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Eigenvalues of a Linear Transformation

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Summary. The article presents well known facts about eigenvalues of linear transformation of a vector space (see [13]). I formalize main dependencies between eigenvalues and the diagram of the matrix of a linear transformation over a finite-dimensional vector space. Finally, I formalize the subspace $\bigcup_{i=0}^{\infty} \mathrm{Ker}(f - \lambda I)^i$ called a generalized eigenspace for the eigenvalue λ and show its basic properties.

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The articles [11], [33], [2], [3], [12], [34], [8], [10], [9], [5], [31], [27], [15], [7], [14], [32], [35], [25], [30], [29], [28], [26], [6], [22], [16], [23], [20], [1], [19], [4], [21], [17], [18], and [24] provide the notation and terminology for this paper.

1. Preliminaries

We adopt the following convention: i, j, m, n denote natural numbers, K denotes a field, and a denotes an element of K.

Next we state several propositions:

(1) Let A, B be matrices over K, n_1 be an element of $\mathbb{N}^n$, and m_1 be an element of $\mathbb{N}^m$. If $\mathrm{rng}\, n_1 \times \mathrm{rng}\, m_1 \subseteq$ the indices of A, then $\mathrm{Segm}(A + B, n_1, m_1) = \mathrm{Segm}(A, n_1, m_1) + \mathrm{Segm}(B, n_1, m_1)$.

(2) For every without zero finite subset P of $\mathbb{N}$ such that $P \subseteq \mathrm{Seg}\, n$ holds $\mathrm{Segm}(I_K^{n \times n}, P, P) = I_K^{\mathrm{card}\, P \times \mathrm{card}\, P}$.

(3) Let A, B be matrices over K and P, Q be without zero finite subsets of $\mathbb{N}$. If $P \times Q \subseteq$ the indices of A, then $\mathrm{Segm}(A + B, P, Q) = \mathrm{Segm}(A, P, Q) + \mathrm{Segm}(B, P, Q)$.

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(4) For all square matrices A, B over K of dimension n such that $i, j \in \operatorname{Seg} n$ holds $\operatorname{Delete}(A+B,i,j) = \operatorname{Delete}(A,i,j) + \operatorname{Delete}(B,i,j)$.

(5) For every square matrix A over K of dimension n such that $i, j \in \operatorname{Seg} n$ holds $\operatorname{Delete}(a \cdot A,i,j) = a \cdot \operatorname{Delete}(A,i,j)$.

(6) If $i \in \operatorname{Seg} n$, then $\operatorname{Delete}(I_K^{n\times n},i,i) = I_K^{(n-'1)\times(n-'1)}$.

(7) Let A, B be square matrices over K of dimension n. Then there exists a polynomial P of K such that $\operatorname{len} P \leq n+1$ and for every element x of K holds $\operatorname{eval}(P,x) = \operatorname{Det}(A + x \cdot B)$.

(8) Let A be a square matrix over K of dimension n. Then there exists a polynomial P of K such that $\operatorname{len} P = n+1$ and for every element x of K holds $\operatorname{eval}(P,x) = \operatorname{Det}(A + x \cdot I_K^{n\times n})$.

Let us consider K. Observe that there exists a vector space over K which is non trivial and finite dimensional.

2. Maps with Eigenvalues

Let R be a non empty double loop structure, let V be a non empty vector space structure over R, and let I_1 be a function from V into V. We say that I_1 has eigenvalues if and only if:

(Def. 1) There exists a vector v of V and there exists a scalar a of R such that $v \neq 0_V$ and $I_1(v) = a \cdot v$.

For simplicity, we follow the rules: V denotes a non trivial vector space over K, V_1, V_2 denote vector spaces over K, f denotes a linear transformation from V_1 to V_1, v, w denote vectors of V, v_1 denotes a vector of V_1, and L denotes a scalar of K.

Let us consider K, V. One can verify that there exists a linear transformation from V to V which has eigenvalues.

Let R be a non empty double loop structure, let V be a non empty vector space structure over R, and let f be a function from V into V. Let us assume that f has eigenvalues. An element of R is called an eigenvalue of f if:

(Def. 2) There exists a vector v of V such that $v \neq 0_V$ and $f(v) = \mathrm{it} \cdot v$.

Let R be a non empty double loop structure, let V be a non empty vector space structure over R, let f be a function from V into V, and let L be a scalar of R. Let us assume that f has eigenvalues and L is an eigenvalue of f. A vector of V is called an eigenvector of f and L if:

(Def. 3) $f(\mathrm{it}) = L \cdot \mathrm{it}$.

We now state several propositions:

(9) Let given a. Suppose $a \neq 0_K$. Let f be a function from V into V with eigenvalues and L be an eigenvalue of f. Then

(i) $a \cdot f$ has eigenvalues,

(ii) $a \cdot L$ is an eigenvalue of $a \cdot f$, and

(iii) w is an eigenvector of f and L iff w is an eigenvector of $a \cdot f$ and $a \cdot L$.

(10) Let f_1, f_2 be functions from V into V with eigenvalues and L_1, L_2 be scalars of K. Suppose that

(i) L_1 is an eigenvalue of f_1,

(ii) L_2 is an eigenvalue of f_2, and

(iii) there exists v such that v is an eigenvector of f_1 and L_1 and an eigenvector of f_2 and L_2 and $v \neq 0_V$.

Then

(iv) $f_1 + f_2$ has eigenvalues,

(v) $L_1 + L_2$ is an eigenvalue of $f_1 + f_2$, and

(vi) for every w such that w is an eigenvector of f_1 and L_1 and an eigenvector of f_2 and L_2 holds w is an eigenvector of $f_1 + f_2$ and $L_1 + L_2$.

(11) id_V has eigenvalues and $\mathbf{1}_K$ is an eigenvalue of id_V and every v is an eigenvector of id_V and $\mathbf{1}_K$.

(12) For every eigenvalue L of id_V holds $L = \mathbf{1}_K$.

(13) If $\ker f$ is non trivial, then f has eigenvalues and 0_K is an eigenvalue of f.

(14) f has eigenvalues and L is an eigenvalue of f iff $\ker f + (-L) \cdot \mathrm{id}_{(V_1)}$ is non trivial.

(15) Let V_1 be a finite dimensional vector space over K, b_1, b_1' be ordered bases of V_1, and f be a linear transformation from V_1 to V_1. Then f has eigenvalues and L is an eigenvalue of f if and only if $\mathrm{Det}\,\mathrm{AutEqMt}(f + (-L) \cdot \mathrm{id}_{(V_1)}, b_1, b_1') = 0_K$.

(16) Let K be an algebraic-closed field and V_1 be a non trivial finite dimensional vector space over K. Then every linear transformation from V_1 to V_1 has eigenvalues.

(17) Let given f, L. Suppose f has eigenvalues and L is an eigenvalue of f. Then v_1 is an eigenvector of f and L if and only if $v_1 \in \ker f + (-L) \cdot \mathrm{id}_{(V_1)}$.

Let S be a 1-sorted structure, let F be a function from S into S, and let n be a natural number. The functor F^n yields a function from S into S and is defined as follows:

(Def. 4) For every element F' of the semigroup of functions onto the carrier of S such that $F' = F$ holds $F^n = \prod(n \mapsto F')$.

In the sequel S denotes a 1-sorted structure and F denotes a function from S into S.

Next we state several propositions:

(18) $F^0 = \mathrm{id}_S$.

(19) $F^1 = F$.

(20) $F^{i+j} = F^i \cdot F^j$.

(21) For all elements s_1, s_2 of S and for all n, m such that $F^m(s_1) = s_2$ and $F^n(s_2) = s_2$ holds $F^{m+i\cdot n}(s_1) = s_2$.

(22) Let K be an add-associative right zeroed right complementable Abelian associative well unital distributive non empty double loop structure, V_1 be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over K, W be a subspace of V_1, f be a function from V_1 into V_1, and f_3 be a function from W into W. If $f_3 = f \restriction W$, then $f^n \restriction W = {f_3}^n$.

Let us consider K, V_1, let f be a linear transformation from V_1 to V_1, and let n be a natural number. Then f^n is a linear transformation from V_1 to V_1.

We now state the proposition

(23) If $f^i(v_1) = 0_{(V_1)}$, then $f^{i+j}(v_1) = 0_{(V_1)}$.

3. Generalized Eigenspace of a Linear Transformation

Let us consider K, V_1, f. The functor UnionKers f yielding a strict subspace of V_1 is defined by:

(Def. 5) The carrier of UnionKers $f = \{v; v$ ranges over vectors of V_1: $\bigvee_n f^n(v) = 0_{(V_1)}\}$.

We now state a number of propositions:

(24) $v_1 \in$ UnionKers f iff there exists n such that $f^n(v_1) = 0_{(V_1)}$.

(25) $\ker f^i$ is a subspace of UnionKers f.

(26) $\ker f^i$ is a subspace of $\ker f^{i+j}$.

(27) Let V be a finite dimensional vector space over K and f be a linear transformation from V to V. Then there exists n such that UnionKers $f = \ker f^n$.

(28) $f \restriction \ker f^n$ is a linear transformation from $\ker f^n$ to $\ker f^n$.

(29) $f \restriction \ker (f + L \cdot \mathrm{id}_{(V_1)})^n$ is a linear transformation from $\ker (f + L \cdot \mathrm{id}_{(V_1)})^n$ to $\ker (f + L \cdot \mathrm{id}_{(V_1)})^n$.

(30) $f \restriction$ UnionKers f is a linear transformation from UnionKers f to UnionKers f.

(31) $f \restriction$ UnionKers$(f + L \cdot \mathrm{id}_{(V_1)})$ is a linear transformation from UnionKers$(f + L \cdot \mathrm{id}_{(V_1)})$ to UnionKers$(f + L \cdot \mathrm{id}_{(V_1)})$.

(32) $f \restriction \mathrm{im}(f^n)$ is a linear transformation from $\mathrm{im}(f^n)$ to $\mathrm{im}(f^n)$.

(33) $f \restriction \mathrm{im}((f + L \cdot \mathrm{id}_{(V_1)})^n)$ is a linear transformation from $\mathrm{im}((f + L \cdot \mathrm{id}_{(V_1)})^n)$ to $\mathrm{im}((f + L \cdot \mathrm{id}_{(V_1)})^n)$.

(34) If UnionKers $f = \ker f^n$, then $\ker f^n \cap \mathrm{im}(f^n) = \mathbf{0}_{(V_1)}$.

(35) Let V be a finite dimensional vector space over K, f be a linear transformation from V to V, and given n. If $\mathrm{UnionKers}\, f = \ker f^n$, then V is the direct sum of $\ker f^n$ and $\mathrm{im}(f^n)$.

(36) For every linear complement I of $\mathrm{UnionKers}\, f$ holds $f{\restriction}I$ is one-to-one.

(37) Let I be a linear complement of $\mathrm{UnionKers}(f + (-L) \cdot \mathrm{id}_{(V_1)})$ and f_4 be a linear transformation from I to I. If $f_4 = f{\restriction}I$, then for every vector v of I such that $f_4(v) = L \cdot v$ holds $v = 0_{(V_1)}$.

(38) Suppose $n \geq 1$. Then there exists a linear transformation h from V_1 to V_1 such that $(f + L \cdot \mathrm{id}_{(V_1)})^n = f \cdot h + (L \cdot \mathrm{id}_{(V_1)})^n$ and for every i holds $f^i \cdot h = h \cdot f^i$.

