structures which are extensions of 1-sorted structure and are systems

⟨ a carrier, an alternative relation ⟩,

where the carrier is a set and the alternative relation is a binary relation on the carrier.

Let P be an alternative relational structure and let p, q be elements of P. The predicate $p \sim q$ is defined by:

(Def. 7) $\langle p, q \rangle \in$ the alternative relation of P.

Let P be an alternative relational structure. We say that P is β-total if and only if:

(Def. 8) The alternative relation of P is total.

We say that P is β-reflexive if and only if:

(Def. 9) The alternative relation of P is reflexive in the carrier of P.

We say that P is β-irreflexive if and only if:

(Def. 10) The alternative relation of P is irreflexive in the carrier of P.

We say that P is β-symmetric if and only if:

(Def. 11) The alternative relation of P is symmetric in the carrier of P.

The alternative relational structure emptyTolStr is defined as follows:

(Def. 12) $\mathrm{emptyTolStr} = \langle \emptyset, \emptyset_{\emptyset,\emptyset} \rangle$.

One can check that emptyTolStr is empty and strict.

The following proposition is true

(2) Let P be an alternative relational structure. If P is empty, then the alternative relational structure of $P = \mathrm{emptyTolStr}$.

One can check that every alternative relational structure which is β-reflexive is also β-total.

Let us note that every alternative relational structure which is empty is also β-reflexive, β-irreflexive, and β-symmetric.

Let us note that there exists an alternative relational structure which is empty.

Let P be a β-total alternative relational structure. Observe that the alternative relation of P is total.

Let P be a β-reflexive alternative relational structure. One can check that the alternative relation of P is reflexive.

Let P be a β-irreflexive alternative relational structure. One can verify that the alternative relation of P is irreflexive.

Let P be a β-symmetric alternative relational structure. One can verify that the alternative relation of P is symmetric.

Let L be a β-total alternative relational structure. Note that the alternative relational structure of L is β-total.

Let P be a β-symmetric alternative relational structure and let p, q be elements of P. Let us note that the predicate $p \sim q$ is symmetric.

Let D be a set. Note that $\langle D, \nabla_D \rangle$ is β-reflexive and β-symmetric.

Let D be a set. Note that $\langle D, \emptyset_{D,D} \rangle$ is β-irreflexive and β-symmetric.

Let us note that there exists an alternative relational structure which is strict, non empty, β-reflexive, and β-symmetric.

One can check that there exists an alternative relational structure which is strict, non empty, β-irreflexive, and β-symmetric.

Let R be a binary relation. We say that R is alternative relational structure yielding if and only if:

(Def. 13) For every set P such that $P \in \operatorname{rng} R$ holds P is an alternative relational structure.

Let f be a function. Let us observe that f is alternative relational structure yielding if and only if:

(Def. 14) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is an alternative relational structure.

Let I be a set and let f be a many sorted set indexed by I. Let us observe that f is alternative relational structure yielding if and only if:

(Def. 15) For every set x such that $x \in I$ holds $f(x)$ is an alternative relational structure.

Let R be a binary relation. We say that R is β-reflexive yielding if and only if:

(Def. 16) For every alternative relational structure S such that $S \in \operatorname{rng} R$ holds S is β-reflexive.

We say that R is β-irreflexive yielding if and only if:

(Def. 17) For every alternative relational structure S such that $S \in \operatorname{rng} R$ holds S is β-irreflexive.

We say that R is β-symmetric yielding if and only if:

(Def. 18) For every alternative relational structure S such that $S \in \operatorname{rng} R$ holds S is β-symmetric.

One can check that every binary relation which is empty is also β-reflexive yielding, β-irreflexive yielding, and β-symmetric yielding.

Let I be a set and let P be an alternative relational structure. Note that $I \longmapsto P$ is alternative relational structure yielding.

Let I be a set and let P be a β-reflexive alternative relational structure. Observe that $I \longmapsto P$ is β-reflexive yielding.

Let I be a set and let P be a β-irreflexive alternative relational structure. One can check that $I \longmapsto P$ is β-irreflexive yielding.

Let I be a set and let P be a β-symmetric alternative relational structure. One can verify that $I \longmapsto P$ is β-symmetric yielding.

Let us observe that every function which is alternative relational structure yielding is also 1-sorted yielding.

Let I be a set. Observe that there exists a many sorted set indexed by I which is β-reflexive yielding, β-symmetric yielding, and alternative relational structure yielding.

Let I be a set. Note that there exists a many sorted set indexed by I which is β-irreflexive yielding, β-symmetric yielding, and alternative relational structure yielding.

Let I be a set. Observe that there exists a many sorted set indexed by I which is alternative relational structure yielding.

Let I be a non empty set, let C be an alternative relational structure yielding many sorted set indexed by I, and let i be an element of I. Then $C(i)$ is an alternative relational structure.

Let I be a set and let C be an alternative relational structure yielding many sorted set indexed by I. The functor pcs-ToleranceRels C yields a many sorted set indexed by I and is defined by the condition (Def. 19).

(Def. 19) Let i be a set. Suppose $i \in I$. Then there exists an alternative relational structure P such that $P = C(i)$ and (pcs-ToleranceRels $C)(i) =$ the alternative relation of P.

Let I be a non empty set and let C be an alternative relational structure yielding many sorted set indexed by I. Then pcs-ToleranceRels C can be characterized by the condition:

(Def. 20) For every element i of I holds (pcs-ToleranceRels $C)(i) =$ the alternative relation of $C(i)$.

Let I be a set and let C be an alternative relational structure yielding many sorted set indexed by I. Note that pcs-ToleranceRels C is binary relation yielding.

Let I be a non empty set, let C be a β-reflexive yielding alternative relational structure yielding many sorted set indexed by I, and let i be an element of I. One can verify that $C(i)$ is β-reflexive.

Let I be a non empty set, let C be a β-irreflexive yielding alternative relational structure yielding many sorted set indexed by I, and let i be an element of I. Note that $C(i)$ is β-irreflexive.

Let I be a non empty set, let C be a β-symmetric yielding alternative relational structure yielding many sorted set indexed by I, and let i be an element of I. Observe that $C(i)$ is β-symmetric.

The following propositions are true:

(3) Let P, Q be alternative relational structures. Suppose that
 (i) the alternative relational structure of $P =$ the alternative relational structure of Q, and
 (ii) P is β-reflexive.
 Then Q is β-reflexive.

(4) Let P, Q be alternative relational structures. Suppose that
 (i) the alternative relational structure of $P =$ the alternative relational structure of Q, and
 (ii) P is β-irreflexive.
 Then Q is β-irreflexive.

(5) Let P, Q be alternative relational structures. Suppose that
 (i) the alternative relational structure of $P =$ the alternative relational structure of Q, and
 (ii) P is β-symmetric.

Then Q is β-symmetric.

Let P, Q be alternative relational structures. The functor $[\hat{}P, Q\hat{}]$ yields an alternative relational structure and is defined by the condition (Def. 21).

(Def. 21) $[\hat{}P, Q\hat{}] = \langle [\!:$ the carrier of P, the carrier of $Q :\!]$, $[\hat{}$the alternative relation of P, the alternative relation of $Q\hat{}]\rangle$.

Let P, Q be alternative relational structures, let p be an element of P, and let q be an element of Q. We introduce $[\hat{}p, q\hat{}]$ as a synonym of $\langle p, q\rangle$.

Let P, Q be non empty alternative relational structures, let p be an element of P, and let q be an element of Q. Then $[\hat{}p, q\hat{}]$ is an element of $[\hat{}P, Q\hat{}]$.

Let P, Q be alternative relational structures and let p be an element of $[\hat{}P, Q\hat{}]$. We introduce $p`1$ as a synonym of $p_{\mathbf{1}}$. We introduce $p`2$ as a synonym of $p_{\mathbf{2}}$.

Let P, Q be non empty alternative relational structures and let p be an element of $[\hat{}P, Q\hat{}]$. Then $p`1$ is an element of $P$. Then $p`2$ is an element of Q.

We now state two propositions:

(6) Let S_1, S_2 be non empty alternative relational structures, a, c be elements of S_1, and b, d be elements of S_2. Then $[\hat{}a, b\hat{}] \sim [\hat{}c, d\hat{}]$ if and only if $a \sim c$ or $b \sim d$.

(7) Let S_1, S_2 be non empty alternative relational structures and x, y be elements of $[\hat{}S_1, S_2\hat{}]$. Then $x \sim y$ if and only if one of the following conditions is satisfied:

(i) $x`1 \sim y`1$, or

(ii) $x`2 \sim y`2$.

Let P be an alternative relational structure and let Q be a β-reflexive alternative relational structure. Note that $[\hat{}P, Q\hat{}]$ is β-reflexive.

Let P be a β-reflexive alternative relational structure and let Q be an alternative relational structure. Observe that $[\hat{}P, Q\hat{}]$ is β-reflexive.

Let P, Q be β-symmetric alternative relational structures. One can check that $[\hat{}P, Q\hat{}]$ is β-symmetric.

4. PCS's

We introduce pcs structures which are extensions of relational structure and alternative relational structure and are systems

$\langle$ a carrier, an internal relation, an alternative relation $\rangle$,

where the carrier is a set, the internal relation is a binary relation on the carrier, and the alternative relation is a binary relation on the carrier.

Let P be a pcs structure. We say that P is compatible if and only if:

(Def. 22) For all elements p, p', q, q' of P such that $p \sim q$ and $p' \leq p$ and $q' \leq q$ holds $p' \sim q'$.

Let P be a pcs structure. We say that P is pcs-like if and only if:

(Def. 23) P is reflexive, transitive, β-reflexive, β-symmetric, and compatible.

We say that P is anti-pcs-like if and only if:

(Def. 24) P is reflexive, transitive, β-irreflexive, β-symmetric, and compatible.

