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Arrow's Impossibility Theorem

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Summary. A formalization of the first proof from [6].

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The terminology and notation used here are introduced in the following articles: [11], [13], [12], [10], [9], [5], [2], [3], [1], [8], [4], and [7].

1. Preliminaries

Let A, B' be non empty sets, let B be a non empty subset of B', let f be a function from A into B, and let x be an element of A. Then $f(x)$ is an element of B.

Next we state two propositions:

(1) For every finite set A such that $\operatorname{card} A \geq 2$ and for every element a of A there exists an element b of A such that $b \neq a$.

(2) Let A be a finite set. Suppose $\operatorname{card} A \geq 3$. Let a, b be elements of A. Then there exists an element c of A such that $c \neq a$ and $c \neq b$.

2. Linear Preorders and Linear Orders

In the sequel A denotes a non empty set and a, b, c denote elements of A.

Let us consider A. The functor $\operatorname{LinPreorders} A$ is defined by the condition (Def. 1).

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(Def. 1) Let R be a set. Then $R \in \mathrm{LinPreorders}\,A$ if and only if the following conditions are satisfied:

(i) R is a binary relation on A,

(ii) for all a, b holds $\langle a, b\rangle \in R$ or $\langle b, a\rangle \in R$, and

(iii) for all a, b, c such that $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$ holds $\langle a, c\rangle \in R$.

Let us consider A. Note that $\mathrm{LinPreorders}\,A$ is non empty.

Let us consider A. The functor $\mathrm{LinOrders}\,A$ yielding a subset of $\mathrm{LinPreorders}\,A$ is defined by:

(Def. 2) For every element R of $\mathrm{LinPreorders}\,A$ holds $R \in \mathrm{LinOrders}\,A$ iff for all a, b such that $\langle a, b\rangle \in R$ and $\langle b, a\rangle \in R$ holds $a = b$.

Let A be a set. One can verify that there exists an order in A which is connected.

Let us consider A. Then $\mathrm{LinOrders}\,A$ can be characterized by the condition:

(Def. 3) For every set R holds $R \in \mathrm{LinOrders}\,A$ iff R is a connected order in A.

Let us consider A. One can verify that $\mathrm{LinOrders}\,A$ is non empty.

In the sequel o, o' are elements of $\mathrm{LinPreorders}\,A$ and o'' is an element of $\mathrm{LinOrders}\,A$.

Let us consider A, o, a, b. The predicate $a \leq_o b$ is defined by:

(Def. 4) $\langle a, b\rangle \in o$.

Let us consider A, o, a, b. We introduce $b \geq_o a$ as a synonym of $a \leq_o b$. We introduce $b <_o a$ as an antonym of $a \leq_o b$. We introduce $a >_o b$ as an antonym of $a \leq_o b$.

We now state a number of propositions:

(3) $a \leq_o a$.

(4) $a \leq_o b$ or $b \leq_o a$.

(5) If $a \leq_o b$ or $a <_o b$ and if $b \leq_o c$ or $b <_o c$, then $a \leq_o c$.

(6) If $a \leq_{o''} b$ and $b \leq_{o''} a$, then $a = b$.

(7) If $a \neq b$ and $b \neq c$ and $a \neq c$, then there exists o such that $a <_o b$ and $b <_o c$.

(8) There exists o such that for every a such that $a \neq b$ holds $b <_o a$.

(9) There exists o such that for every a such that $a \neq b$ holds $a <_o b$.

(10) If $a \neq b$ and $a \neq c$, then there exists o such that $a <_o b$ and $a <_o c$ and $b <_o c$ iff $b <_{o'} c$ and $c <_o b$ iff $c <_{o'} b$.

(11) If $a \neq b$ and $a \neq c$, then there exists o such that $b <_o a$ and $c <_o a$ and $b <_o c$ iff $b <_{o'} c$ and $c <_o b$ iff $c <_{o'} b$.

(12) Let o, o' be elements of $\mathrm{LinOrders}\,A$. Then $a <_o b$ iff $a <_{o'} b$ and $b <_o a$ iff $b <_{o'} a$ if and only if $a <_o b$ iff $a <_{o'} b$.

(13) Let o be an element of $\mathrm{LinOrders}\,A$ and o' be an element of $\mathrm{LinPreorders}\,A$. Then for all a, b such that $a <_o b$ holds $a <_{o'} b$ if and only

if for all a, b holds $a <_o b$ iff $a <_{o'} b$.

3. Arrow's Theorem

For simplicity, we follow the rules: A, N are finite non empty sets, a, b are elements of A, i, n are elements of N, p, p' are elements of $(\text{LinPreorders}\,A)^N$, and f is a function from $(\text{LinPreorders}\,A)^N$ into $\text{LinPreorders}\,A$.

We now state the proposition

(14) Suppose that

(i) for all p, a, b such that for every i holds $a <_{p(i)} b$ holds $a <_{f(p)} b$,

(ii) for all p, p', a, b such that for every i holds $a <_{p(i)} b$ iff $a <_{p'(i)} b$ and $b <_{p(i)} a$ iff $b <_{p'(i)} a$ holds $a <_{f(p)} b$ iff $a <_{f(p')} b$, and

(iii) $\text{card}\,A \geq 3$.

Then there exists n such that for all p, a, b such that $a <_{p(n)} b$ holds $a <_{f(p)} b$.

In the sequel p, p' denote elements of $(\text{LinOrders}\,A)^N$ and f denotes a function from $(\text{LinOrders}\,A)^N$ into $\text{LinPreorders}\,A$.

One can prove the following proposition

(15) Suppose that

(i) for all p, a, b such that for every i holds $a <_{p(i)} b$ holds $a <_{f(p)} b$,

(ii) for all p, p', a, b such that for every i holds $a <_{p(i)} b$ iff $a <_{p'(i)} b$ holds $a <_{f(p)} b$ iff $a <_{f(p')} b$, and

(iii) $\text{card}\,A \geq 3$.

Then there exists n such that for all p, a, b holds $a <_{p(n)} b$ iff $a <_{f(p)} b$.

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