(39) Let L_1, L_2 be scalars of K. Suppose f has eigenvalues and $L_1 \neq L_2$ and L_1 is an eigenvalue of f and L_2 is an eigenvalue of f. Let I be a linear complement of $\mathrm{UnionKers}(f + (-L_1) \cdot \mathrm{id}_{(V_1)})$ and f_4 be a linear transformation from I to I. Suppose $f_4 = f{\restriction}I$. Then f_4 has eigenvalues and L_1 is not an eigenvalue of f_4 and L_2 is an eigenvalue of f_4 and $\mathrm{UnionKers}(f + (-L_2) \cdot \mathrm{id}_{(V_1)})$ is a subspace of I.

(40) Let U be a finite subset of V_1. Suppose U is linearly independent. Let u be a vector of V_1. Suppose $u \in U$. Let L be a linear combination of $U \setminus \{u\}$. Then $\overline{\overline{U}} = \overline{\overline{(U \setminus \{u\}) \cup \{u + \sum L\}}}$ and $(U \setminus \{u\}) \cup \{u + \sum L\}$ is linearly independent.

(41) Let A be a subset of V_1, L be a linear combination of V_2, and f be a linear transformation from V_1 to V_2. Suppose the support of $L \subseteq f^\circ A$. Then there exists a linear combination M of A such that $f(\sum M) = \sum L$.

(42) Let f be a linear transformation from V_1 to V_2, A be a subset of V_1, and B be a subset of V_2. If $f^\circ A = B$, then f°(the carrier of $\mathrm{Lin}(A)$) = the carrier of $\mathrm{Lin}(B)$.

(43) Let L be a linear combination of V_1, F be a finite sequence of elements of V_1, and f be a linear transformation from V_1 to V_2. Suppose $f{\restriction}$((the support of L) $\cap \mathrm{rng}\, F$) is one-to-one and $\mathrm{rng}\, F \subseteq$ the support of L. Then there exists a linear combination L_3 of V_2 such that

(i) the support of $L_3 = f^\circ$((the support of L) $\cap \mathrm{rng}\, F$),

(ii) $f \cdot (L\, F) = L_3\, (f \cdot F)$, and

(iii) for every v_1 such that $v_1 \in$ (the support of L) $\cap \mathrm{rng}\, F$ holds $L(v_1) = L_3(f(v_1))$.

(44) Let A, B be subsets of V_1 and L be a linear combination of V_1. Suppose the support of $L \subseteq A \cup B$ and $\sum L = 0_{(V_1)}$. Let f be a linear function from V_1 into V_2. Suppose $f{\restriction}B$ is one-to-one and $f^\circ B$ is a linearly independent subset of V_2 and $f^\circ A \subseteq \{0_{(V_2)}\}$. Then the support of $L \subseteq A$.

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Jordan Matrix Decomposition

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Summary. In this paper I present the Jordan Matrix Decomposition Theorem which states that an arbitrary square matrix M over an algebraically closed field can be decomposed into the form

$$M = SJS^{-1}$$

where S is an invertible matrix and J is a matrix in a Jordan canonical form, i.e. a special type of block diagonal matrix in which each block consists of Jordan blocks (see [13]).

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The terminology and notation used here are introduced in the following articles: [11], [2], [3], [12], [34], [7], [10], [8], [4], [28], [33], [30], [18], [6], [14], [15], [36], [23], [37], [35], [9], [29], [32], [31], [5], [19], [24], [22], [17], [1], [21], [20], [16], [25], [27], and [26].

1. Jordan Blocks

We follow the rules: i, j, m, n, k denote natural numbers, K denotes a field, and a, λ denote elements of K.

Let us consider K, λ, n. The Jordan block of λ and n yields a matrix over K and is defined by the conditions (Def. 1).

(Def. 1)(i) len (the Jordan block of λ and n) $= n$,

(ii) width (the Jordan block of λ and n) $= n$, and

(iii) for all i, j such that $\langle i, j\rangle \in$ the indices of the Jordan block of λ and n holds if $i = j$, then (the Jordan block of λ and $n)_{i,j} = \lambda$ and if $i + 1 = j$, then (the Jordan block of λ and $n)_{i,j} = \mathbf{1}_K$ and if $i \neq j$ and $i + 1 \neq j$, then (the Jordan block of λ and $n)_{i,j} = 0_K$.

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Let us consider K, λ, n. Then the Jordan block of λ and n is an upper triangular matrix over K of dimension n.

The following propositions are true:

(1) The diagonal of the Jordan block of λ and $n = n \mapsto \lambda$.

(2) Det (the Jordan block of λ and n) $= \mathrm{power}_K(\lambda, n)$.

(3) The Jordan block of λ and n is invertible iff $n = 0$ or $\lambda \neq 0_K$.

(4) If $i \in \mathrm{Seg}\, n$ and $i \neq n$, then Line(the Jordan block of λ and n, i) $= \lambda \cdot \mathrm{Line}(I_K^{n \times n}, i) + \mathrm{Line}(I_K^{n \times n}, i+1)$.

(5) Line(the Jordan block of λ and n, n) $= \lambda \cdot \mathrm{Line}(I_K^{n \times n}, n)$.

(6) Let F be an element of (the carrier of K)^n such that $i \in \mathrm{Seg}\, n$. Then

(i) if $i \neq n$, then Line(the Jordan block of λ and n, i) $\cdot F = \lambda \cdot F_i + F_{i+1}$, and

(ii) if $i = n$, then Line(the Jordan block of λ and n, i) $\cdot F = \lambda \cdot F_i$.

(7) Let F be an element of (the carrier of K)^n such that $i \in \mathrm{Seg}\, n$. Then

(i) if $i = 1$, then (the Jordan block of λ and n)$_{\Box,i} \cdot F = \lambda \cdot F_i$, and

(ii) if $i \neq 1$, then (the Jordan block of λ and n)$_{\Box,i} \cdot F = \lambda \cdot F_i + F_{i-1}$.

(8) Suppose $\lambda \neq 0_K$. Then there exists a square matrix M over K of dimension n such that

(i) (the Jordan block of λ and n)$^\smile = M$, and

(ii) for all i, j such that $\langle i, j \rangle \in$ the indices of M holds if $i > j$, then $M_{i,j} = 0_K$ and if $i \leq j$, then $M_{i,j} = -\mathrm{power}_K(-\lambda^{-1}, (j -' i) + 1)$.

(9) (The Jordan block of λ and n) $+ a \cdot I_K^{n \times n} =$ the Jordan block of $\lambda + a$ and n.

2. Finite Sequences of Jordan Blocks

Let us consider K and let G be a finite sequence of elements of ((the carrier of K)*)*. We say that G is Jordan-block-yielding if and only if:

(Def. 2) For every i such that $i \in \mathrm{dom}\, G$ there exist λ, n such that $G(i) =$ the Jordan block of λ and n.

Let us consider K. Observe that there exists a finite sequence of elements of ((the carrier of K)*)* which is Jordan-block-yielding.

Let us consider K. One can verify that every finite sequence of elements of ((the carrier of K)*)* which is Jordan-block-yielding is also square-matrix-yielding.

Let us consider K. A finite sequence of Jordan blocks of K is a Jordan-block-yielding finite sequence of elements of ((the carrier of K)*)*.

Let us consider K, λ. A finite sequence of Jordan blocks of K is said to be a finite sequence of Jordan blocks of λ and K if:

(Def. 3) For every i such that $i \in \operatorname{dom} \mathrm{it}$ there exists n such that $\mathrm{it}(i) =$ the Jordan block of λ and n.

Next we state two propositions:

(10) $\emptyset$ is a finite sequence of Jordan blocks of λ and K.

(11) $\langle$the Jordan block of λ and $n\rangle$ is a finite sequence of Jordan blocks of λ and K.

Let us consider K, λ. Observe that there exists a finite sequence of Jordan blocks of λ and K which is non-empty.

Let us consider K. Note that there exists a finite sequence of Jordan blocks of K which is non-empty.

Next we state the proposition

(12) Let J be a finite sequence of Jordan blocks of λ and K. Then $J \oplus \operatorname{len} J \mapsto a \bullet I_K^{\operatorname{Len} J \times \operatorname{Len} J}$ is a finite sequence of Jordan blocks of $\lambda + a$ and K.

Let us consider K and let J_1, J_2 be fininte sequences of Jordan blocks of K. Then $J_1 \frown J_2$ is a finite sequence of Jordan blocks of K.

Let us consider K, let J be a finite sequence of Jordan blocks of K, and let us consider n. Then $J \restriction n$ is a finite sequence of Jordan blocks of K. Then $J_{\lfloor n}$ is a finite sequence of Jordan blocks of K.

Let us consider K, λ and let J_1, J_2 be finite sequences of Jordan blocks of λ and K. Then $J_1 \frown J_2$ is a finite sequence of Jordan blocks of λ and K.

Let us consider K, λ, let J be a finite sequence of Jordan blocks of λ and K, and let us consider n. Then $J \restriction n$ is a finite sequence of Jordan blocks of λ and K. Then $J_{\lfloor n}$ is a finite sequence of Jordan blocks of λ and K.

3. Nilpotent Transformations

Let K be a double loop structure, let V be a non empty vector space structure over K, and let f be a function from V into V. We say that f is nilpotent if and only if:

(Def. 4) There exists n such that for every vector v of V holds $f^n(v) = 0_V$.

We now state the proposition

(13) Let K be a double loop structure, V be a non empty vector space structure over K, and f be a function from V into V. Then f is nilpotent if and only if there exists n such that $f^n = \mathrm{ZeroMap}(V, V)$.

Let K be a double loop structure and let V be a non empty vector space structure over K. Observe that there exists a function from V into V which is nilpotent.

Let R be a ring and let V be a left module over R. Observe that there exists a function from V into V which is nilpotent and linear.

Next we state the proposition

(14) Let V be a vector space over K and f be a linear transformation from V to V. Then $f \restriction \ker f^n$ is a nilpotent linear transformation from $\ker f^n$ to $\ker f^n$.

Let K be a double loop structure, let V be a non empty vector space structure over K, and let f be a nilpotent function from V into V. The degree of nilpotence of f yielding a natural number is defined by the conditions (Def. 5).

(Def. 5)(i) $f^{\text{the degree of nilpotence of } f} = \mathrm{ZeroMap}(V, V)$, and

(ii) for every k such that $f^k = \mathrm{ZeroMap}(V, V)$ holds the degree of nilpotence of $f \leq k$.

Let K be a double loop structure, let V be a non empty vector space structure over K, and let f be a nilpotent function from V into V. We introduce $\deg f$ as a synonym of the degree of nilpotence of f.

One can prove the following propositions:

(15) Let K be a double loop structure, V be a non empty vector space structure over K, and f be a nilpotent function from V into V. Then $\deg f = 0$ if and only if $\Omega_V = \{0_V\}$.

(16) Let K be a double loop structure, V be a non empty vector space structure over K, and f be a nilpotent function from V into V. Then there exists a vector v of V such that for every i such that $i < \deg f$ holds $f^i(v) \neq 0_V$.

(17) Let K be a field, V be a vector space over K, W be a subspace of V, and f be a nilpotent function from V into V. Suppose $f \restriction W$ is a function from W into W. Then $f \restriction W$ is a nilpotent function from W into W.

(18) Let K be a field, V be a vector space over K, W be a subspace of V, f be a nilpotent linear transformation from V to V, and f_1 be a nilpotent function from $\mathrm{im}(f^n)$ into $\mathrm{im}(f^n)$. If $f_1 = f \restriction \mathrm{im}(f^n)$ and $n \leq \deg f$, then $\deg f_1 + n = \deg f$.