One can verify the following observations:

* every pcs structure which is pcs-like is also reflexive, transitive, β-reflexive, β-symmetric, and compatible,
* every pcs structure which is reflexive, transitive, β-reflexive, β-symmetric, and compatible is also pcs-like,
* every pcs structure which is anti-pcs-like is also reflexive, transitive, β-irreflexive, β-symmetric, and compatible, and
* every pcs structure which is reflexive, transitive, β-irreflexive, β-symmetric, and compatible is also anti-pcs-like.

Let D be a set. The functor TotalPCS D yields a pcs structure and is defined as follows:

(Def. 25) $\mathrm{TotalPCS}\, D = \langle D, \nabla_D, \nabla_D \rangle$.

Let D be a set. Observe that TotalPCS D is strict.

Let D be a non empty set. One can verify that TotalPCS D is non empty.

Let D be a set. One can check that TotalPCS D is reflexive, transitive, β-reflexive, and β-symmetric.

Let D be a set. Note that TotalPCS D is pcs-like.

Let D be a set. One can verify that $\langle D, \nabla_D, \emptyset_{D,D} \rangle$ is anti-pcs-like.

One can verify that there exists a pcs structure which is strict, non empty, and pcs-like and there exists a pcs structure which is strict, non empty, and anti-pcs-like.

A pcs is a pcs-like pcs structure. An anti-pcs is an anti-pcs-like pcs structure.

The pcs structure EmptyPCS is defined by:

(Def. 26) $\mathrm{EmptyPCS} = \mathrm{TotalPCS}\, 0$.

Let us mention that EmptyPCS is strict, empty, and pcs-like.

Let p be a set. The functor SingletonPCS p yielding a pcs structure is defined by:

(Def. 27) $\mathrm{SingletonPCS}\, p = \mathrm{TotalPCS}\{p\}$.

Let p be a set. Observe that SingletonPCS p is strict, non empty, and pcs-like.

Let R be a binary relation. We say that R is pcs structure yielding if and only if:

(Def. 28) For every set P such that $P \in \mathrm{rng}\, R$ holds P is a pcs structure.

We say that R is pcs-yielding if and only if:

(Def. 29) For every set P such that $P \in \mathrm{rng}\, R$ holds P is a pcs.

Let f be a function. Let us observe that f is pcs structure yielding if and only if:

(Def. 30) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is a pcs structure.

Let us observe that f is pcs-yielding if and only if:

(Def. 31) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is a pcs.

Let I be a set and let f be a many sorted set indexed by I. Let us observe that f is pcs structure yielding if and only if:

(Def. 32) For every set x such that $x \in I$ holds $f(x)$ is a pcs structure.

Let us observe that f is pcs-yielding if and only if:

(Def. 33) For every set x such that $x \in I$ holds $f(x)$ is a pcs.

One can verify the following observations:

* every binary relation which is pcs structure yielding is also alternative relational structure yielding and relational structure yielding,
* every binary relation which is pcs-yielding is also pcs structure yielding, and
* every binary relation which is pcs-yielding is also reflexive-yielding, transitive-yielding, β-reflexive yielding, and β-symmetric yielding.

Let I be a set and let P be a pcs. Note that $I \longmapsto P$ is pcs-yielding.

Let I be a set. Observe that there exists a many sorted set indexed by I which is pcs-yielding.

Let I be a non empty set, let C be a pcs structure yielding many sorted set indexed by I, and let i be an element of I. Then $C(i)$ is a pcs structure.

Let I be a non empty set, let C be a pcs-yielding many sorted set indexed by I, and let i be an element of I. Then $C(i)$ is a pcs.

Let P, Q be pcs structures. The predicate $P \subseteq Q$ is defined by the conditions (Def. 34).

(Def. 34)(i) The carrier of $P \subseteq$ the carrier of Q,

(ii) the internal relation of $P \subseteq$ the internal relation of Q, and

(iii) the alternative relation of $P \subseteq$ the alternative relation of Q.

Let us note that the predicate $P \subseteq Q$ is reflexive.

Next we state two propositions:

(8) Let P, Q be relational structures, p, q be elements of P, and p_1, q_1 be elements of Q. Suppose the internal relation of $P \subseteq$ the internal relation of Q and $p = p_1$ and $q = q_1$ and $p \leq q$. Then $p_1 \leq q_1$.

(9) Let P, Q be pcs structures, p, q be elements of P, and p_1, q_1 be elements of Q. Suppose the alternative relation of $P \subseteq$ the alternative relation of Q and $p = p_1$ and $q = q_1$ and $p \sim q$. Then $p_1 \sim q_1$.

Let C be a binary relation. We say that C is chain-like if and only if:

(Def. 35) For all pcs structures P, Q such that $P \in \operatorname{rng} C$ and $Q \in \operatorname{rng} C$ holds $P \subseteq Q$ or $Q \subseteq P$.

Let I be a set and let P be a pcs structure. Observe that $I \longmapsto P$ is chain-like.

Let us note that there exists a function which is chain-like and pcs-yielding.

Let I be a set. Note that there exists a many sorted set indexed by I which is chain-like and pcs-yielding.

Let I be a set. A pcs-chain of I is a chain-like pcs-yielding many sorted set indexed by I.

Let I be a set and let C be a pcs structure yielding many sorted set indexed by I. The functor $\bigcup C$ yielding a strict pcs structure is defined by the conditions (Def. 36).

(Def. 36)(i) The carrier of $\bigcup C = \bigcup$ (the support of C),

(ii) the internal relation of $\bigcup C = \bigcup$ pcs-InternalRels C, and

(iii) the alternative relation of $\bigcup C = \bigcup$ pcs-ToleranceRels C.

We now state four propositions:

(10) Let I be a set, C be a pcs structure yielding many sorted set indexed by I, and p, q be elements of $\bigcup C$. Then $p \leq q$ if and only if there exists a set i and there exists a pcs structure P and there exist elements p', q' of P such that $i \in I$ and $P = C(i)$ and $p' = p$ and $q' = q$ and $p' \leq q'$.

(11) Let I be a non empty set, C be a pcs structure yielding many sorted set indexed by I, and p, q be elements of $\bigcup C$. Then $p \leq q$ if and only if there exists an element i of I and there exist elements p', q' of $C(i)$ such that $p' = p$ and $q' = q$ and $p' \leq q'$.

(12) Let I be a set, C be a pcs structure yielding many sorted set indexed by I, and p, q be elements of $\bigcup C$. Then $p \sim q$ if and only if there exists a set i and there exists a pcs structure P and there exist elements p', q' of P such that $i \in I$ and $P = C(i)$ and $p' = p$ and $q' = q$ and $p' \sim q'$.

(13) Let I be a non empty set, C be a pcs structure yielding many sorted set indexed by I, and p, q be elements of $\bigcup C$. Then $p \sim q$ if and only if there exists an element i of I and there exist elements p', q' of $C(i)$ such that $p' = p$ and $q' = q$ and $p' \sim q'$.

Let I be a set and let C be a reflexive-yielding pcs structure yielding many sorted set indexed by I. Observe that $\bigcup C$ is reflexive.

Let I be a set and let C be a β-reflexive yielding pcs structure yielding many sorted set indexed by I. Observe that $\bigcup C$ is β-reflexive.

Let I be a set and let C be a β-symmetric yielding pcs structure yielding many sorted set indexed by I. Note that $\bigcup C$ is β-symmetric.

Let I be a set and let C be a pcs-chain of I. One can check that $\bigcup C$ is transitive and compatible.

Let p, q be sets. The functor $\mathrm{MSSet}(p, q)$ yielding a many sorted set indexed by $\{0, 1\}$ is defined by:

(Def. 37) $\mathrm{MSSet}(p, q) = [0 \longmapsto p, 1 \longmapsto q]$.

Let P, Q be 1-sorted structures. One can check that $\mathrm{MSSet}(P, Q)$ is 1-sorted yielding.

Let P, Q be relational structures. Observe that $\mathrm{MSSet}(P, Q)$ is relational structure yielding.

Let P, Q be alternative relational structures. Observe that $\mathrm{MSSet}(P, Q)$ is alternative relational structure yielding.

Let P, Q be pcs structures. Note that $\mathrm{MSSet}(P, Q)$ is pcs structure yielding.

Let P, Q be reflexive pcs structures. Observe that $\mathrm{MSSet}(P, Q)$ is reflexive-yielding.

Let P, Q be transitive pcs structures. One can check that $\mathrm{MSSet}(P, Q)$ is transitive-yielding.

Let P, Q be β-reflexive pcs structures. Note that $\mathrm{MSSet}(P, Q)$ is β-reflexive yielding.

Let P, Q be β-symmetric pcs structures. Observe that $\mathrm{MSSet}(P, Q)$ is β-symmetric yielding.

Let P, Q be pcs's. Observe that $\mathrm{MSSet}(P, Q)$ is pcs-yielding.

Let P, Q be pcs structures. The functor $P \oplus Q$ yielding a pcs structure is defined by:

(Def. 38) $P \oplus Q = \bigcup \mathrm{MSSet}(P, Q)$.

One can prove the following four propositions:

(14) Let P, Q be pcs structures. Then

(i) the carrier of $P \oplus Q =$ (the carrier of P) $\cup$ (the carrier of Q),

(ii) the internal relation of $P \oplus Q =$ (the internal relation of P) $\cup$ (the internal relation of Q), and

(iii) the alternative relation of $P \oplus Q =$ (the alternative relation of P) $\cup$ (the alternative relation of Q).

(15) Let P, Q be pcs structures. Then $P \oplus Q = \langle$(the carrier of P) $\cup$ (the carrier of Q), (the internal relation of P) $\cup$ (the internal relation of Q), (the alternative relation of P) $\cup$ (the alternative relation of Q)$\rangle$.