For simplicity, we adopt the following convention: V_1, V_2 denote finite dimensional vector spaces over K, W_1, W_2 denote subspaces of V_1, U_1, U_2 denote subspaces of V_2, b_1 denotes an ordered basis of V_1, B_1 denotes a finite sequence of elements of V_1, b_2 denotes an ordered basis of V_2, B_2 denotes a finite sequence of elements of V_2, b_3 denotes an ordered basis of W_1, b_4 denotes an ordered basis of W_2, B_3 denotes a finite sequence of elements of U_1, and B_4 denotes a finite sequence of elements of U_2.

Next we state a number of propositions:

(19) Let M be a matrix over K of dimension $\mathrm{len}\, b_1 \times \mathrm{len}\, B_2$, M_1 be a matrix over K of dimension $\mathrm{len}\, b_3 \times \mathrm{len}\, B_3$, and M_2 be a matrix over K of dimension $\mathrm{len}\, b_4 \times \mathrm{len}\, B_4$ such that $b_1 = b_3 \frown b_4$ and $B_2 = B_3 \frown B_4$ and $M =$ the 0_K-block diagonal of $\langle M_1, M_2 \rangle$ and $\mathrm{width}\, M_1 = \mathrm{len}\, B_3$ and $\mathrm{width}\, M_2 = \mathrm{len}\, B_4$. Then

(i) if $i \in \operatorname{dom} b_3$, then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_i) = (\mathrm{Mx2Tran}(M_1, b_3, B_3))((b_3)_i)$, and
(ii) if $i \in \operatorname{dom} b_4$, then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_{i+\operatorname{len} b_3}) = (\mathrm{Mx2Tran}(M_2, b_4, B_4))((b_4)_i)$.

(20) Let M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} B_2$ and F be a finite sequence of matrices over K. Suppose $M =$ the 0_K-block diagonal of F. Let given i, m. Suppose $i \in \operatorname{dom} b_1$ and $m = \min(\operatorname{Len} F, i)$. Then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \sum \mathrm{lmlt}(\mathrm{Line}(F(m), i -' \sum \mathrm{Len}(F{\restriction}(m -' 1))), (B_2{\restriction} \sum \mathrm{Width}(F{\restriction}m))_{\downharpoonleft \sum \mathrm{Width}(F{\restriction}(m-'1))})$ and $\operatorname{len}((B_2{\restriction} \sum \mathrm{Width}(F{\restriction}m))_{\downharpoonleft \sum \mathrm{Width}(F{\restriction}(m-'1))}) = \operatorname{width} F(m)$.

(21) If $\operatorname{len} B_1 \in \operatorname{dom} B_1$, then $\sum \mathrm{lmlt}(\mathrm{Line}(\text{the Jordan block of } \lambda \text{ and } \operatorname{len} B_1, \operatorname{len} B_1), B_1) = \lambda \cdot (B_1)_{\operatorname{len} B_1}$.

(22) If $i \in \operatorname{dom} B_1$ and $i \neq \operatorname{len} B_1$, then $\sum \mathrm{lmlt}(\mathrm{Line}(\text{the Jordan block of } \lambda \text{ and } \operatorname{len} B_1, i), B_1) = \lambda \cdot (B_1)_i + (B_1)_{i+1}$.

(23) Let M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} B_2$. Suppose $M =$ the Jordan block of λ and n. Let given i such that $i \in \operatorname{dom} b_1$. Then
(i) if $i = \operatorname{len} b_1$, then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \lambda \cdot (B_2)_i$, and
(ii) if $i \neq \operatorname{len} b_1$, then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \lambda \cdot (B_2)_i + (B_2)_{i+1}$.

(24) Let J be a finite sequence of Jordan blocks of λ and K and M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} B_2$. Suppose $M =$ the 0_K-block diagonal of J. Let given i, m such that $i \in \operatorname{dom} b_1$ and $m = \min(\operatorname{Len} J, i)$. Then
(i) if $i = \sum \mathrm{Len}(J{\restriction}m)$, then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \lambda \cdot (B_2)_i$, and
(ii) if $i \neq \sum \mathrm{Len}(J{\restriction}m)$, then $(\mathrm{Mx2Tran}(M, b_1, B_2))((b_1)_i) = \lambda \cdot (B_2)_i + (B_2)_{i+1}$.

(25) Let J be a finite sequence of Jordan blocks of 0_K and K and M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_1$. Suppose $M =$ the 0_K-block diagonal of J. Let given m. If for every i such that $i \in \operatorname{dom} J$ holds $\operatorname{len} J(i) \leq m$, then $(\mathrm{Mx2Tran}(M, b_1, b_1))^m = \mathrm{ZeroMap}(V_1, V_1)$.

(26) Let J be a finite sequence of Jordan blocks of λ and K and M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_1$. Suppose $M =$ the 0_K-block diagonal of J. Then $\mathrm{Mx2Tran}(M, b_1, b_1)$ is nilpotent if and only if $\operatorname{len} b_1 = 0$ or $\lambda = 0_K$.

(27) Let J be a finite sequence of Jordan blocks of 0_K and K and M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_1$. Suppose $M =$ the 0_K-block diagonal of J and $\operatorname{len} b_1 > 0$. Let F be a nilpotent function from V_1 into V_1. Suppose $F = \mathrm{Mx2Tran}(M, b_1, b_1)$. Then there exists i such that $i \in \operatorname{dom} J$ and $\operatorname{len} J(i) = \deg F$ and for every i such that $i \in \operatorname{dom} J$ holds $\operatorname{len} J(i) \leq \deg F$.

(28) Let given V_1, V_2, b_1, b_2, λ. Suppose $\operatorname{len} b_1 = \operatorname{len} b_2$. Let F be a linear

transformation from V_1 to V_2. Suppose that for every i such that $i \in \operatorname{dom} b_1$ holds $F((b_1)_i) = \lambda \cdot (b_2)_i$ or $i+1 \in \operatorname{dom} b_1$ and $F((b_1)_i) = \lambda \cdot (b_2)_i + (b_2)_{i+1}$. Then there exists a non-empty finite sequence J of Jordan blocks of λ and K such that $\operatorname{AutMt}(F, b_1, b_2) =$ the 0_K-block diagonal of J.

(29) Let V_1 be a finite dimensional vector space over K and F be a nilpotent linear transformation from V_1 to V_1. Then there exists a non-empty finite sequence J of Jordan blocks of 0_K and K and there exists an ordered basis b_1 of V_1 such that $\operatorname{AutMt}(F, b_1, b_1) =$ the 0_K-block diagonal of J.

(30) Let V be a vector space over K, F be a linear transformation from V to V, V_1 be a finite dimensional subspace of V, and F_1 be a linear transformation from V_1 to V_1. Suppose $V_1 = \ker (F + (-\lambda) \cdot \mathrm{id}_V)^n$ and $F \restriction V_1 = F_1$. Then there exists a non-empty finite sequence J of Jordan blocks of λ and K and there exists an ordered basis b_1 of V_1 such that $\operatorname{AutMt}(F_1, b_1, b_1) =$ the 0_K-block diagonal of J.

4. The Main Theorem

The following two propositions are true:

(31) Let K be an algebraic-closed field, V be a non trivial finite dimensional vector space over K, and F be a linear transformation from V to V. Then there exists a non-empty finite sequence J of Jordan blocks of K and there exists an ordered basis b_1 of V such that

(i) $\operatorname{AutMt}(F, b_1, b_1) =$ the 0_K-block diagonal of J, and

(ii) for every scalar λ of K holds λ is an eigenvalue of F iff there exists i such that $i \in \operatorname{dom} J$ and $J(i) =$ the Jordan block of λ and $\operatorname{len} J(i)$.

(32) Let K be an algebraic-closed field and M be a square matrix over K of dimension n. Then there exists a non-empty finite sequence J of Jordan blocks of K and there exists a square matrix P over K of dimension n such that $\sum \operatorname{Len} J = n$ and P is invertible and $M = P \cdot$ the 0_K-block diagonal of $J \cdot P^{\smile}$.

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Fatou's Lemma and the Lebesgue's Convergence Theorem

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Summary. In this article we prove the Fatou's Lemma and Lebesgue's Convergence Theorem [10].

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The articles [15], [1], [16], [14], [11], [5], [12], [2], [3], [4], [8], [9], [13], [6], [7], and [17] provide the terminology and notation for this paper.

1. Fatou's Lemma

For simplicity, we adopt the following rules: X denotes a non empty set, F denotes a sequence of partial functions from X into $\overline{\mathbb{R}}$ with the same dom, s_1, s_2, s_3 denote sequences of extended reals, x denotes an element of X, a, r denote extended real numbers, and n, m, k denote natural numbers.

We now state several propositions:

(1) If for every natural number n holds $s_2(n) \leq s_3(n)$, then $\inf \operatorname{rng} s_2 \leq \inf \operatorname{rng} s_3$.

(2) Suppose that for every natural number n holds $s_2(n) \leq s_3(n)$. Then

(i) (the inferior real sequence of s_2)$(k) \leq$ (the inferior real sequence of s_3)(k), and

(ii) (the superior real sequence of s_2)$(k) \leq$ (the superior real sequence of s_3)(k).

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(3) If for every natural number n holds $s_2(n) \leq s_3(n)$, then $\liminf s_2 \leq \liminf s_3$ and $\limsup s_2 \leq \limsup s_3$.

(4) If for every natural number n holds $s_1(n) \geq a$, then $\inf s_1 \geq a$.

(5) If for every natural number n holds $s_1(n) \leq a$, then $\sup s_1 \leq a$.

(6) For every element n of $\mathbb{N}$ such that $x \in \operatorname{dom} \inf(F \uparrow n)$ holds $(\inf(F \uparrow n))(x) = \inf((F\#x) \uparrow n)$.

In the sequel S is a σ-field of subsets of X, M is a σ-measure on S, and E is an element of S.

We now state the proposition

(7) Suppose $E = \operatorname{dom} F(0)$ and for every n holds $F(n)$ is non-negative and $F(n)$ is measurable on E. Then there exists a sequence I of extended reals such that for every n holds $I(n) = \int F(n)\, \mathrm{d}M$ and $\int \liminf F\, \mathrm{d}M \leq \liminf I$.

2. Lebesgue's Convergence Theorem

We now state three propositions:

(8) For all non empty subsets X, Y of $\overline{\mathbb{R}}$ and for every extended real number r such that $X = \{r\}$ and $r \in \mathbb{R}$ holds $\sup(X + Y) = \sup X + \sup Y$.

(9) For all non empty subsets X, Y of $\overline{\mathbb{R}}$ and for every extended real number r such that $X = \{r\}$ and $r \in \mathbb{R}$ holds $\inf(X + Y) = \inf X + \inf Y$.

(10) If $r \in \mathbb{R}$ and for every natural number n holds $s_2(n) = r + s_3(n)$, then $\liminf s_2 = r + \liminf s_3$ and $\limsup s_2 = r + \limsup s_3$.

We follow the rules: F_1, F_2 are sequences of partial functions from X into $\overline{\mathbb{R}}$ and f, g, P are partial functions from X to $\overline{\mathbb{R}}$.

We now state several propositions:

(11) Suppose that
 (i) $\operatorname{dom} F_1(0) = \operatorname{dom} F_2(0)$,
 (ii) F_1 has the same dom,
 (iii) F_2 has the same dom,
 (iv) $f^{-1}(\{+\infty\}) = \emptyset$,
 (v) $f^{-1}(\{-\infty\}) = \emptyset$, and
 (vi) for every natural number n holds $F_1(n) = f + F_2(n)$.

Then $\liminf F_1 = f + \liminf F_2$ and $\limsup F_1 = f + \limsup F_2$.

(12) $s_1 \uparrow 0 = s_1$.

(13) If f is integrable on M and g is integrable on M, then $f - g$ is integrable on M.

(14) Suppose f is integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int f - g\, \mathrm{d}M = \int f{\restriction}E\, \mathrm{d}M + \int(-g){\restriction}E\, \mathrm{d}M$.

(15) If for every natural number n holds $s_2(n) = -s_3(n)$, then $\liminf s_3 = -\limsup s_2$ and $\limsup s_3 = -\liminf s_2$.

(16) Suppose $\operatorname{dom} F_1(0) = \operatorname{dom} F_2(0)$ and F_1 has the same dom and F_2 has the same dom and for every natural number n holds $F_1(n) = -F_2(n)$. Then $\liminf F_1 = -\limsup F_2$ and $\limsup F_1 = -\liminf F_2$.

(17) Suppose that
- (i) $E = \operatorname{dom} F(0)$,
- (ii) $E = \operatorname{dom} P$,
- (iii) for every natural number n holds $F(n)$ is measurable on E,
- (iv) P is integrable on M,
- (v) P is non-negative, and
- (vi) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$.

Then
- (vii) for every natural number n holds $|F(n)|$ is integrable on M,
- (viii) $|\liminf F|$ is integrable on M, and
- (ix) $|\limsup F|$ is integrable on M.

(18) Suppose that
- (i) $E = \operatorname{dom} F(0)$,
- (ii) $E = \operatorname{dom} P$,
- (iii) for every natural number n holds $F(n)$ is measurable on E,
- (iv) P is integrable on M,
- (v) P is non-negative, and
- (vi) for every element x of X and for every natural number n such that $x \in E$ holds $|F(n)|(x) \leq P(x)$.

Then there exists a sequence I of extended reals such that
- (vii) for every natural number n holds $I(n) = \int F(n)\,\mathrm{d}M$,
- (viii) $\liminf I \geq \int \liminf F\,\mathrm{d}M$,
- (ix) $\limsup I \leq \int \limsup F\,\mathrm{d}M$, and
- (x) if for every element x of X such that $x \in E$ holds $F\#x$ is convergent, then I is convergent and $\lim I = \int \lim F\,\mathrm{d}M$.

(19) Suppose that
- (i) $E = \operatorname{dom} F(0)$,
- (ii) for every n holds $F(n)$ is non-negative and $F(n)$ is measurable on E,
- (iii) for all x, n, m such that $x \in E$ and $n \leq m$ holds $F(n)(x) \geq F(m)(x)$, and
- (iv) $\int F(0){\restriction}E\,\mathrm{d}M < +\infty$.

Then there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n)\,\mathrm{d}M$ and I is convergent and $\lim I = \int \lim F\,\mathrm{d}M$.

Let X be a set and let F be a sequence of partial functions from X into $\overline{\mathbb{R}}$. We say that F is uniformly bounded if and only if:

(Def. 1) There exists a real number K such that for every natural number n and for every set x such that $x \in \mathrm{dom}\, F(0)$ holds $|F(n)(x)| \leq K$.

Next we state the proposition

(20) Suppose that

(i) $M(E) < +\infty$,

(ii) $E = \mathrm{dom}\, F(0)$,

(iii) for every natural number n holds $F(n)$ is measurable on E,

(iv) F is uniformly bounded, and

(v) for every element x of X such that $x \in E$ holds $F\#x$ is convergent.

Then

(vi) for every natural number n holds $F(n)$ is integrable on M,

(vii) $\lim F$ is integrable on M, and

(viii) there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n)\,\mathrm{d}M$ and I is convergent and $\lim I = \int \lim F\,\mathrm{d}M$.

Let X be a set, let F be a sequence of partial functions from X into $\overline{\mathbb{R}}$, and let f be a partial function from X to $\overline{\mathbb{R}}$. We say that F is uniformly convergent to f if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) F has the same dom,

(ii) $\mathrm{dom}\, F(0) = \mathrm{dom}\, f$, and

(iii) for every real number e such that $e > 0$ there exists a natural number N such that for every natural number n and for every set x such that $n \geq N$ and $x \in \mathrm{dom}\, F(0)$ holds $|F(n)(x) - f(x)| < e$.

One can prove the following two propositions:

(21) Suppose F_1 is uniformly convergent to f. Let x be an element of X. If $x \in \mathrm{dom}\, F_1(0)$, then $F_1\#x$ is convergent and $\lim(F_1\#x) = f(x)$.

(22) Suppose that

(i) $M(E) < +\infty$,

(ii) $E = \mathrm{dom}\, F(0)$,

(iii) for every natural number n holds $F(n)$ is integrable on M, and

(iv) F is uniformly convergent to f.

Then

(v) f is integrable on M, and

(vi) there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n)\,\mathrm{d}M$ and I is convergent and $\lim I = \int f\,\mathrm{d}M$.

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Extended Riemann Integral of Functions of Real Variable and One-sided Laplace Transform[1]

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Summary. In this article, we defined a variety of extended Riemann integrals and proved that such integration is linear. Furthermore, we defined the one-sided Laplace transform and proved the linearity of that operator.

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The papers [11], [1], [5], [12], [10], [2], [7], [6], [8], [9], [3], [4], and [13] provide the terminology and notation for this paper.

1. Preliminaries

In this paper a, b, r are elements of $\mathbb{R}$.

We now state three propositions:

(1) For all real numbers a, b, g_1, M such that $a < b$ and $0 < g_1$ and $0 < M$ there exists r such that $a < r < b$ and $(b - r) \cdot M < g_1$.

(2) For all real numbers a, b, g_1, M such that $a < b$ and $0 < g_1$ and $0 < M$ there exists r such that $a < r < b$ and $(r - a) \cdot M < g_1$.

(3) $\exp b - \exp a = \int\limits_a^b (\text{the function } \exp)(x)dx$.

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2. The Definition of Extended Riemann Integral

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let a, b be real numbers. We say that f is right extended Riemann integrable on a, b if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) For every real number d such that $a \leq d < b$ holds f is integrable on $[a, d]$ and $f \restriction [a, d]$ is bounded, and

(ii) there exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathrm{dom}\,\mathcal{I} = [a, b[$ and for every real number x such that $x \in \mathrm{dom}\,\mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_a^x f(x)dx$ and $\mathcal{I}$ is left convergent in b.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let a, b be real numbers. We say that f is left extended Riemann integrable on a, b if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) For every real number d such that $a < d \leq b$ holds f is integrable on $[d, b]$ and $f \restriction [d, b]$ is bounded, and

(ii) there exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathrm{dom}\,\mathcal{I} =]a, b]$ and for every real number x such that $x \in \mathrm{dom}\,\mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_x^b f(x)dx$ and $\mathcal{I}$ is right convergent in a.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let a, b be real numbers. Let us assume that f is right extended Riemann integrable on a, b. The functor $(R^{>})\int\limits_a^b f(x)dx$ yielding a real number is defined by the condition (Def. 3).

(Def. 3) There exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathrm{dom}\,\mathcal{I} = [a, b[$ and for every real number x such that $x \in \mathrm{dom}\,\mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_a^x f(x)dx$ and $\mathcal{I}$ is left convergent in b and $(R^{>})\int\limits_a^b f(x)dx = \lim_{b^-} \mathcal{I}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let a, b be real numbers. Let us assume that f is left extended Riemann integrable on a, b. The functor $(R^{<})\int\limits_a^b f(x)dx$ yielding a real number is defined by the condition (Def. 4).

(Def. 4) There exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathrm{dom}\,\mathcal{I} =]a, b]$ and for every real number x such that $x \in \mathrm{dom}\,\mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_x^b f(x)dx$

and $\mathcal{I}$ is right convergent in a and $(R^{<})\int\limits_{a}^{b} f(x)dx = \lim\limits_{a^{+}} \mathcal{I}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let a be a real number. We say that f is extended Riemann integrable on a, $+\infty$ if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) For every real number b such that $a \leq b$ holds f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded, and

(ii) there exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} \mathcal{I} = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} \mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_{a}^{x} f(x)dx$ and $\mathcal{I}$ is convergent in $+\infty$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let b be a real number. We say that f is extended Riemann integrable on $-\infty$, b if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) For every real number a such that $a \leq b$ holds f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded, and

(ii) there exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} \mathcal{I} =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} \mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_{x}^{b} f(x)dx$ and $\mathcal{I}$ is convergent in $-\infty$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let a be a real number. Let us assume that f is extended Riemann integrable on a, $+\infty$. The functor $(R^{>})\int\limits_{a}^{+\infty} f(x)dx$ yielding a real number is defined by the condition (Def. 7).

(Def. 7) There exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} \mathcal{I} = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} \mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_{a}^{x} f(x)dx$ and $\mathcal{I}$ is convergent in $+\infty$ and $(R^{>})\int\limits_{a}^{+\infty} f(x)dx = \lim\limits_{+\infty} \mathcal{I}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let b be a real number. Let us assume that f is extended Riemann integrable on $-\infty$, b. The functor $(R^{<})\int\limits_{-\infty}^{b} f(x)dx$ yields a real number and is defined by the condition (Def. 8).

(Def. 8) There exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} \mathcal{I} =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} \mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_{x}^{b} f(x)dx$

and $\mathcal{I}$ is convergent in $-\infty$ and $(R^{<})\int\limits_{-\infty}^{b} f(x)dx = \lim\limits_{-\infty}\mathcal{I}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. We say that f is ∞-extended Riemann integrable if and only if:

(Def. 9) f is extended Riemann integrable on 0, $+\infty$ and extended Riemann integrable on $-\infty$, 0.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $(R)\int\limits_{-\infty}^{+\infty} f(x)dx$ yields a real number and is defined by:

(Def. 10) $$(R)\int\limits_{-\infty}^{+\infty} f(x)dx = (R^{>})\int\limits_{0}^{+\infty} f(x)dx + (R^{<})\int\limits_{-\infty}^{0} f(x)dx.$$

3. Linearity of Extended Riemann Integral

One can prove the following propositions:

(4) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. Suppose that

(i) $a < b$,

(ii) $[a, b] \subseteq \operatorname{dom} f$,

(iii) $[a, b] \subseteq \operatorname{dom} g$,

(iv) f is right extended Riemann integrable on a, b, and

(v) g is right extended Riemann integrable on a, b.

Then $f + g$ is right extended Riemann integrable on a, b and $(R^{>})\int\limits_{a}^{b}(f+g)(x)dx = (R^{>})\int\limits_{a}^{b} f(x)dx + (R^{>})\int\limits_{a}^{b} g(x)dx$.

(5) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b. Let r be a real number. Then $r\, f$ is right extended Riemann integrable on a, b and $(R^{>})\int\limits_{a}^{b}(r\, f)(x)dx = r \cdot (R^{>})\int\limits_{a}^{b} f(x)dx$.

(6) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. Suppose that

(i) $a < b$,

(ii) $[a, b] \subseteq \operatorname{dom} f$,

(iii) $[a, b] \subseteq \operatorname{dom} g$,

(iv) f is left extended Riemann integrable on a, b, and

(v) g is left extended Riemann integrable on a, b.

Then $f+g$ is left extended Riemann integrable on a, b and $(R^{<})\int\limits_{a}^{b}(f+g)(x)dx = (R^{<})\int\limits_{a}^{b} f(x)dx + (R^{<})\int\limits_{a}^{b} g(x)dx$.

(7) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b. Let r be a real number. Then $r\,f$ is left extended Riemann integrable on a, b and $(R^{<})\int\limits_{a}^{b}(r\,f)(x)dx = r\cdot(R^{<})\int\limits_{a}^{b} f(x)dx$.