(16) Let P, Q be pcs structures and p, q be elements of $P \oplus Q$. Then $p \leq q$ if and only if one of the following conditions is satisfied:

(i) there exist elements p', q' of P such that $p' = p$ and $q' = q$ and $p' \leq q'$, or

(ii) there exist elements p', q' of Q such that $p' = p$ and $q' = q$ and $p' \leq q'$.

(17) Let P, Q be pcs structures and p, q be elements of $P \oplus Q$. Then $p \sim q$ if and only if one of the following conditions is satisfied:

(i) there exist elements p', q' of P such that $p' = p$ and $q' = q$ and $p' \sim q'$, or

(ii) there exist elements p', q' of Q such that $p' = p$ and $q' = q$ and $p' \sim q'$.

Let P, Q be reflexive pcs structures. Observe that $P \oplus Q$ is reflexive.

Let P, Q be β-reflexive pcs structures. One can verify that $P \oplus Q$ is β-reflexive.

Let P, Q be β-symmetric pcs structures. Observe that $P \oplus Q$ is β-symmetric.

The following three propositions are true:

(18) For all pcs's P, Q such that P misses Q holds the internal relation of $P \oplus Q$ is transitive.

(19) For all pcs's P, Q such that P misses Q holds $P \oplus Q$ is compatible.

(20) For all pcs's P, Q such that P misses Q holds $P \oplus Q$ is a pcs.

Let P be a pcs structure and let a be a set. The functor P_a yields a strict pcs structure and is defined by the conditions (Def. 39).

(Def. 39)(i) The carrier of $P_a = \{a\} \cup$ the carrier of P,

(ii) the internal relation of $P_a = [\!:\{a\}$, the carrier of $P_a :\!] \cup$ the internal relation of P, and

(iii) the alternative relation of $P_a = [\!:\{a\}$, the carrier of $P_a :\!] \cup [\!:$ the carrier of P_a, $\{a\}:\!] \cup$ the alternative relation of P.

Let P be a pcs structure and let a be a set. Observe that P_a is non empty.

The following propositions are true:

(21) Let P be a pcs structure and a be a set. Then

(i) the carrier of $P \subseteq$ the carrier of P_a,

(ii) the internal relation of $P \subseteq$ the internal relation of P_a, and

(iii) the alternative relation of $P \subseteq$ the alternative relation of P_a.

(22) For every pcs structure P and for every set a and for all elements p, q of P_a such that $p = a$ holds $p \leq q$.

(23) Let P be a pcs structure, a be a set, p, q be elements of P, and p_1, q_1 be elements of P_a. If $p = p_1$ and $q = q_1$ and $p \leq q$, then $p_1 \leq q_1$.

(24) Let P be a pcs structure, a be a set, p be an element of P, and p_1, q_1 be elements of P_a. Suppose $p = p_1$ and $p \neq a$ and $p_1 \leq q_1$ and $a \notin$ the carrier of P. Then $q_1 \in$ the carrier of P and $q_1 \neq a$.

(25) Let P be a pcs structure, a be a set, and p be an element of P_a. If $p \neq a$, then $p \in$ the carrier of P.

(26) Let P be a pcs structure, a be a set, p, q be elements of P, and p_1, q_1 be elements of P_a. If $p = p_1$ and $q = q_1$ and $p \neq a$ and $p_1 \leq q_1$, then $p \leq q$.

(27) For every pcs structure P and for every set a and for all elements p, q of P_a such that $p = a$ holds $p \sim q$ and $q \sim p$.

(28) Let P be a pcs structure, a be a set, p, q be elements of P, and p_1, q_1 be elements of P_a. If $p = p_1$ and $q = q_1$ and $p \sim q$, then $p_1 \sim q_1$.

(29) Let P be a pcs structure, a be a set, p, q be elements of P, and p_1, q_1 be elements of P_a. If $p = p_1$ and $q = q_1$ and $p \neq a$ and $q \neq a$ and $p_1 \sim q_1$, then $p \sim q$.

Let P be a reflexive pcs structure and let a be a set. Observe that P_a is reflexive.

The following proposition is true

(30) For every transitive pcs structure P and for every set a such that $a \notin$ the carrier of P holds P_a is transitive.

Let P be a β-reflexive pcs structure and let a be a set. One can verify that P_a is β-reflexive.

Let P be a β-symmetric pcs structure and let a be a set. One can check that P_a is β-symmetric.

Next we state two propositions:

(31) For every compatible pcs structure P and for every set a such that $a \notin$ the carrier of P holds P_a is compatible.

(32) For every pcs P and for every set a such that $a \notin$ the carrier of P holds P_a is a pcs.

Let P be a pcs structure. The functor $\updownarrow P$ yields a strict pcs structure and is defined by the conditions (Def. 40).

(Def. 40)(i) The carrier of $\updownarrow P =$ the carrier of P,

(ii) the internal relation of $\updownarrow P =$ (the internal relation of $P)^{\smile}$, and

(iii) the alternative relation of $\updownarrow P =$ (the alternative relation of $P)^{\mathrm{c}}$.

Let P be a non empty pcs structure. One can check that $\updownarrow P$ is non empty.

Next we state three propositions:

(33) Let P be a pcs structure, p, q be elements of P, and p', q' be elements of $\updownarrow P$. If $p = p'$ and $q = q'$, then $p \leq q$ iff $q' \leq p'$.

(34) Let P be a pcs structure, p, q be elements of P, and p', q' be elements of $\updownarrow P$. If $p = p'$ and $q = q'$, then if $p \sim q$, then $p' \not\sim q'$.

(35) Let P be a non empty pcs structure, p, q be elements of P, and p', q' be elements of $\updownarrow P$. If $p = p'$ and $q = q'$, then if $p' \not\sim q'$, then $p \sim q$.

Let P be a reflexive pcs structure. One can check that $\updownarrow P$ is reflexive.

Let P be a transitive pcs structure. Observe that $\updownarrow P$ is transitive.

Let P be a β-reflexive pcs structure. One can verify that $\updownarrow P$ is β-irreflexive.

Let P be a β-irreflexive pcs structure. One can check that $\updownarrow P$ is β-reflexive.

Let P be a β-symmetric pcs structure. One can verify that $\updownarrow P$ is β-symmetric.

Let P be a compatible pcs structure. Note that $\updownarrow P$ is compatible.

Let P, Q be pcs structures. The functor $P \otimes Q$ yielding a pcs structure is defined by the condition (Def. 41).

(Def. 41) $P \otimes Q = \langle [\!:$ the carrier of P, the carrier of $Q :\!]$, (the internal relation of P) $\times$ (the internal relation of Q), [^the alternative relation of P, the alternative relation of Q^]$\rangle$.

Let P, Q be pcs structures. One can check that $P \otimes Q$ is strict.

Let P, Q be non empty pcs structures. Note that $P \otimes Q$ is non empty.

One can prove the following propositions:

(36) Let P, Q be pcs structures, p, q be elements of $P \otimes Q$, p_1, p_2 be elements of P, and q_1, q_2 be elements of Q. If $p = \langle p_1, q_1 \rangle$ and $q = \langle p_2, q_2 \rangle$, then $p \leq q$ iff $p_1 \leq p_2$ and $q_1 \leq q_2$.

(37) Let P, Q be pcs structures, p, q be elements of $P \otimes Q$, p_1, p_2 be elements of P, and q_1, q_2 be elements of Q. If $p = \langle p_1, q_1 \rangle$ and $q = \langle p_2, q_2 \rangle$, then if $p \sim q$, then $p_1 \sim p_2$ or $q_1 \sim q_2$.

(38) Let P, Q be non empty pcs structures, p, q be elements of $P \otimes Q$, p_1, p_2 be elements of P, and q_1, q_2 be elements of Q. If $p = \langle p_1, q_1 \rangle$ and $q = \langle p_2, q_2 \rangle$, then if $p_1 \sim p_2$ or $q_1 \sim q_2$, then $p \sim q$.

Let P, Q be reflexive pcs structures. Observe that $P \otimes Q$ is reflexive.

Let P, Q be transitive pcs structures. One can check that $P \otimes Q$ is transitive.

Let P be a pcs structure and let Q be a β-reflexive pcs structure. One can check that $P \otimes Q$ is β-reflexive.

Let P be a β-reflexive pcs structure and let Q be a pcs structure. One can check that $P \otimes Q$ is β-reflexive.

Let P, Q be β-symmetric pcs structures. One can verify that $P \otimes Q$ is β-symmetric.

Let P, Q be compatible pcs structures. Observe that $P \otimes Q$ is compatible.

Let P, Q be pcs structures. The functor $P \longmapsto Q$ yielding a pcs structure is defined as follows:

(Def. 42) $P \longmapsto Q = \updownarrow P \otimes Q$.

Let P, Q be pcs structures. One can check that $P \longmapsto Q$ is strict.

Let P, Q be non empty pcs structures. Note that $P \longmapsto Q$ is non empty.

Next we state three propositions:

(39) Let P, Q be pcs structures, p, q be elements of $P \longmapsto Q$, p_1, p_2 be elements of P, and q_1, q_2 be elements of Q. If $p = \langle p_1, q_1 \rangle$ and $q = \langle p_2, q_2 \rangle$, then $p \leq q$ iff $p_2 \leq p_1$ and $q_1 \leq q_2$.

(40) Let P, Q be pcs structures, p, q be elements of $P \longmapsto Q$, p_1, p_2 be elements of P, and q_1, q_2 be elements of Q. If $p = \langle p_1, q_1 \rangle$ and $q = \langle p_2, q_2 \rangle$, then if $p \sim q$, then $p_1 \not\sim p_2$ or $q_1 \sim q_2$.

(41) Let P, Q be non empty pcs structures, p, q be elements of $P \longmapsto Q$, p_1, p_2 be elements of P, and q_1, q_2 be elements of Q. If $p = \langle p_1, q_1 \rangle$ and $q = \langle p_2, q_2 \rangle$, then if $p_1 \not\sim p_2$ or $q_1 \sim q_2$, then $p \sim q$.