(8) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and a be a real number. Suppose that

(i) $[a, +\infty[\ \subseteq \operatorname{dom} f$,

(ii) $[a, +\infty[\ \subseteq \operatorname{dom} g$,

(iii) f is extended Riemann integrable on a, $+\infty$, and

(iv) g is extended Riemann integrable on a, $+\infty$.

Then $f+g$ is extended Riemann integrable on a, $+\infty$ and $(R^{>})\int\limits_{a}^{+\infty}(f+g)(x)dx = (R^{>})\int\limits_{a}^{+\infty} f(x)dx + (R^{>})\int\limits_{a}^{+\infty} g(x)dx$.

(9) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a be a real number. Suppose $[a, +\infty[\ \subseteq \operatorname{dom} f$ and f is extended Riemann integrable on a, $+\infty$. Let r be a real number. Then $r\,f$ is extended Riemann integrable on a, $+\infty$ and $(R^{>})\int\limits_{a}^{+\infty}(r\,f)(x)dx = r\cdot(R^{>})\int\limits_{a}^{+\infty} f(x)dx$.

(10) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and b be a real number. Suppose that

(i) $]-\infty, b] \subseteq \operatorname{dom} f$,

(ii) $]-\infty, b] \subseteq \operatorname{dom} g$,

(iii) f is extended Riemann integrable on $-\infty$, b, and

(iv) g is extended Riemann integrable on $-\infty$, b.

Then $f+g$ is extended Riemann integrable on $-\infty$, b and $(R^{<})\int\limits_{-\infty}^{b}(f+g)(x)dx = (R^{<})\int\limits_{-\infty}^{b} f(x)dx + (R^{<})\int\limits_{-\infty}^{b} g(x)dx$.

(11) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and b be a real number. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and f is extended Riemann integrable on $-\infty$, b. Let r be a real number. Then $r\,f$ is extended Riemann integrable on $-\infty$, b and $(R^{<})\int\limits_{-\infty}^{b}(r\,f)(x)dx = r\cdot(R^{<})\int\limits_{-\infty}^{b} f(x)dx$.

(12) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers.

Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded. Then $(R^{>}) \int\limits_{a}^{b} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

(13) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded. Then $(R^{<}) \int\limits_{a}^{b} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

4. The Definition of One-sided Laplace Transform

Let s be a real number. The functor $e^{-s \cdot \square}$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by:

(Def. 11) For every real number t holds $e^{-s \cdot \square}(t) = (\text{the function exp})(-s \cdot t)$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The one-sided Laplace transform of f yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by the conditions (Def. 12).

(Def. 12)(i) dom (the one-sided Laplace transform of f) $=]0, +\infty[$, and

(ii) for every real number s such that $s \in$ dom (the one-sided Laplace transform of f) holds (the one-sided Laplace transform of f)(s) $=$ $(R^{>}) \int\limits_{0}^{+\infty} (f\, e^{-s \cdot \square})(x)dx$.

5. Linearity of One-sided Laplace Transform

Next we state two propositions:

(14) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that

(i) $\operatorname{dom} f = [0, +\infty[$,

(ii) $\operatorname{dom} g = [0, +\infty[$,

(iii) for every real number s such that $s \in]0, +\infty[$ holds $f\, e^{-s \cdot \square}$ is extended Riemann integrable on 0, $+\infty$, and

(iv) for every real number s such that $s \in]0, +\infty[$ holds $g\, e^{-s \cdot \square}$ is extended Riemann integrable on 0, $+\infty$.

Then

(v) for every real number s such that $s \in]0, +\infty[$ holds $(f + g)\, e^{-s \cdot \square}$ is extended Riemann integrable on 0, $+\infty$, and

(vi) the one-sided Laplace transform of $f + g$ $=$ (the one-sided Laplace transform of f) $+$ (the one-sided Laplace transform of g).

(15) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a be a real number. Suppose $\operatorname{dom} f = [0, +\infty[$ and for every real number s such that $s \in]0, +\infty[$ holds $f\, e^{-s \cdot \square}$ is extended Riemann integrable on 0, $+\infty$. Then

(i) for every real number s such that $s \in]0, +\infty[$ holds $a\, f\, e^{-s \cdot \square}$ is extended Riemann integrable on 0, $+\infty$, and

(ii) the one-sided Laplace transform of $a\, f = a$ the one-sided Laplace transform of f.

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Integral of Complex-Valued Measurable Function

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Summary. In this article, we formalized the notion of the integral of a complex-valued function considered as a sum of its real and imaginary parts. Then we defined the measurability and integrability in this context, and proved the linearity and several other basic properties of complex-valued measurable functions. The set of properties showed in this paper is based on [15], where the case of real-valued measurable functions is considered.

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The notation and terminology used here are introduced in the following papers: [17], [1], [11], [18], [6], [19], [7], [2], [12], [14], [16], [5], [4], [3], [9], [10], [13], [8], and [15].

1. Definitions for Complex-valued Functions

One can prove the following proposition

(1) For all real numbers a, b holds $\overline{\mathbb{R}}(a)+\overline{\mathbb{R}}(b) = a+b$ and $-\overline{\mathbb{R}}(a) = -a$ and $\overline{\mathbb{R}}(a) - \overline{\mathbb{R}}(b) = a - b$ and $\overline{\mathbb{R}}(a) \cdot \overline{\mathbb{R}}(b) = a \cdot b$.

Let X be a non empty set and let f be a partial function from X to $\mathbb{C}$. The functor $\Re(f)$ yields a partial function from X to $\mathbb{R}$ and is defined as follows:

(Def. 1) $\operatorname{dom} \Re(f) = \operatorname{dom} f$ and for every element x of X such that $x \in \operatorname{dom} \Re(f)$ holds $\Re(f)(x) = \Re(f(x))$.

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The functor $\Im(f)$ yields a partial function from X to $\mathbb{R}$ and is defined as follows:

(Def. 2) $\operatorname{dom}\Im(f) = \operatorname{dom} f$ and for every element x of X such that $x \in \operatorname{dom}\Im(f)$ holds $\Im(f)(x) = \Im(f(x))$.

2. The Measurability of Complex-valued Functions

For simplicity, we use the following convention: X is a non empty set, Y is a set, S is a σ-field of subsets of X, M is a σ-measure on S, f, g are partial functions from X to $\mathbb{C}$, r is a real number, c is a complex number, and E, A, B are elements of S.

Let X be a non empty set, let S be a σ-field of subsets of X, let f be a partial function from X to $\mathbb{C}$, and let E be an element of S. We say that f is measurable on E if and only if:

(Def. 3) $\Re(f)$ is measurable on E and $\Im(f)$ is measurable on E.

One can prove the following propositions:

(2) $r\,\Re(f) = \Re(r\,f)$ and $r\,\Im(f) = \Im(r\,f)$.

(3) $\Re(c\,f) = \Re(c)\,\Re(f) - \Im(c)\,\Im(f)$ and $\Im(c\,f) = \Im(c)\,\Re(f) + \Re(c)\,\Im(f)$.

(4) $-\Im(f) = \Re(i\,f)$ and $\Re(f) = \Im(i\,f)$.

(5) $\Re(f+g) = \Re(f) + \Re(g)$ and $\Im(f+g) = \Im(f) + \Im(g)$.

(6) $\Re(f-g) = \Re(f) - \Re(g)$ and $\Im(f-g) = \Im(f) - \Im(g)$.

(7) $\Re(f)\restriction A = \Re(f\restriction A)$ and $\Im(f)\restriction A = \Im(f\restriction A)$.

(8) $f = \Re(f) + i\,\Im(f)$.

(9) If $B \subseteq A$ and f is measurable on A, then f is measurable on B.

(10) If f is measurable on A and f is measurable on B, then f is measurable on $A \cup B$.

(11) If f is measurable on A and g is measurable on A, then $f+g$ is measurable on A.

(12) If f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom} g$, then $f - g$ is measurable on A.

(13) If $Y \subseteq \operatorname{dom}(f+g)$, then $\operatorname{dom}(f\restriction Y + g\restriction Y) = Y$ and $(f+g)\restriction Y = f\restriction Y + g\restriction Y$.

(14) If f is measurable on B and $A = \operatorname{dom} f \cap B$, then $f\restriction B$ is measurable on A.

(15) If $\operatorname{dom} f, \operatorname{dom} g \in S$, then $\operatorname{dom}(f+g) \in S$.

(16) If $\operatorname{dom} f = A$, then f is measurable on B iff f is measurable on $A \cap B$.

(17) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $c\,f$ is measurable on A.

(18) Given an element A of S such that $\operatorname{dom} f = A$. Let c be a complex number and B be an element of S. If f is measurable on B, then $c\,f$ is measurable on B.

3. The Integral of a Complex-valued Measurable Function

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{C}$. We say that f is integrable on M if and only if:

(Def. 4) $\Re(f)$ is integrable on M and $\Im(f)$ is integrable on M.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{C}$. Let us assume that f is integrable on M. The functor $\int f \, \mathrm{d}M$ yielding a complex number is defined by:

(Def. 5) There exist real numbers R, I such that $R = \int \Re(f) \, \mathrm{d}M$ and $I = \int \Im(f) \, \mathrm{d}M$ and $\int f \, \mathrm{d}M = R + I \cdot i$.

We now state several propositions:

(19) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\overline{\mathbb{R}}$, and A be an element of S. Suppose there exists an element E of S such that $E = \mathrm{dom}\, f$ and f is measurable on E and $M(A) = 0$. Then $f{\restriction}A$ is integrable on M.

(20) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, f be a partial function from X to $\mathbb{R}$, and E, A be elements of S. Suppose there exists an element E of S such that $E = \mathrm{dom}\, f$ and f is measurable on E and $M(A) = 0$. Then $f{\restriction}A$ is integrable on M.

(21) Suppose there exists an element E of S such that $E = \mathrm{dom}\, f$ and f is measurable on E and $M(A) = 0$. Then $f{\restriction}A$ is integrable on M and $\int f{\restriction}A \, \mathrm{d}M = 0$.

(22) If $E = \mathrm{dom}\, f$ and f is integrable on M and $M(A) = 0$, then $\int f{\restriction}(E \setminus A) \, \mathrm{d}M = \int f \, \mathrm{d}M$.

(23) If f is integrable on M, then $f{\restriction}A$ is integrable on M.

(24) If f is integrable on M and A misses B, then $\int f{\restriction}(A \cup B) \, \mathrm{d}M = \int f{\restriction}A \, \mathrm{d}M + \int f{\restriction}B \, \mathrm{d}M$.

(25) If f is integrable on M and $B = \mathrm{dom}\, f \setminus A$, then $f{\restriction}A$ is integrable on M and $\int f \, \mathrm{d}M = \int f{\restriction}A \, \mathrm{d}M + \int f{\restriction}B \, \mathrm{d}M$.

Let k be a real number, let X be a non empty set, and let f be a partial function from X to $\mathbb{R}$. The functor f^k yields a partial function from X to $\mathbb{R}$ and is defined as follows:

(Def. 6) $\mathrm{dom}(f^k) = \mathrm{dom}\, f$ and for every element x of X such that $x \in \mathrm{dom}(f^k)$ holds $f^k(x) = f(x)^k$.

Let us consider X. Observe that there exists a partial function from X to $\mathbb{R}$ which is non-negative.

Let k be a non negative real number, let us consider X, and let f be a non-negative partial function from X to $\mathbb{R}$. Observe that f^k is non-negative.

We now state a number of propositions:

(26) Let k be a real number, given X, S, E, and f be a partial function from X to $\mathbb{R}$. If f is non-negative and $0 \leq k$, then f^k is non-negative.

(27) Let x be a set, given X, S, E, and f be a partial function from X to $\mathbb{R}$. If f is non-negative, then $f(x)^{\frac{1}{2}} = \sqrt{f(x)}$.