Let P, Q be reflexive pcs structures. One can check that $P \longmapsto Q$ is reflexive.

Let P, Q be transitive pcs structures. Observe that $P \longmapsto Q$ is transitive.

Let P be a pcs structure and let Q be a β-reflexive pcs structure. Note that $P \longmapsto Q$ is β-reflexive.

Let P be a β-irreflexive pcs structure and let Q be a pcs structure. One can verify that $P \longmapsto Q$ is β-reflexive.

Let P, Q be β-symmetric pcs structures. Note that $P \longmapsto Q$ is β-symmetric.

Let P, Q be compatible pcs structures. Note that $P \longmapsto Q$ is compatible.

Let P, Q be pcs's. Note that $P \longmapsto Q$ is pcs-like.

Let P be a pcs structure and let S be a subset of P. We say that S is self-coherent if and only if:

(Def. 43) For all elements x, y of P such that $x \in S$ and $y \in S$ holds $x \sim y$.

Let P be a pcs structure. Observe that every subset of P which is empty is also self-coherent.

Let P be a pcs structure. One can check that there exists a subset of P which is empty.

Let P be a pcs structure and let F be a family of subsets of P. We say that F is self-coherent-membered if and only if:

(Def. 44) For every subset S of P such that $S \in F$ holds S is self-coherent.

Let P be a pcs structure. Observe that there exists a family of subsets of P which is non empty and self-coherent-membered.

Let P be a pcs structure and let D be a set. The functor $\mathcal{P}_{\mathrm{IR}}(P, D)$ yields a binary relation on D and is defined by the condition (Def. 45).

(Def. 45) Let A, B be sets. Then $\langle A, B \rangle \in \mathcal{P}_{\mathrm{IR}}(P, D)$ if and only if the following conditions are satisfied:

- (i) $A \in D$,
- (ii) $B \in D$, and
- (iii) for every set a such that $a \in A$ there exists a set b such that $b \in B$ and $\langle a, b \rangle \in$ the internal relation of P.

The functor $\mathcal{P}_{\mathrm{TR}}(P, D)$ yielding a binary relation on D is defined by the condition (Def. 46).

(Def. 46) Let A, B be sets. Then $\langle A, B \rangle \in \mathcal{P}_{\mathrm{TR}}(P, D)$ if and only if the following conditions are satisfied:

- (i) $A \in D$,
- (ii) $B \in D$, and
- (iii) for all sets a, b such that $a \in A$ and $b \in B$ holds $\langle a, b \rangle \in$ the alternative relation of P.

Next we state two propositions:

(42) Let P be a pcs structure, D be a family of subsets of P, and A, B be sets. Then $\langle A, B \rangle \in \mathcal{P}_{\mathrm{IR}}(P, D)$ if and only if the following conditions are satisfied:

(i) $A \in D$,

(ii) $B \in D$, and

(iii) for every element a of P such that $a \in A$ there exists an element b of P such that $b \in B$ and $a \leq b$.

(43) Let P be a pcs structure, D be a family of subsets of P, and A, B be sets. Then $\langle A, B \rangle \in \mathcal{P}_{\mathrm{TR}}(P, D)$ if and only if the following conditions are satisfied:

(i) $A \in D$,

(ii) $B \in D$, and

(iii) for all elements a, b of P such that $a \in A$ and $b \in B$ holds $a \sim b$.

Let P be a pcs structure and let D be a set. The functor $\mathcal{P}(P, D)$ yielding a pcs structure is defined by:

(Def. 47) $\mathcal{P}(P, D) = \langle D, \mathcal{P}_{\mathrm{IR}}(P, D), \mathcal{P}_{\mathrm{TR}}(P, D) \rangle$.

Let P be a pcs structure and let D be a family of subsets of P. We introduce $\mathcal{P}(D)$ as a synonym of $\mathcal{P}(P, D)$.

Let P be a pcs structure and let D be a non empty set. Observe that $\mathcal{P}(P, D)$ is non empty.

Next we state four propositions:

(44) Let P be a pcs structure, D be a set, and p, q be elements of $\mathcal{P}(P, D)$. Suppose $p \leq q$. Let p' be an element of P. If $p' \in p$, then there exists an element q' of P such that $q' \in q$ and $p' \leq q'$.

(45) Let P be a pcs structure, D be a non empty family of subsets of P, and p, q be elements of $\mathcal{P}(D)$. Suppose that for every element p' of P such that $p' \in p$ there exists an element q' of P such that $q' \in q$ and $p' \leq q'$. Then $p \leq q$.

(46) Let P be a pcs structure, D be a set, and p, q be elements of $\mathcal{P}(P, D)$. Suppose $p \sim q$. Let p', q' be elements of P. If $p' \in p$ and $q' \in q$, then $p' \sim q'$.

(47) Let P be a pcs structure, D be a non empty family of subsets of P, and p, q be elements of $\mathcal{P}(D)$. Suppose that for all elements p', q' of P such that $p' \in p$ and $q' \in q$ holds $p' \sim q'$. Then $p \sim q$.

Let P be a pcs structure and let D be a set. One can check that $\mathcal{P}(P, D)$ is strict.

Let P be a reflexive pcs structure and let D be a family of subsets of P. Note that $\mathcal{P}(D)$ is reflexive.

Let P be a transitive pcs structure and let D be a set. One can check that $\mathcal{P}(P, D)$ is transitive.

Let P be a β-reflexive pcs structure and let D be a self-coherent-membered family of subsets of P. One can check that $\mathcal{P}(D)$ is β-reflexive.

Let P be a β-symmetric pcs structure and let D be a family of subsets of P. Observe that $\mathcal{P}(D)$ is β-symmetric.

Let P be a compatible pcs structure and let D be a family of subsets of P. Note that $\mathcal{P}(D)$ is compatible.

Let P be a pcs structure. The functor pcs-coherent-power P yields a set and is defined as follows:

(Def. 48) pcs-coherent-power $P = \{X; X$ ranges over subsets of P: X is self-coherent$\}$.

We now state the proposition

(48) For every pcs structure P and for every set X such that $X \in$ pcs-coherent-power P holds X is a self-coherent subset of P.

Let P be a pcs structure. Note that pcs-coherent-power P is non empty.

Let P be a pcs structure. Then pcs-coherent-power P is a family of subsets of P.

Let P be a pcs structure. Observe that pcs-coherent-power P is self-coherent-membered.

Let P be a pcs structure. The functor $\mathcal{P}(P)$ yielding a pcs structure is defined by:

(Def. 49) $\mathcal{P}(P) = \mathcal{P}(\text{pcs-coherent-power}\, P)$.

Let P be a pcs structure. Note that $\mathcal{P}(P)$ is strict.

Let P be a pcs structure. Note that $\mathcal{P}(P)$ is non empty.

Let P be a reflexive pcs structure. One can verify that $\mathcal{P}(P)$ is reflexive.

Let P be a transitive pcs structure. One can check that $\mathcal{P}(P)$ is transitive.

Let P be a β-reflexive pcs structure. Note that $\mathcal{P}(P)$ is β-reflexive.

Let P be a β-symmetric pcs structure. Note that $\mathcal{P}(P)$ is β-symmetric.

Let P be a compatible pcs structure. Note that $\mathcal{P}(P)$ is compatible.

Let P be a pcs. Observe that $\mathcal{P}(P)$ is pcs-like.

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Inferior Limit, Superior Limit and Convergence of Sequences of Extended Real Numbers

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Summary. In this article, we extended properties of sequences of real numbers to sequences of extended real numbers. We also introduced basic properties of the inferior limit, superior limit and convergence of sequences of extended real numbers.

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The notation and terminology used in this paper are introduced in the following articles: [18], [19], [1], [17], [20], [5], [21], [6], [7], [16], [2], [3], [8], [15], [13], [14], [12], [11], [22], [4], [10], and [9].

We adopt the following convention: n, m, k are elements of $\mathbb{N}$, X is a non empty subset of $\overline{\mathbb{R}}$, and Y is a non empty subset of $\mathbb{R}$.

Next we state four propositions:

(1) If $X = Y$ and Y is upper bounded, then X is upper bounded and $\sup X = \sup Y$.

(2) If $X = Y$ and X is upper bounded, then Y is upper bounded and $\sup X = \sup Y$.

(3) If $X = Y$ and Y is lower bounded, then X is lower bounded and $\inf X = \inf Y$.

(4) If $X = Y$ and X is lower bounded, then Y is lower bounded and $\inf X = \inf Y$.

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Let s_1 be a sequence of extended reals. The functor $\sup s_1$ yields an element of $\overline{\mathbb{R}}$ and is defined by:

(Def. 1) $\sup s_1 = \sup \operatorname{rng} s_1$.

The functor $\inf s_1$ yields an element of $\overline{\mathbb{R}}$ and is defined as follows:

(Def. 2) $\inf s_1 = \inf \operatorname{rng} s_1$.

Let s_1 be a sequence of extended reals. We say that s_1 is lower bounded if and only if:

(Def. 3) $\operatorname{rng} s_1$ is lower bounded.

We say that s_1 is upper bounded if and only if:

(Def. 4) $\operatorname{rng} s_1$ is upper bounded.

Let s_1 be a sequence of extended reals. We say that s_1 is bounded if and only if:

(Def. 5) s_1 is upper bounded and lower bounded.

In the sequel s_1 is a sequence of extended reals.

One can prove the following proposition

(5) For all s_1, n holds $\{s_1(k); k$ ranges over elements of $\mathbb{N}$: $n \leq k\}$ is a non empty subset of $\overline{\mathbb{R}}$.

Let s_1 be a sequence of extended reals. The inferior realsequence s_1 yields a sequence of extended reals and is defined by the condition (Def. 6).