(28) For every partial function f from X to $\mathbb{R}$ and for every real number a such that $A \subseteq \operatorname{dom} f$ holds $A \cap \text{LE-dom}(f, a) = A \setminus A \cap \text{GTE-dom}(f, a)$.

(29) Let k be a real number, given X, S, E, and f be a partial function from X to $\mathbb{R}$. Suppose f is non-negative and $0 \leq k$ and $E \subseteq \operatorname{dom} f$ and f is measurable on E. Then f^k is measurable on E.

(30) If f is measurable on A and $A \subseteq \operatorname{dom} f$, then $|f|$ is measurable on A.

(31) Given an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A. Then f is integrable on M if and only if $|f|$ is integrable on M.

(32) If f is integrable on M and g is integrable on M, then $\operatorname{dom}(f + g) \in S$.

(33) If f is integrable on M and g is integrable on M, then $f + g$ is integrable on M.

(34) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\mathbb{R}$. Suppose f is integrable on M and g is integrable on M. Then $f - g$ is integrable on M.

(35) If f is integrable on M and g is integrable on M, then $f - g$ is integrable on M.

(36) Suppose f is integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int f + g \, \mathrm{d}M = \int f {\restriction} E \, \mathrm{d}M + \int g {\restriction} E \, \mathrm{d}M$.

(37) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f, g be partial functions from X to $\mathbb{R}$. Suppose f is integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int f - g \, \mathrm{d}M = \int f {\restriction} E \, \mathrm{d}M + \int (-g) {\restriction} E \, \mathrm{d}M$.

(38) If f is integrable on M, then $r\, f$ is integrable on M and $\int r\, f \, \mathrm{d}M = r \cdot \int f \, \mathrm{d}M$.

(39) If f is integrable on M, then $i\, f$ is integrable on M and $\int i\, f \, \mathrm{d}M = i \cdot \int f \, \mathrm{d}M$.

(40) If f is integrable on M, then $c\, f$ is integrable on M and $\int c\, f \, \mathrm{d}M = c \cdot \int f \, \mathrm{d}M$.

(41) For every partial function f from X to $\mathbb{R}$ and for all Y, r holds $(r\, f){\restriction}Y = r\,(f{\restriction}Y)$.

(42) Let f, g be partial functions from X to $\mathbb{R}$. Suppose that

(i) there exists an element A of S such that $A = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and f is measurable on A and g is measurable on A,

(ii) f is integrable on M,

(iii) g is integrable on M, and

(iv) $g - f$ is non-negative.

Then there exists an element E of S such that $E = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and $\int f{\restriction}E \,\mathrm{d}M \leq \int g{\restriction}E \,\mathrm{d}M$.

(43) Suppose there exists an element A of S such that $A = \mathrm{dom}\, f$ and f is measurable on A and f is integrable on M. Then $|\int f \,\mathrm{d}M| \leq \int |f| \,\mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, let f be a partial function from X to $\mathbb{C}$, and let B be an element of S. The functor $\int\limits_B f \,\mathrm{d}M$ yields a complex number and is defined by:

(Def. 7) $\int\limits_B f \,\mathrm{d}M = \int f{\restriction}B \,\mathrm{d}M$.

Next we state two propositions:

(44) Suppose f is integrable on M and g is integrable on M and $B \subseteq \mathrm{dom}(f + g)$. Then $f + g$ is integrable on M and $\int\limits_B f + g \,\mathrm{d}M = \int\limits_B f \,\mathrm{d}M + \int\limits_B g \,\mathrm{d}M$.

(45) If f is integrable on M and f is measurable on B, then $\int\limits_B c\, f \,\mathrm{d}M = c \cdot \int\limits_B f \,\mathrm{d}M$.

4. Several Properties of Real-valued Measurable Functions

In the sequel f denotes a partial function from X to $\mathbb{R}$ and a denotes a real number.

One can prove the following four propositions:

(46) If $A \subseteq \mathrm{dom}\, f$, then $A \cap \text{GTE-dom}(f, a) = A \setminus A \cap \text{LE-dom}(f, a)$.

(47) If $A \subseteq \mathrm{dom}\, f$, then $A \cap \text{GT-dom}(f, a) = A \setminus A \cap \text{LEQ-dom}(f, a)$.

(48) If $A \subseteq \mathrm{dom}\, f$, then $A \cap \text{LEQ-dom}(f, a) = A \setminus A \cap \text{GT-dom}(f, a)$.

(49) $A \cap \text{EQ-dom}(f, a) = A \cap \text{GTE-dom}(f, a) \cap \text{LEQ-dom}(f, a)$.

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Introduction to Matroids[1]

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Summary. The paper includes elements of the theory of matroids [23]. The formalization is done according to [12].

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The articles [7], [22], [17], [15], [8], [5], [6], [19], [9], [3], [2], [4], [1], [21], [11], [20], [18], [16], [10], [13], and [14] provide the terminology and notation for this paper.

1. Definition by Independent Sets

A subset family structure is a topological structure.

Let M be a subset family structure and let A be a subset of M. We introduce A is independent as a synonym of A is open. We introduce A is dependent as an antonym of A is open.

Let M be a subset family structure. The family of M yielding a family of subsets of M is defined as follows:

(Def. 1) The family of $M =$ the topology of M.

Let M be a subset family structure and let A be a subset of M. Let us observe that A is independent if and only if:

(Def. 2) $A \in$ the family of M.

Let M be a subset family structure. We say that M is subset-closed if and only if:

(Def. 3) The family of M is subset-closed.

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We say that M has exchange property if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let A, B be finite subsets of M. Suppose $A \in$ the family of M and $B \in$ the family of M and $\operatorname{card} B = \operatorname{card} A + 1$. Then there exists an element e of M such that $e \in B \setminus A$ and $A \cup \{e\} \in$ the family of M.

One can check that there exists a subset family structure which is strict, non empty, non void, finite, and subset-closed and has exchange property.

Let M be a non void subset family structure. One can verify that there exists a subset of M which is independent.

Let M be a subset-closed subset family structure. One can verify that the family of M is subset-closed.

We now state the proposition

(1) Let M be a non void subset-closed subset family structure, A be an independent subset of M, and B be a set. If $B \subseteq A$, then B is an independent subset of M.

Let M be a non void subset-closed subset family structure. Note that there exists a subset of M which is finite and independent.

A matroid is a non empty non void subset-closed subset family structure with exchange property.

One can prove the following proposition

(2) For every subset-closed subset family structure M holds M is non void iff $\emptyset \in$ the family of M.

Let M be a non void subset-closed subset family structure. Note that $\emptyset_{\text{the carrier of } M}$ is independent.

The following proposition is true

(3) Let M be a non void subset family structure. Then M is subset-closed if and only if for all subsets A, B of M such that A is independent and $B \subseteq A$ holds B is independent.

Let M be a non void subset-closed subset family structure, let A be an independent subset of M, and let B be a set. One can check the following observations:

* $A \cap B$ is independent,
* $B \cap A$ is independent, and
* $A \setminus B$ is independent.

Next we state the proposition

(4) Let M be a non void non empty subset family structure. Then M has exchange property if and only if for all finite subsets A, B of M such that A is independent and B is independent and $\operatorname{card} B = \operatorname{card} A + 1$ there exists an element e of M such that $e \in B \setminus A$ and $A \cup \{e\}$ is independent.

Let A be a set. We introduce A is finite-membered as a synonym of A has finite elements.

Let A be a set. Let us observe that A is finite-membered if and only if:

(Def. 5) For every set B such that $B \in A$ holds B is finite.

Let M be a subset family structure. We say that M is finite-membered if and only if:

(Def. 6) The family of M is finite-membered.

Let M be a subset family structure. We say that M is finite-degree if and only if the conditions (Def. 7) are satisfied.

(Def. 7)(i) M is finite-membered, and

(ii) there exists a natural number n such that for every finite subset A of M such that A is independent holds $\operatorname{card} A \leq n$.

Let us note that every subset family structure which is finite-degree is also finite-membered and every subset family structure which is finite is also finite-degree.

2. Examples

Let us note that there exists a set which is mutually-disjoint and non empty and has non empty elements.

The following propositions are true:

(5) For all finite sets A, B such that $\operatorname{card} A < \operatorname{card} B$ there exists a set x such that $x \in B \setminus A$.

(6) For every mutually-disjoint non empty set P with non empty elements holds every choice function of P is one-to-one.

Let us mention that every discrete subset family structure is non void and subset-closed and has exchange property.

Next we state the proposition

(7) Every non empty discrete topological structure is a matroid.

Let P be a set. The functor $\operatorname{ProdMatroid} P$ yields a strict subset family structure and is defined by the conditions (Def. 8).

(Def. 8)(i) The carrier of $\operatorname{ProdMatroid} P = \bigcup P$, and

(ii) the family of $\operatorname{ProdMatroid} P = \{A \subseteq \bigcup P : \bigwedge_{D:\mathrm{set}} (D \in P \Rightarrow \bigvee_{d:\mathrm{set}} A \cap D \subseteq \{d\})\}$.

Let P be a non empty set with non empty elements. One can verify that $\operatorname{ProdMatroid} P$ is non empty.

Next we state the proposition

(8) Let P be a set and A be a subset of $\operatorname{ProdMatroid} P$. Then A is independent if and only if for every element D of P there exists an element d of D such that $A \cap D \subseteq \{d\}$.

Let P be a set. One can verify that ProdMatroid P is non void and subset-closed.

Next we state two propositions:

(9) Let P be a mutually-disjoint set and x be a subset of ProdMatroid P. Then there exists a function f from x into P such that for every set a such that $a \in x$ holds $a \in f(a)$.

(10) Let P be a mutually-disjoint set, x be a subset of ProdMatroid P, and f be a function from x into P. Suppose that for every set a such that $a \in x$ holds $a \in f(a)$. Then x is independent if and only if f is one-to-one.

Let P be a mutually-disjoint set. Observe that ProdMatroid P has exchange property.

Let X be a finite set and let P be a subset of 2^X. One can check that ProdMatroid P is finite.

Let X be a set. Observe that every partition of X is mutually-disjoint.

One can check that there exists a matroid which is finite and strict.

Let M be a finite-membered non void subset family structure. Observe that every independent subset of M is finite.

Let F be a field and let V be a vector space over F. The matroid of linearly independent subsets of V is a strict subset family structure and is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of the matroid of linearly independent subsets of V = the carrier of V, and

(ii) the family of the matroid of linearly independent subsets of $V = \{A \subseteq V: A$ is linearly independent$\}$.

Let F be a field and let V be a vector space over F. Note that the matroid of linearly independent subsets of V is non empty, non void, and subset-closed.

Let F be a field and let V be a vector space over F. Observe that there exists a subset of V which is linearly independent and empty.

The following three propositions are true:

(11) Let F be a field, V be a vector space over F, and A be a subset of the matroid of linearly independent subsets of V. Then A is independent if and only if A is a linearly independent subset of V.

(12) Let F be a field, V be a vector space over F, and A, B be finite subsets of V. Suppose $B \subseteq A$. Let v be a vector of V. Suppose $v \in \mathrm{Lin}(A)$ and $v \notin \mathrm{Lin}(B)$. Then there exists a vector w of V such that $w \in A \setminus B$ and $w \in \mathrm{Lin}((A \setminus \{w\}) \cup \{v\})$.

(13) Let F be a field, V be a vector space over F, and A be a subset of V. Suppose A is linearly independent. Let a be an element of V. If $a \notin$ the carrier of $\mathrm{Lin}(A)$, then $A \cup \{a\}$ is linearly independent.