(Def. 6) Let n be an element of $\mathbb{N}$. Then there exists a non empty subset Y of $\overline{\mathbb{R}}$ such that $Y = \{s_1(k); k$ ranges over elements of $\mathbb{N}$: $n \leq k\}$ and (the inferior realsequence $s_1)(n) = \inf Y$.

Let s_1 be a sequence of extended reals. The superior realsequence s_1 yields a sequence of extended reals and is defined by the condition (Def. 7).

(Def. 7) Let n be an element of $\mathbb{N}$. Then there exists a non empty subset Y of $\overline{\mathbb{R}}$ such that $Y = \{s_1(k); k$ ranges over elements of $\mathbb{N}$: $n \leq k\}$ and (the superior realsequence $s_1)(n) = \sup Y$.

We now state the proposition

(6) If s_1 is finite, then s_1 is a sequence of real numbers.

Let f be a partial function from $\mathbb{N}$ to $\overline{\mathbb{R}}$. We say that f is increasing if and only if:

(Def. 8) For all m, n such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $m < n$ holds $f(m) < f(n)$.

We say that f is decreasing if and only if:

(Def. 9) For all m, n such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $m < n$ holds $f(m) > f(n)$.

We say that f is non-decreasing if and only if:

(Def. 10) For all m, n such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $m \leq n$ holds $f(m) \leq f(n)$.

We say that f is non-increasing if and only if:

(Def. 11) For all m, n such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $m \leq n$ holds $f(m) \geq f(n)$.

One can prove the following two propositions:

(7)(i) s_1 is increasing iff for all elements n, m of $\mathbb{N}$ such that $m < n$ holds $s_1(m) < s_1(n)$,

(ii) s_1 is decreasing iff for all elements n, m of $\mathbb{N}$ such that $m < n$ holds $s_1(n) < s_1(m)$,

(iii) s_1 is non-decreasing iff for all elements n, m of $\mathbb{N}$ such that $m \leq n$ holds $s_1(m) \leq s_1(n)$, and

(iv) s_1 is non-increasing iff for all elements n, m of $\mathbb{N}$ such that $m \leq n$ holds $s_1(n) \leq s_1(m)$.

(8) (The inferior realsequence $s_1)(n) \leq s_1(n)$ and $s_1(n) \leq$ (the superior realsequence $s_1)(n)$.

Let us consider s_1. Observe that the superior realsequence s_1 is non-increasing and the inferior realsequence s_1 is non-decreasing.

Let s_1 be a sequence of extended reals. The functor $\limsup s_1$ yields an element of $\overline{\mathbb{R}}$ and is defined by:

(Def. 12) $\limsup s_1 = \inf$ (the superior realsequence s_1).

The functor $\liminf s_1$ yields an element of $\overline{\mathbb{R}}$ and is defined by:

(Def. 13) $\liminf s_1 = \sup$ (the inferior realsequence s_1).

In the sequel r_1 is a sequence of real numbers.

The following propositions are true:

(9) If $s_1 = r_1$ and r_1 is bounded, then the superior realsequence $s_1 =$ the superior realsequence r_1 and $\limsup s_1 = \limsup r_1$.

(10) If $s_1 = r_1$ and r_1 is bounded, then the inferior realsequence $s_1 =$ the inferior realsequence r_1 and $\liminf s_1 = \liminf r_1$.

(11) If s_1 is bounded, then s_1 is a sequence of real numbers.

(12) If $s_1 = r_1$, then s_1 is upper bounded iff r_1 is upper bounded.

(13) If $s_1 = r_1$, then s_1 is lower bounded iff r_1 is lower bounded.

(14) If $s_1 = r_1$ and r_1 is convergent, then s_1 is convergent to finite number and convergent and $\lim s_1 = \lim r_1$.

(15) If $s_1 = r_1$ and s_1 is convergent to finite number, then r_1 is convergent and $\lim s_1 = \lim r_1$.

(16) If $s_1 \uparrow k$ is convergent to finite number, then s_1 is convergent to finite number and convergent and $\lim s_1 = \lim(s_1 \uparrow k)$.

(17) If $s_1 \uparrow k$ is convergent, then s_1 is convergent and $\lim s_1 = \lim(s_1 \uparrow k)$.

(18) If $\limsup s_1 = \liminf s_1$ and $\liminf s_1 \in \mathbb{R}$, then there exists k such that $s_1 \uparrow k$ is bounded.

(19) If s_1 is convergent to finite number, then there exists k such that $s_1 \uparrow k$ is bounded.

(20) Suppose s_1 is convergent to finite number. Then $s_1 \uparrow k$ is convergent to finite number and $s_1 \uparrow k$ is convergent and $\lim s_1 = \lim(s_1 \uparrow k)$.

(21) If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim s_1 = \lim(s_1 \uparrow k)$.

(22) If s_1 is upper bounded, then $s_1 \uparrow k$ is upper bounded and if s_1 is lower bounded, then $s_1 \uparrow k$ is lower bounded.

(23) $\inf s_1 \leq s_1(n)$ and $s_1(n) \leq \sup s_1$.

(24) $\inf s_1 \leq \sup s_1$.

(25) If s_1 is non-increasing, then $s_1 \uparrow k$ is non-increasing and $\inf s_1 = \inf(s_1 \uparrow k)$.

(26) If s_1 is non-decreasing, then $s_1 \uparrow k$ is non-decreasing and $\sup s_1 = \sup(s_1 \uparrow k)$.

(27) (The superior realsequence $s_1)(n) = \sup(s_1 \uparrow n)$ and (the inferior realsequence $s_1)(n) = \inf(s_1 \uparrow n)$.

(28) Let s_1 be a sequence of extended reals and j be an element of $\mathbb{N}$. Then the superior realsequence $s_1 \uparrow j =$ (the superior realsequence $s_1) \uparrow j$ and $\limsup(s_1 \uparrow j) = \limsup s_1$.

(29) Let s_1 be a sequence of extended reals and j be an element of $\mathbb{N}$. Then the inferior realsequence $s_1 \uparrow j =$ (the inferior realsequence $s_1) \uparrow j$ and $\liminf(s_1 \uparrow j) = \liminf s_1$.

(30) Let s_1 be a sequence of extended reals and k be an element of $\mathbb{N}$. Suppose s_1 is non-increasing and $-\infty < s_1(k)$ and $s_1(k) < +\infty$. Then $s_1 \uparrow k$ is upper bounded and $\sup(s_1 \uparrow k) = s_1(k)$.

(31) Let s_1 be a sequence of extended reals and k be an element of $\mathbb{N}$. Suppose s_1 is non-decreasing and $-\infty < s_1(k)$ and $s_1(k) < +\infty$. Then $s_1 \uparrow k$ is lower bounded and $\inf(s_1 \uparrow k) = s_1(k)$.

(32) Let s_1 be a sequence of extended reals. Suppose that for every element n of $\mathbb{N}$ holds $+\infty \leq s_1(n)$. Then s_1 is convergent to $+\infty$.

(33) Let s_1 be a sequence of extended reals. Suppose that for every element n of $\mathbb{N}$ holds $s_1(n) \leq -\infty$. Then s_1 is convergent to $-\infty$.

(34) Let s_1 be a sequence of extended reals. Suppose s_1 is non-increasing and $-\infty = \inf s_1$. Then s_1 is convergent to $-\infty$ and $\lim s_1 = -\infty$.

(35) Let s_1 be a sequence of extended reals. Suppose s_1 is non-decreasing and $+\infty = \sup s_1$. Then s_1 is convergent to $+\infty$ and $\lim s_1 = +\infty$.

(36) For every sequence s_1 of extended reals such that s_1 is non-increasing holds s_1 is convergent and $\lim s_1 = \inf s_1$.

(37) For every sequence s_1 of extended reals such that s_1 is non-decreasing holds s_1 is convergent and $\lim s_1 = \sup s_1$.

(38) Let s_2, s_3 be sequences of extended reals. Suppose s_2 is convergent and s_3 is convergent and for every element n of $\mathbb{N}$ holds $s_2(n) \leq s_3(n)$. Then $\lim s_2 \leq \lim s_3$.

(39) For every sequence s_1 of extended reals holds $\liminf s_1 \leq \limsup s_1$.

(40) For every sequence s_1 of extended reals holds s_1 is convergent iff $\liminf s_1 = \limsup s_1$.

(41) For every sequence s_1 of extended reals such that s_1 is convergent holds $\lim s_1 = \liminf s_1$ and $\lim s_1 = \limsup s_1$.

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Several Classes of BCK-algebras and their Properties

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Summary. In this article the general theory of Commutative BCK-algebras and BCI-algebras and several classes of BCK-algebras are given according to [2].

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The articles [3] and [1] provide the notation and terminology for this paper.

1. The Basics of General Theory of Commutative BCK-algebras

Let I_1 be a non empty BCI structure with 0. We say that I_1 is commutative if and only if:

(Def. 1) For all elements x, y of I_1 holds $x \setminus (x \setminus y) = y \setminus (y \setminus x)$.

Let us observe that BCI-EXAMPLE is commutative.

Let us note that there exists a BCK-algebra which is commutative.

In the sequel X denotes a BCK-algebra and I_1 denotes a non empty subset of X.

We now state a number of propositions:

(1) X is a commutative BCK-algebra iff for all elements x, y of X holds $x \setminus (x \setminus y) \leq y \setminus (y \setminus x)$.

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(2) For every BCK-algebra X and for all elements x, y of X holds $x \setminus (x \setminus y) \leq y$ and $x \setminus (x \setminus y) \leq x$.

(3) X is a commutative BCK-algebra iff for all elements x, y of X holds $x \setminus y = x \setminus (y \setminus (y \setminus x))$.

(4) X is a commutative BCK-algebra iff for all elements x, y of X holds $x \setminus (x \setminus y) = y \setminus (y \setminus (x \setminus (x \setminus y)))$.

(5) X is a commutative BCK-algebra iff for all elements x, y of X such that $x \leq y$ holds $x = y \setminus (y \setminus x)$.

(6) Let X be a non empty BCI structure with 0. Then X is a commutative BCK-algebra if and only if for all elements x, y, z of X holds $x \setminus (0_X \setminus y) = x$ and $(x \setminus z) \setminus (x \setminus y) = y \setminus z \setminus (y \setminus x)$.

(7) If X is a commutative BCK-algebra, then for all elements x, y of X such that $x \setminus y = x$ holds $y \setminus x = y$.

(8) If X is a commutative BCK-algebra, then for all elements x, y, a of X such that $y \leq a$ holds $a \setminus x \setminus (a \setminus y) = y \setminus x$.

(9) If X is a commutative BCK-algebra, then for all elements x, y of X holds $x \setminus y = x$ iff $y \setminus (y \setminus x) = 0_X$.

(10) If X is a commutative BCK-algebra, then for all elements x, y of X holds $x \setminus (y \setminus (y \setminus x)) = x \setminus y$ and $x \setminus y \setminus (x \setminus y \setminus x) = x \setminus y$.

(11) Suppose X is a commutative BCK-algebra. Let x, y, a be elements of X. If $x \leq a$, then $(a \setminus y) \setminus (a \setminus y \setminus (a \setminus x)) = a \setminus y \setminus (x \setminus y)$.

Let X be a BCI-algebra and let a be an element of X. We say that a is greatest if and only if:

(Def. 2) For every element x of X holds $x \setminus a = 0_X$.

We say that a is positive if and only if:

(Def. 3) $0_X \setminus a = 0_X$.

2. The Basics of General Theory of Commutative BCI-algebras

Let I_1 be a BCI-algebra. We say that I_1 is BCI-commutative if and only if:

(Def. 4) For all elements x, y of I_1 such that $x \setminus y = 0_{(I_1)}$ holds $x = y \setminus (y \setminus x)$.

We say that I_1 is BCI-weakly-commutative if and only if:

(Def. 5) For all elements x, y of I_1 holds $(x \setminus (x \setminus y)) \setminus (0_{(I_1)} \setminus (x \setminus y)) = y \setminus (y \setminus x)$.

One can check that BCI-EXAMPLE is BCI-commutative and BCI-weakly-commutative.

Let us note that there exists a BCI-algebra which is BCI-commutative and BCI-weakly-commutative.

The following propositions are true:

(12) For every BCI-algebra X such that there exists an element of X which is greatest holds X is a BCK-algebra.

(13) Let X be a BCI-algebra. Suppose X is p-semisimple. Then X is BCI-commutative and BCI-weakly-commutative.

(14) Every commutative BCK-algebra is a BCI-commutative BCI-algebra and a BCI-weakly-commutative BCI-algebra.

(15) If X is a BCI-weakly-commutative BCI-algebra, then X is BCI-commutative.

(16) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for all elements x, y of X holds $x \setminus (x \setminus y) = y \setminus (y \setminus (x \setminus (x \setminus y)))$.

(17) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for all elements x, y of X holds $(x \setminus (x \setminus y)) \setminus (y \setminus (y \setminus x)) = 0_X \setminus (x \setminus y)$.

(18) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for every element a of $\mathrm{AtomSet}\, X$ and for all elements x, y of $\mathrm{BranchV}\, a$ holds $x \setminus (x \setminus y) = y \setminus (y \setminus x)$.

(19) Let X be a non empty BCI structure with 0. Then X is a BCI-commutative BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) \setminus (z \setminus y) = 0_X$ and $x \setminus 0_X = x$ and $x \setminus (x \setminus y) = y \setminus (y \setminus (x \setminus (x \setminus y)))$.

(20) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for all elements x, y, z of X such that $x \leq z$ and $z \setminus y \leq z \setminus x$ holds $x \leq y$.

(21) Let X be a BCI-algebra. Then X is BCI-commutative if and only if for all elements x, y, z of X such that $x \leq y$ and $x \leq z$ holds $x \leq y \setminus (y \setminus z)$.

3. Bounded BCK-algebras

Let I_1 be a BCK-algebra. We say that I_1 is bounded if and only if:

(Def. 6) There exists an element of I_1 which is greatest.

Let us note that BCI-EXAMPLE is bounded.

One can verify that there exists a BCK-algebra which is bounded and commutative.

Let I_1 be a bounded BCK-algebra. We say that I_1 is involutory if and only if:

(Def. 7) For every element a of I_1 such that a is greatest and for every element x of I_1 holds $a \setminus (a \setminus x) = x$.

Next we state three propositions:

(22) Let X be a bounded BCK-algebra. Then X is involutory if and only if for every element a of X such that a is greatest and for all elements x, y of X holds $x \setminus y = a \setminus y \setminus (a \setminus x)$.

(23) Let X be a bounded BCK-algebra. Then X is involutory if and only if for every element a of X such that a is greatest and for all elements x, y of X holds $x \setminus (a \setminus y) = y \setminus (a \setminus x)$.

(24) Let X be a bounded BCK-algebra. Then X is involutory if and only if for every element a of X such that a is greatest and for all elements x, y of X such that $x \leq a \setminus y$ holds $y \leq a \setminus x$.

Let I_1 be a BCK-algebra and let a be an element of I_1. We say that a is Iseki if and only if:

(Def. 8) For every element x of I_1 holds $x \setminus a = 0_{(I_1)}$ and $a \setminus x = a$.

Let I_1 be a BCK-algebra. We say that I_1 is Iseki-extension if and only if:

(Def. 9) There exists an element of I_1 which is Iseki.

Let us observe that BCI-EXAMPLE is Iseki-extension.

Let X be a BCK-algebra. A non empty subset of X is said to be a commutative-ideal of X if:

(Def. 10) $0_X \in$ it and for all elements x, y, z of X such that $x \setminus y \setminus z \in$ it and $z \in$ it holds $x \setminus (y \setminus (y \setminus x)) \in$ it.

The following three propositions are true:

(25) If I_1 is a commutative-ideal of X, then for all elements x, y of X such that $x \setminus y \in I_1$ holds $x \setminus (y \setminus (y \setminus x)) \in I_1$.

(26) For every BCK-algebra X such that I_1 is a commutative-ideal of X holds I_1 is an ideal of X.

(27) If I_1 is a commutative-ideal of X, then for all elements x, y of X such that $x \setminus (x \setminus y) \in I_1$ holds $y \setminus (y \setminus x) \setminus (x \setminus y) \in I_1$.

4. Implicative and Positive-Implicative BCK-algebras

Let I_1 be a BCK-algebra. We say that I_1 is BCK-positive-implicative if and only if:

(Def. 11) For all elements x, y, z of I_1 holds $(x \setminus y) \setminus z = x \setminus z \setminus (y \setminus z)$.

We say that I_1 is BCK-implicative if and only if:

(Def. 12) For all elements x, y of I_1 holds $x \setminus (y \setminus x) = x$.

Let us observe that BCI-EXAMPLE is BCK-positive-implicative and BCK-implicative.

Let us mention that there exists a BCK-algebra which is Iseki-extension, BCK-positive-implicative, BCK-implicative, bounded, and commutative.

The following propositions are true:

(28) X is a BCK-positive-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus y = x \setminus y \setminus y$.

(29) X is a BCK-positive-implicative BCK-algebra if and only if for all elements x, y of X holds $(x \setminus (x \setminus y)) \setminus (y \setminus x) = x \setminus (x \setminus (y \setminus (y \setminus x)))$.

(30) X is a BCK-positive-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus y = x \setminus y \setminus (x \setminus (x \setminus y))$.

(31) X is a BCK-positive-implicative BCK-algebra if and only if for all elements x, y, z of X holds $x \setminus z \setminus (y \setminus z) \leq (x \setminus y) \setminus z$.

(32) X is a BCK-positive-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus y \leq x \setminus y \setminus y$.

(33) X is a BCK-positive-implicative BCK-algebra if and only if for all elements x, y of X holds $x \setminus (x \setminus (y \setminus (y \setminus x))) \leq (x \setminus (x \setminus y)) \setminus (y \setminus x)$.

(34) X is a BCK-implicative BCK-algebra if and only if X is a commutative BCK-algebra and a BCK-positive-implicative BCK-algebra.

(35) X is a BCK-implicative BCK-algebra iff for all elements x, y of X holds $(x \setminus (x \setminus y)) \setminus (x \setminus y) = y \setminus (y \setminus x)$.

(36) Let X be a non empty BCI structure with 0. Then X is a BCK-implicative BCK-algebra if and only if for all elements x, y, z of X holds $x \setminus (0_X \setminus y) = x$ and $(x \setminus z) \setminus (x \setminus y) = y \setminus z \setminus (y \setminus x) \setminus (x \setminus y)$.

(37) Let X be a bounded BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if X is involutory and BCK-positive-implicative.

(38) X is a BCK-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus (x \setminus (y \setminus x)) = 0_X$.

(39) X is a BCK-implicative BCK-algebra iff for all elements x, y of X holds $(x \setminus (x \setminus y)) \setminus (x \setminus y) = y \setminus (y \setminus (x \setminus (x \setminus y)))$.

(40) X is a BCK-implicative BCK-algebra iff for all elements x, y, z of X holds $(x \setminus z) \setminus (x \setminus y) = y \setminus z \setminus (y \setminus x \setminus z)$.

(41) X is a BCK-implicative BCK-algebra iff for all elements x, y, z of X holds $x \setminus (x \setminus (y \setminus z)) = (y \setminus z) \setminus (y \setminus z \setminus (x \setminus z))$.

(42) X is a BCK-implicative BCK-algebra iff for all elements x, y of X holds $x \setminus (x \setminus y) = (y \setminus (y \setminus x)) \setminus (x \setminus y)$.

(43) Let X be a bounded commutative BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if for every element x of X holds $a \setminus x \setminus (a \setminus x \setminus x) = 0_X$.

(44) Let X be a bounded commutative BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if for every element x of X holds $x \setminus (a \setminus x) = x$.