Let F be a field and let V be a vector space over F. Observe that the matroid of linearly independent subsets of V has exchange property.

Let F be a field and let V be a finite dimensional vector space over F. Note that the matroid of linearly independent subsets of V is finite-membered.

3. Maximal Independent Subsets, Ranks, and Basis

Let M be a subset family structure and let A, C be subsets of M. We say that A is maximal independent in C if and only if:

(Def. 10) A is independent and $A \subseteq C$ and for every subset B of M such that B is independent and $B \subseteq C$ and $A \subseteq B$ holds $A = B$.

The following propositions are true:

(14) Let M be a non void finite-degree subset family structure and C, A be subsets of M. Suppose $A \subseteq C$ and A is independent. Then there exists an independent subset B of M such that $A \subseteq B$ and B is maximal independent in C.

(15) Let M be a non void finite-degree subset-closed subset family structure and C be a subset of M. Then there exists an independent subset of M which is maximal independent in C.

(16) Let M be a non empty non void subset-closed finite-degree subset family structure. Then M is a matroid if and only if for every subset C of M and for all independent subsets A, B of M such that A is maximal independent in C and B is maximal independent in C holds $\operatorname{card} A = \operatorname{card} B$.

Let M be a finite-degree matroid and let C be a subset of M. The functor $\operatorname{Rnk} C$ yields a natural number and is defined by:

(Def. 11) $\operatorname{Rnk} C = \bigcup\{\operatorname{card} A; A$ ranges over independent subsets of M: $A \subseteq C\}$.

One can prove the following propositions:

(17) Let M be a finite-degree matroid, C be a subset of M, and A be an independent subset of M. If $A \subseteq C$, then $\operatorname{card} A \leq \operatorname{Rnk} C$.

(18) Let M be a finite-degree matroid and C be a subset of M. Then there exists an independent subset A of M such that $A \subseteq C$ and $\operatorname{card} A = \operatorname{Rnk} C$.

(19) Let M be a finite-degree matroid, C be a subset of M, and A be an independent subset of M. Then A is maximal independent in C if and only if $A \subseteq C$ and $\operatorname{card} A = \operatorname{Rnk} C$.

(20) For every finite-degree matroid M and for every finite subset C of M holds $\operatorname{Rnk} C \leq \operatorname{card} C$.

(21) Let M be a finite-degree matroid and C be a finite subset of M. Then C is independent if and only if $\operatorname{card} C = \operatorname{Rnk} C$.

Let M be a finite-degree matroid. The functor $\operatorname{Rnk} M$ yielding a natural number is defined by:

(Def. 12) $\operatorname{Rnk} M = \operatorname{Rnk}(\Omega_M)$.

Let M be a non void finite-degree subset family structure. An independent subset of M is said to be a basis of M if:

(Def. 13) It is maximal independent in Ω_M.

One can prove the following propositions:

(22) For every finite-degree matroid M and for all bases B_1, B_2 of M holds $\operatorname{card} B_1 = \operatorname{card} B_2$.

(23) For every finite-degree matroid M and for every independent subset A of M there exists a basis B of M such that $A \subseteq B$.

We follow the rules: M is a finite-degree matroid, A, B, C are subsets of M, and e, f are elements of M.

Next we state four propositions:

(24) If $A \subseteq B$, then $\operatorname{Rnk} A \leq \operatorname{Rnk} B$.

(25) $\operatorname{Rnk}(A \cup B) + \operatorname{Rnk}(A \cap B) \leq \operatorname{Rnk} A + \operatorname{Rnk} B$.

(26) $\operatorname{Rnk} A \leq \operatorname{Rnk}(A \cup B)$ and $\operatorname{Rnk}(A \cup \{e\}) \leq \operatorname{Rnk} A + 1$.

(27) If $\operatorname{Rnk}(A \cup \{e\}) = \operatorname{Rnk}(A \cup \{f\})$ and $\operatorname{Rnk}(A \cup \{f\}) = \operatorname{Rnk} A$, then $\operatorname{Rnk}(A \cup \{e, f\}) = \operatorname{Rnk} A$.

4. Dependence on a Set, Spans, and Cycles

Let M be a finite-degree matroid, let e be an element of M, and let A be a subset of M. We say that e is dependent on A if and only if:

(Def. 14) $\operatorname{Rnk}(A \cup \{e\}) = \operatorname{Rnk} A$.

We now state two propositions:

(28) If $e \in A$, then e is dependent on A.

(29) If $A \subseteq B$ and e is dependent on A, then e is dependent on B.

Let M be a finite-degree matroid and let A be a subset of M. The functor $\operatorname{Span} A$ yielding a subset of M is defined as follows:

(Def. 15) $\operatorname{Span} A = \{e \in M \colon e \text{ is dependent on } A\}$.

Next we state several propositions:

(30) $e \in \operatorname{Span} A$ iff $\operatorname{Rnk}(A \cup \{e\}) = \operatorname{Rnk} A$.

(31) $A \subseteq \operatorname{Span} A$.

(32) If $A \subseteq B$, then $\operatorname{Span} A \subseteq \operatorname{Span} B$.

(33) $\operatorname{Rnk} \operatorname{Span} A = \operatorname{Rnk} A$.

(34) If e is dependent on $\operatorname{Span} A$, then e is dependent on A.

(35) $\operatorname{Span} \operatorname{Span} A = \operatorname{Span} A$.

(36) If $f \notin \operatorname{Span} A$ and $f \in \operatorname{Span}(A \cup \{e\})$, then $e \in \operatorname{Span}(A \cup \{f\})$.

Let M be a subset family structure and let A be a subset of M. We say that A is cycle if and only if:

(Def. 16) A is dependent and for every element e of M such that $e \in A$ holds $A \setminus \{e\}$ is independent.

Next we state the proposition

(37) If A is cycle, then A is non empty and finite.

Let us consider M. Note that every subset of M which is cycle is also non empty and finite.

One can prove the following propositions:

(38) A is cycle iff A is non empty and for every e such that $e \in A$ holds $A \setminus \{e\}$ is maximal independent in A.

(39) If A is cycle, then $\operatorname{Rnk} A + 1 = \overline{\overline{A}}$.

(40) If A is cycle and $e \in A$, then e is dependent on $A \setminus \{e\}$.

(41) If A is cycle and B is cycle and $A \subseteq B$, then $A = B$.

(42) If for every B such that $B \subseteq A$ holds B is not cycle, then A is independent.

(43) If A is cycle and B is cycle and $A \neq B$ and $e \in A \cap B$, then there exists C such that C is cycle and $C \subseteq (A \cup B) \setminus \{e\}$.

(44) If A is independent and B is cycle and C is cycle and $B \subseteq A \cup \{e\}$ and $C \subseteq A \cup \{e\}$, then $B = C$.

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Partial Differentiation of Real Binary Functions

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Summary. In this article, we define two single-variable functions SVF1 and SVF2, then discuss partial differentiation of real binary functions by dint of one variable function SVF1 and SVF2. The main properties of partial differentiation are shown [7].

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The articles [14], [4], [15], [5], [1], [8], [10], [9], [2], [3], [13], [6], [12], [11], and [7] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: x, x_0, y, y_0, r are real numbers, z, z_0 are elements of $\mathcal{R}^2$, Z is a subset of $\mathcal{R}^2$, f, f_1, f_2 are partial functions from $\mathcal{R}^2$ to $\mathbb{R}$, R is a rest, and L is a linear function.

Next we state two propositions:

(1) $\operatorname{dom}\operatorname{proj}(1,2) = \mathcal{R}^2$ and $\operatorname{rng}\operatorname{proj}(1,2) = \mathbb{R}$ and for all elements x, y of $\mathbb{R}$ holds $(\operatorname{proj}(1,2))(\langle x, y\rangle) = x$.

(2) $\operatorname{dom}\operatorname{proj}(2,2) = \mathcal{R}^2$ and $\operatorname{rng}\operatorname{proj}(2,2) = \mathbb{R}$ and for all elements x, y of $\mathbb{R}$ holds $(\operatorname{proj}(2,2))(\langle x, y\rangle) = y$.

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2. Partial Differentiation of Real Binary Functions

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. The functor $\mathrm{SVF1}(f, z)$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by:

(Def. 1) $\mathrm{SVF1}(f, z) = f \cdot \mathrm{reproj}(1, z)$.

The functor $\mathrm{SVF2}(f, z)$ yields a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined as follows:

(Def. 2) $\mathrm{SVF2}(f, z) = f \cdot \mathrm{reproj}(2, z)$.

Next we state two propositions:

(3) If $z = \langle x, y\rangle$ and f is partially differentiable in z w.r.t. 1 coordinate, then $\mathrm{SVF1}(f, z)$ is differentiable in x.

(4) If $z = \langle x, y\rangle$ and f is partially differentiable in z w.r.t. 2 coordinate, then $\mathrm{SVF2}(f, z)$ is differentiable in y.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. We say that f is partial differentiable on 1st coordinate in z if and only if:

(Def. 3) There exist real numbers x_0, y_0 such that $z = \langle x_0, y_0\rangle$ and $\mathrm{SVF1}(f, z)$ is differentiable in x_0.

We say that f is partial differentiable on 2nd coordinate in z if and only if:

(Def. 4) There exist real numbers x_0, y_0 such that $z = \langle x_0, y_0\rangle$ and $\mathrm{SVF2}(f, z)$ is differentiable in y_0.

Next we state two propositions:

(5) Suppose $z = \langle x_0, y_0\rangle$ and f is partial differentiable on 1st coordinate in z. Then there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(f, z)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(f, z))(x) - (\mathrm{SVF1}(f, z))(x_0) = L(x - x_0) + R(x - x_0)$.

(6) Suppose $z = \langle x_0, y_0\rangle$ and f is partial differentiable on 2nd coordinate in z. Then there exists a neighbourhood N of y_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF2}(f, z)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF2}(f, z))(y) - (\mathrm{SVF2}(f, z))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. Let us observe that f is partial differentiable on 1st coordinate in z if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exist real numbers x_0, y_0 such that

(i) $z = \langle x_0, y_0\rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(f, z)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(f, z))(x) - (\mathrm{SVF1}(f, z))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. Let us observe that f is partial differentiable on 2nd coordinate in z if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exist real numbers x_0, y_0 such that

(i) $z = \langle x_0, y_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF2}(f, z)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF2}(f, z))(y) - (\mathrm{SVF2}(f, z))(y_0) = L(y - y_0) + R(y - y_0)$.

Next we state two propositions:

(7) Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and z be an element of $\mathcal{R}^2$. Then f is partial differentiable on 1st coordinate in z if and only if f is partially differentiable in z w.r.t. 1 coordinate.

(8) Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and z be an element of $\mathcal{R}^2$. Then f is partial differentiable on 2nd coordinate in z if and only if f is partially differentiable in z w.r.t. 2 coordinate.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. The functor partdiff1(f, z) yielding a real number is defined by:

(Def. 7) $\mathrm{partdiff1}(f, z) = \mathrm{partdiff}(f, z, 1)$.

The functor partdiff2(f, z) yielding a real number is defined as follows:

(Def. 8) $\mathrm{partdiff2}(f, z) = \mathrm{partdiff}(f, z, 2)$.

One can prove the following propositions:

(9) Suppose $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st coordinate in z. Then $r = \mathrm{partdiff1}(f, z)$ if and only if there exist real numbers x_0, y_0 such that $z = \langle x_0, y_0 \rangle$ and there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(f, z)$ and there exist L, R such that $r = L(1)$ and for every x such that $x \in N$ holds $(\mathrm{SVF1}(f, z))(x) - (\mathrm{SVF1}(f, z))(x_0) = L(x - x_0) + R(x - x_0)$.