(45) Let X be a bounded commutative BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if for every element x of X holds $a \setminus x \setminus x = a \setminus x$.

(46) Let X be a bounded commutative BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if for all elements x, y of X holds $a \setminus y \setminus (a \setminus y \setminus x) = x \setminus y$.

(47) Let X be a bounded commutative BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if for all elements x, y of X holds $y \setminus (y \setminus x) = x \setminus (a \setminus y)$.

(48) Let X be a bounded commutative BCK-algebra and a be an element of X. Suppose a is greatest. Then X is BCK-implicative if and only if for all elements x, y, z of X holds $(x \setminus (y \setminus z)) \setminus (x \setminus y) \leq x \setminus (a \setminus z)$.

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Several Differentiation Formulas of Special Functions. Part VI

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Summary. In this article, we prove a series of differentiation identities [3] involving the secant and cosecant functions and specific combinations of special functions including trigonometric, exponential and logarithmic functions.

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The papers [11], [13], [1], [15], [2], [8], [9], [16], [5], [12], [10], [4], [6], [7], and [14] provide the notation and terminology for this paper.

In this paper x denotes a real number and Z denotes an open subset of $\mathbb{R}$.

One can prove the following propositions:

(1) Suppose $Z \subseteq \operatorname{dom}((\text{the function tan}) \cdot (\text{the function cot}))$. Then

(i) (the function tan) $\cdot$(the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function tan}) \cdot (\text{the function cot}))'_{\restriction Z}(x) = \frac{1}{(\text{the function cos})((\text{the function cot})(x))^{\mathbf{2}}} \cdot -\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(2) Suppose $Z \subseteq \operatorname{dom}((\text{the function tan}) \cdot (\text{the function tan}))$. Then

(i) (the function tan) $\cdot$(the function tan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function tan}) \cdot (\text{the function tan}))'_{\restriction Z}(x) = \frac{1}{(\text{the function cos})((\text{the function tan})(x))^{\mathbf{2}}} \cdot \frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(3) Suppose $Z \subseteq \operatorname{dom}((\text{the function cot}) \cdot (\text{the function cot}))$. Then

(i) (the function cot) $\cdot$(the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function cot}) \cdot (\text{the function cot}))'_{\restriction Z}(x) = \frac{1}{(\text{the function sin})((\text{the function cot})(x))^{\mathbf{2}}} \cdot \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(4) Suppose $Z \subseteq \operatorname{dom}((\text{the function cot}) \cdot (\text{the function tan}))$. Then

(i) (the function cot) $\cdot$(the function tan) is differentiable on Z, and

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(ii) for every x such that $x \in Z$ holds ((the function cot) $\cdot$(the function tan))$'_{\restriction Z}(x) = (-\frac{1}{\text{(the function sin)((the function tan)}(x))^{\mathbf{2}}}) \cdot \frac{1}{\text{(the function cos)}(x)^{\mathbf{2}}}$.

(5) Suppose $Z \subseteq$ dom((the function tan)$-$(the function cot)). Then

(i) (the function tan)$-$(the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function tan)$-$(the function cot))$'_{\restriction Z}(x) = \frac{1}{\text{(the function cos)}(x)^{\mathbf{2}}} + \frac{1}{\text{(the function sin)}(x)^{\mathbf{2}}}$.

(6) Suppose $Z \subseteq$ dom((the function tan)+(the function cot)). Then

(i) (the function tan)+(the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function tan)+(the function cot))$'_{\restriction Z}(x) = \frac{1}{\text{(the function cos)}(x)^{\mathbf{2}}} - \frac{1}{\text{(the function sin)}(x)^{\mathbf{2}}}$.

(7)(i) (The function sin) $\cdot$(the function sin) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) $\cdot$(the function sin))$'_{\restriction Z}(x) =$ (the function cos)((the function sin)(x)) $\cdot$ (the function cos)(x).

(8)(i) (The function sin) $\cdot$(the function cos) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) $\cdot$(the function cos))$'_{\restriction Z}(x) = -$(the function cos)((the function cos)(x)) $\cdot$ (the function sin) (x).

(9)(i) (The function cos) $\cdot$(the function sin) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) $\cdot$(the function sin))$'_{\restriction Z}(x) = -$(the function sin)((the function sin)(x)) $\cdot$ (the function cos) (x).

(10)(i) (The function cos) $\cdot$(the function cos) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) $\cdot$(the function cos))$'_{\restriction Z}(x) =$ (the function sin)((the function cos)(x)) $\cdot$ (the function sin)(x).

(11) Suppose $Z \subseteq$ dom((the function cos) (the function cot)). Then

(i) (the function cos) (the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) (the function cot))$'_{\restriction Z}(x) = -$(the function cos)$(x) - \frac{\text{(the function cos)}(x)}{\text{(the function sin)}(x)^{\mathbf{2}}}$.

(12) Suppose $Z \subseteq$ dom((the function sin) (the function tan)). Then

(i) (the function sin) (the function tan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) (the function tan))$'_{\restriction Z}(x) =$ (the function sin)$(x) + \frac{\text{(the function sin)}(x)}{\text{(the function cos)}(x)^{\mathbf{2}}}$.

(13) Suppose $Z \subseteq$ dom((the function sin) (the function cot)). Then

(i) (the function sin) (the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) (the function cot))$'_{\restriction Z}(x) =$ (the function cos)$(x) \cdot$ (the function cot)$(x) - \frac{1}{\text{(the function sin)}(x)}$.

(14) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos})\ (\text{the function tan}))$. Then

(i) (the function cos) (the function tan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) (the function tan))$'_{\restriction Z}(x) = -\frac{(\text{the function sin})(x)^{\mathbf{2}}}{(\text{the function cos})(x)} + \frac{1}{(\text{the function cos})(x)}$.

(15) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin})\ (\text{the function cos}))$. Then

(i) (the function sin) (the function cos) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) (the function cos))$'_{\restriction Z}(x) = (\text{the function cos})(x)^{\mathbf{2}} - (\text{the function sin})(x)^{\mathbf{2}}$.

(16) Suppose $Z \subseteq \mathrm{dom}((\text{the function ln})\ (\text{the function sin}))$. Then

(i) (the function ln) (the function sin) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function ln) (the function sin))$'_{\restriction Z}(x) = \frac{(\text{the function sin})(x)}{x} + (\text{the function ln})(x) \cdot (\text{the function cos})(x)$.

(17) Suppose $Z \subseteq \mathrm{dom}((\text{the function ln})\ (\text{the function cos}))$. Then

(i) (the function ln) (the function cos) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function ln) (the function cos))$'_{\restriction Z}(x) = \frac{(\text{the function cos})(x)}{x} - (\text{the function ln})(x) \cdot (\text{the function sin})(x)$.

(18) Suppose $Z \subseteq \mathrm{dom}((\text{the function ln})\ (\text{the function exp}))$. Then

(i) (the function ln) (the function exp) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function ln) (the function exp))$'_{\restriction Z}(x) = \frac{(\text{the function exp})(x)}{x} + (\text{the function ln})(x) \cdot (\text{the function exp})(x)$.

(19) Suppose $Z \subseteq \mathrm{dom}((\text{the function ln}) \cdot (\text{the function ln}))$ and for every x such that $x \in Z$ holds $x > 0$. Then

(i) (the function ln) $\cdot$(the function ln) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function ln) $\cdot$(the function ln))$'_{\restriction Z}(x) = \frac{1}{(\text{the function ln})(x) \cdot x}$.

(20) Suppose $Z \subseteq \mathrm{dom}((\text{the function exp}) \cdot (\text{the function exp}))$. Then

(i) (the function exp) $\cdot$(the function exp) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) $\cdot$(the function exp))$'_{\restriction Z}(x) = (\text{the function exp})((\text{the function exp})(x)) \cdot (\text{the function exp})(x)$.

(21) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin}) \cdot (\text{the function tan}))$. Then

(i) (the function sin) $\cdot$(the function tan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) $\cdot$(the function tan))$'_{\restriction Z}(x) = \frac{\cos(\text{the function tan})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(22) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin}) \cdot (\text{the function cot}))$. Then

(i) (the function sin) $\cdot$(the function cot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) $\cdot$(the function cot))$'_{\restriction Z}(x) = -\frac{\cos(\text{the function cot})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(23) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos}) \cdot (\text{the function tan}))$. Then
(i) (the function cos) $\cdot$(the function tan) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function cos}) \cdot (\text{the function tan}))'_{\restriction Z}(x) = -\frac{\sin(\text{the function tan})(x)}{(\text{the function cos})(x)^{2}}$.

(24) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos}) \cdot (\text{the function cot}))$. Then
(i) (the function cos) $\cdot$(the function cot) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function cos}) \cdot (\text{the function cot}))'_{\restriction Z}(x) = \frac{\sin(\text{the function cot})(x)}{(\text{the function sin})(x)^{2}}$.

(25) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin})\ ((\text{the function tan})+(\text{the function cot})))$. Then
(i) (the function sin) ((the function tan)+(the function cot)) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function sin})\ ((\text{the function tan})+(\text{the function cot})))'_{\restriction Z}(x) = (\text{the function cos})(x) \cdot ((\text{the function tan})(x) + (\text{the function cot})(x)) + (\text{the function sin})(x) \cdot (\frac{1}{(\text{the function cos})(x)^{2}} - \frac{1}{(\text{the function sin})(x)^{2}})$.

(26) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos})\ ((\text{the function tan})+(\text{the function cot})))$. Then
(i) (the function cos) ((the function tan)+(the function cot)) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function cos})\ ((\text{the function tan})+(\text{the function cot})))'_{\restriction Z}(x) = -(\text{the function sin})(x) \cdot ((\text{the function tan})(x) + (\text{the function cot})(x)) + (\text{the function cos})(x) \cdot (\frac{1}{(\text{the function cos})(x)^{2}} - \frac{1}{(\text{the function sin})(x)^{2}})$.

(27) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin})\ ((\text{the function tan})-(\text{the function cot})))$. Then
(i) (the function sin) ((the function tan)−(the function cot)) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function sin})\ ((\text{the function tan})-(\text{the function cot})))'_{\restriction Z}(x) = (\text{the function cos})(x) \cdot ((\text{the function tan})(x) - (\text{the function cot})(x)) + (\text{the function sin})(x) \cdot (\frac{1}{(\text{the function cos})(x)^{2}} + \frac{1}{(\text{the function sin})(x)^{2}})$.

(28) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos})\ ((\text{the function tan})-(\text{the function cot})))$. Then
(i) (the function cos) ((the function tan)−(the function cot)) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\text{the function cos})\ ((\text{the function tan})-(\text{the function cot})))'_{\restriction Z}(x) = -(\text{the function sin})(x) \cdot ((\text{the function tan})(x) - (\text{the function cot})(x)) + (\text{the function cos})(x) \cdot (\frac{1}{(\text{the function cos})(x)^{2}} + \frac{1}{(\text{the function sin})(x)^{2}})$.

(29) Suppose $Z \subseteq$ dom((the function exp) ((the function tan)+(the function cot))). Then

(i) (the function exp) ((the function tan)+(the function cot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) ((the function tan)+(the function cot)))$'_{\restriction Z}(x)$ = (the function exp)(x) $\cdot$ ((the function tan)(x) + (the function cot)(x)) + (the function exp)(x) $\cdot$ $(\frac{1}{(\text{the function cos})(x)^{\mathbf{2}}} - \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}})$.

(30) Suppose $Z \subseteq$ dom((the function exp) ((the function tan)$-$(the function cot))). Then

(i) (the function exp) ((the function tan)$-$(the function cot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) ((the function tan)$-$(the function cot)))$'_{\restriction Z}(x)$ = (the function exp)(x) $\cdot$ ((the function tan)(x) $-$ (the function cot)(x)) + (the function exp)(x) $\cdot$ $(\frac{1}{(\text{the function cos})(x)^{\mathbf{2}}} + \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}})$.

(31) Suppose $Z \subseteq$ dom((the function sin) ((the function sin)+(the function cos))). Then

(i) (the function sin) ((the function sin)+(the function cos)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) ((the function sin)+(the function cos)))$'_{\restriction Z}(x)$ = ((the function cos)$(x)^{\mathbf{2}}$ + $2 \cdot$ (the function sin)(x) $\cdot$ (the function cos)(x)) $-$ (the function sin)$(x)^{\mathbf{2}}$.

(32) Suppose $Z \subseteq$ dom((the function sin) ((the function sin)$-$(the function cos))). Then

(i) (the function sin) ((the function sin)$-$(the function cos)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) ((the function sin)$-$(the function cos)))$'_{\restriction Z}(x)$ = ((the function sin)$(x)^{\mathbf{2}}$ + $2 \cdot$ (the function sin)(x) $\cdot$ (the function cos)(x)) $-$ (the function cos)$(x)^{\mathbf{2}}$.

(33) Suppose $Z \subseteq$ dom((the function cos) ((the function sin)$-$(the function cos))). Then

(i) (the function cos) ((the function sin)$-$(the function cos)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) ((the function sin)$-$(the function cos)))$'_{\restriction Z}(x)$ = ((the function cos)$(x)^{\mathbf{2}}$ + $2 \cdot$ (the function sin)(x) $\cdot$ (the function cos)(x)) $-$ (the function sin)$(x)^{\mathbf{2}}$.

(34) Suppose $Z \subseteq$ dom((the function cos) ((the function sin)+(the function cos))). Then

(i) (the function cos) ((the function sin)+(the function cos)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) ((the function sin)+(the function cos)))$'_{\restriction Z}(x)$ = (the function cos)$(x)^{\mathbf{2}} - 2 \cdot$ (the function sin)$(x) \cdot$ (the function cos)$(x) -$ (the function sin)$(x)^{\mathbf{2}}$.

(35) Suppose $Z \subseteq$ dom((the function sin) $\cdot$((the function tan)+(the function cot))). Then

(i) (the function sin) $\cdot$((the function tan)+(the function cot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) $\cdot$((the function tan)+(the function cot)))$'_{\restriction Z}(x)$ = (the function cos)((the function tan)(x) + (the function cot)$(x)) \cdot (\frac{1}{\text{(the function cos)}(x)^{\mathbf{2}}} - \frac{1}{\text{(the function sin)}(x)^{\mathbf{2}}})$.

(36) Suppose $Z \subseteq$ dom((the function sin) $\cdot$((the function tan)$-$(the function cot))). Then

(i) (the function sin) $\cdot$((the function tan)$-$(the function cot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) $\cdot$((the function tan)$-$(the function cot)))$'_{\restriction Z}(x)$ = (the function cos)((the function tan)(x) $-$ (the function cot)$(x)) \cdot (\frac{1}{\text{(the function cos)}(x)^{\mathbf{2}}} + \frac{1}{\text{(the function sin)}(x)^{\mathbf{2}}})$.

(37) Suppose $Z \subseteq$ dom((the function cos) $\cdot$((the function tan)$-$(the function cot))). Then

(i) (the function cos) $\cdot$((the function tan)$-$(the function cot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) $\cdot$((the function tan)$-$(the function cot)))$'_{\restriction Z}(x)$ = $-$(the function sin)((the function tan)(x) $-$ (the function cot)$(x)) \cdot (\frac{1}{\text{(the function cos)}(x)^{\mathbf{2}}} + \frac{1}{\text{(the function sin)}(x)^{\mathbf{2}}})$.

(38) Suppose $Z \subseteq$ dom((the function cos) $\cdot$((the function tan)+(the function cot))). Then

(i) (the function cos) $\cdot$((the function tan)+(the function cot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) $\cdot$((the function tan)+(the function cot)))$'_{\restriction Z}(x)$ = $-$(the function sin)((the function tan) (x) + (the function cot)$(x)) \cdot (\frac{1}{\text{(the function cos)}(x)^{\mathbf{2}}} - \frac{1}{\text{(the function sin)}(x)^{\mathbf{2}}})$.

(39) Suppose $Z \subseteq$ dom((the function exp) $\cdot$((the function tan)+(the function cot))). Then

(i) (the function exp) $\cdot$((the function tan)+(the function cot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) $\cdot$((the function tan)+(the function cot)))$'_{\restriction Z}(x)$ = (the function exp)((the function tan)(x) + (the function cot)$(x)) \cdot (\frac{1}{\text{(the function cos)}(x)^{\mathbf{2}}} - \frac{1}{\text{(the function sin)}(x)^{\mathbf{2}}})$.

(40) Suppose $Z \subseteq$ dom((the function exp) $\cdot$((the function tan)$-$(the function cot))). Then

(i) (the function exp) $\cdot$((the function tan)$-$(the function cot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) $\cdot$((the function tan)$-$(the function cot)))$'_{\restriction Z}(x)$ = (the function exp)((the function tan)(x) $-$ (the function cot)(x)) $\cdot$ $(\frac{1}{(\text{the function cos})(x)^{\mathbf{2}}} + \frac{1}{(\text{the function sin})(x)^{\mathbf{2}}})$.

(41) Suppose $Z \subseteq \mathrm{dom}(\frac{(\text{the function tan})-(\text{the function cot})}{\text{the function exp}})$. Then

(i) $\frac{(\text{the function tan})-(\text{the function cot})}{\text{the function exp}}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{(\text{the function tan})-(\text{the function cot})}{\text{the function exp}})'_{\restriction Z}(x) = \frac{((\frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}+\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}})-(\text{the function tan})(x))+(\text{the function cot})(x)}{(\text{the function exp})(x)}$.

(42) Suppose $Z \subseteq \mathrm{dom}(\frac{(\text{the function tan})+(\text{the function cot})}{\text{the function exp}})$. Then

(i) $\frac{(\text{the function tan})+(\text{the function cot})}{\text{the function exp}}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{(\text{the function tan})+(\text{the function cot})}{\text{the function exp}})'_{\restriction Z}(x) = \frac{\frac{1}{(\text{the function cos})(x)^{\mathbf{2}}}-\frac{1}{(\text{the function sin})(x)^{\mathbf{2}}}-(\text{the function tan})(x)-(\text{the function cot})(x)}{(\text{the function exp})(x)}$.

(43) Suppose $Z \subseteq \mathrm{dom}$((the function sin) $\cdot$ sec). Then

(i) (the function sin) $\cdot$ sec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) $\cdot$ sec)$'_{\restriction Z}(x)$ = $\frac{(\text{the function cos})((\text{sec})(x))\cdot(\text{the function sin})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(44) Suppose $Z \subseteq \mathrm{dom}$((the function cos) $\cdot$ sec). Then

(i) (the function cos) $\cdot$ sec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) $\cdot$ sec)$'_{\restriction Z}(x)$ = $-\frac{(\text{the function sin})((\text{sec})(x))\cdot(\text{the function sin})(x)}{(\text{the function cos})(x)^{\mathbf{2}}}$.

(45) Suppose $Z \subseteq \mathrm{dom}$((the function sin) $\cdot$ cosec). Then

(i) (the function sin) $\cdot$ cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) $\cdot$ cosec)$'_{\restriction Z}(x)$ = $-\frac{(\text{the function cos})((\text{cosec})(x))\cdot(\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$.

(46) Suppose $Z \subseteq \mathrm{dom}$((the function cos) $\cdot$ cosec). Then

(i) (the function cos) $\cdot$ cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) $\cdot$ cosec)$'_{\restriction Z}(x)$ = $\frac{(\text{the function sin})((\text{cosec})(x))\cdot(\text{the function cos})(x)}{(\text{the function sin})(x)^{\mathbf{2}}}$.

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Index of MML Identifiers