(10) Suppose $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd coordinate in z. Then $r = \mathrm{partdiff2}(f, z)$ if and only if there exist real numbers x_0, y_0 such that $z = \langle x_0, y_0 \rangle$ and there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF2}(f, z)$ and there exist L, R such that $r = L(1)$ and for every y such that $y \in N$ holds $(\mathrm{SVF2}(f, z))(y) - (\mathrm{SVF2}(f, z))(y_0) = L(y - y_0) + R(y - y_0)$.

(11) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st coordinate in z, then $\mathrm{partdiff1}(f, z) = (\mathrm{SVF1}(f, z))'(x_0)$.

(12) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd coordinate in z, then $\mathrm{partdiff2}(f, z) = (\mathrm{SVF2}(f, z))'(y_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. We say that f is partial differentiable w.r.t. 1st coordinate on Z if and only if:

(Def. 9) $Z \subseteq \operatorname{dom} f$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f \restriction Z$ is partial differentiable on 1st coordinate in z.

We say that f is partial differentiable w.r.t. 2nd coordinate on Z if and only if:

(Def. 10) $Z \subseteq \operatorname{dom} f$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{\restriction}Z$ is partial differentiable on 2nd coordinate in z.

One can prove the following two propositions:

(13) Suppose f is partial differentiable w.r.t. 1st coordinate on Z. Then $Z \subseteq \operatorname{dom} f$ and for every z such that $z \in Z$ holds f is partial differentiable on 1st coordinate in z.

(14) Suppose f is partial differentiable w.r.t. 2nd coordinate on Z. Then $Z \subseteq \operatorname{dom} f$ and for every z such that $z \in Z$ holds f is partial differentiable on 2nd coordinate in z.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. Let us assume that f is partial differentiable w.r.t. 1st coordinate on Z. The functor $f^{1\mathrm{st}}_{\restriction Z}$ yielding a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ is defined as follows:

(Def. 11) $\operatorname{dom}(f^{1\mathrm{st}}_{\restriction Z}) = Z$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f^{1\mathrm{st}}_{\restriction Z}(z) = \mathrm{partdiff1}(f, z)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. Let us assume that f is partial differentiable w.r.t. 2nd coordinate on Z. The functor $f^{2\mathrm{nd}}_{\restriction Z}$ yielding a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ is defined as follows:

(Def. 12) $\operatorname{dom}(f^{2\mathrm{nd}}_{\restriction Z}) = Z$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f^{2\mathrm{nd}}_{\restriction Z}(z) = \mathrm{partdiff2}(f, z)$.

3. Main Properties of Partial Differentiation of Real Binary Functions

We now state a number of propositions:

(15) Let z_0 be an element of $\mathcal{R}^2$ and N be a neighbourhood of $(\mathrm{proj}(1,2))(z_0)$. Suppose f is partial differentiable on 1st coordinate in z_0 and $N \subseteq \operatorname{dom} \mathrm{SVF1}(f, z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\mathrm{proj}(1,2))(z_0)\}$ and $\operatorname{rng}(h+c) \subseteq N$. Then $h^{-1}\,(\mathrm{SVF1}(f, z_0) \cdot (h+c) - \mathrm{SVF1}(f, z_0) \cdot c)$ is convergent and $\mathrm{partdiff1}(f, z_0) = \lim(h^{-1}\,(\mathrm{SVF1}(f, z_0) \cdot (h+c) - \mathrm{SVF1}(f, z_0) \cdot c))$.

(16) Let z_0 be an element of $\mathcal{R}^2$ and N be a neighbourhood of $(\mathrm{proj}(2,2))(z_0)$. Suppose f is partial differentiable on 2nd coordinate in z_0 and $N \subseteq \operatorname{dom} \mathrm{SVF2}(f, z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\mathrm{proj}(2,2))(z_0)\}$ and $\operatorname{rng}(h+c) \subseteq N$. Then $h^{-1}\,(\mathrm{SVF2}(f, z_0) \cdot (h+c) - \mathrm{SVF2}(f, z_0) \cdot c)$ is convergent and $\mathrm{partdiff2}(f, z_0) = \lim(h^{-1}\,(\mathrm{SVF2}(f, z_0) \cdot (h+c) - \mathrm{SVF2}(f, z_0) \cdot c))$.

(17) Suppose f_1 is partial differentiable on 1st coordinate in z_0 and f_2 is partial differentiable on 1st coordinate in z_0. Then $f_1 + f_2$ is par-

tial differentiable on 1st coordinate in z_0 and partdiff1$(f_1 + f_2, z_0)$ = partdiff1(f_1, z_0) + partdiff1(f_2, z_0).

(18) Suppose f_1 is partial differentiable on 2nd coordinate in z_0 and f_2 is partial differentiable on 2nd coordinate in z_0. Then $f_1 + f_2$ is partial differentiable on 2nd coordinate in z_0 and partdiff2$(f_1 + f_2, z_0)$ = partdiff2(f_1, z_0) + partdiff2(f_2, z_0).

(19) Suppose f_1 is partial differentiable on 1st coordinate in z_0 and f_2 is partial differentiable on 1st coordinate in z_0. Then $f_1 - f_2$ is partial differentiable on 1st coordinate in z_0 and partdiff1$(f_1 - f_2, z_0)$ = partdiff1(f_1, z_0) − partdiff1(f_2, z_0).

(20) Suppose f_1 is partial differentiable on 2nd coordinate in z_0 and f_2 is partial differentiable on 2nd coordinate in z_0. Then $f_1 - f_2$ is partial differentiable on 2nd coordinate in z_0 and partdiff2$(f_1 - f_2, z_0)$ = partdiff2(f_1, z_0) − partdiff2(f_2, z_0).

(21) Suppose f is partial differentiable on 1st coordinate in z_0. Then $r\,f$ is partial differentiable on 1st coordinate in z_0 and partdiff1$(r\,f, z_0)$ = $r \cdot$ partdiff1(f, z_0).

(22) Suppose f is partial differentiable on 2nd coordinate in z_0. Then $r\,f$ is partial differentiable on 2nd coordinate in z_0 and partdiff2$(r\,f, z_0)$ = $r \cdot$ partdiff2(f, z_0).

(23) Suppose f_1 is partial differentiable on 1st coordinate in z_0 and f_2 is partial differentiable on 1st coordinate in z_0. Then $f_1\,f_2$ is partial differentiable on 1st coordinate in z_0.

(24) Suppose f_1 is partial differentiable on 2nd coordinate in z_0 and f_2 is partial differentiable on 2nd coordinate in z_0. Then $f_1\,f_2$ is partial differentiable on 2nd coordinate in z_0.

(25) Let z_0 be an element of $\mathcal{R}^2$. Suppose f is partial differentiable on 1st coordinate in z_0. Then SVF1(f, z_0) is continuous in (proj$(1,2)$)(z_0).

(26) Let z_0 be an element of $\mathcal{R}^2$. Suppose f is partial differentiable on 2nd coordinate in z_0. Then SVF2(f, z_0) is continuous in (proj$(2,2)$)(z_0).

(27) If f is partial differentiable on 1st coordinate in z_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

(28) If f is partial differentiable on 2nd coordinate in z_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

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Model Checking. Part III

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Summary. This text includes verification of the basic algorithm in Simple On-the-fly Automatic Verification of Linear Temporal Logic (LTL). LTL formula can be transformed to Buchi automaton, and this transforming algorithm is mainly used at Simple On-the-fly Automatic Verification. In this article, we verified the transforming algorithm itself. At first, we prepared some definitions and operations for transforming. And then, we defined the Buchi automaton and verified the transforming algorithm.

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The notation and terminology used in this paper are introduced in the following articles: [5], [14], [6], [7], [1], [15], [3], [16], [2], [13], [4], [12], [10], [11], [8], and [9].

1. Definition of Basic Operations to Build an Automaton for LTL and Properties

For simplicity, we adopt the following rules: k, n, m, i, j are elements of $\mathbb{N}$, x, y, X are sets, L, L_1, L_2 are finite sequences, F, H are LTL-formulae, W, W_1, W_2 are subsets of Subformulae H, and v is an LTL-formula.

Let us consider F. Then Subformulae F is a subset of $\mathrm{WFF_{LTL}}$.

Let us consider H. The functor $\mathrm{LTLNew}_1\, H$ yields a subset of Subformulae H and is defined as follows:

$$\text{(Def. 1)}\quad \mathrm{LTLNew}_1\, H = \begin{cases} \{\mathrm{LeftArg}(H), \mathrm{RightArg}(H)\}, & \text{if } H \text{ is conjunctive}, \\ \{\mathrm{LeftArg}(H)\}, & \text{if } H \text{ is disjunctive}, \\ \emptyset, & \text{if } H \text{ has } \textit{next} \text{ operator}, \\ \{\mathrm{LeftArg}(H)\}, & \text{if } H \text{ has } \textit{until} \text{ operator}, \\ \{\mathrm{RightArg}(H)\}, & \text{if } H \text{ has } \textit{release} \text{ operator}, \\ \emptyset, & \text{otherwise}. \end{cases}$$

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The functor $\mathrm{LTLNew}_2\,H$ yields a subset of $\mathrm{Subformulae}\,H$ and is defined as follows:

$$\text{(Def. 2)}\quad \mathrm{LTLNew}_2\,H = \begin{cases} \emptyset, \text{ if } H \text{ is conjunctive,} \\ \{\mathrm{RightArg}(H)\}, \text{ if } H \text{ is disjunctive,} \\ \emptyset, \text{ if } H \text{ has } \textit{next} \text{ operator,} \\ \{\mathrm{RightArg}(H)\}, \text{ if } H \text{ has } \textit{until} \text{ operator,} \\ \{\mathrm{LeftArg}(H), \mathrm{RightArg}(H)\}, \text{ if } H \text{ has } \textit{release} \text{ operator,} \\ \emptyset, \text{ otherwise.} \end{cases}$$

The functor $\mathrm{LTLNext}\,H$ yielding a subset of $\mathrm{Subformulae}\,H$ is defined as follows:

$$\text{(Def. 3)}\quad \mathrm{LTLNext}\,H = \begin{cases} \emptyset, \text{ if } H \text{ is conjunctive,} \\ \emptyset, \text{ if } H \text{ is disjunctive,} \\ \{\mathrm{Arg}(H)\}, \text{ if } H \text{ has } \textit{next} \text{ operator,} \\ \{H\}, \text{ if } H \text{ has } \textit{until} \text{ operator,} \\ \{H\}, \text{ if } H \text{ has } \textit{release} \text{ operator,} \\ \emptyset, \text{ otherwise.} \end{cases}$$

Let us consider v. We consider LTL-nodes over v as systems

⟨ an old-component, a new-component, a next-component ⟩,

where the old-component, the new-component, and the next-component are subsets of $\mathrm{Subformulae}\,v$.

Let us consider v, let N be an LTL-node over v, and let us consider H. Let us assume that $H \in$ the new-component of N. The functor $\mathrm{SuccNode}_1(H, N)$ yielding a strict LTL-node over v is defined by the conditions (Def. 4).

(Def. 4)(i) The old-component of $\mathrm{SuccNode}_1(H, N)$ = (the old-component of $N) \cup \{H\}$,

(ii) the new-component of $\mathrm{SuccNode}_1(H, N)$ = ((the new-component of $N) \setminus \{H\}) \cup (\mathrm{LTLNew}_1\,H \setminus$ the old-component of N), and

(iii) the next-component of $\mathrm{SuccNode}_1(H, N)$ = (the next-component of $N) \cup \mathrm{LTLNext}\,H$.

Let us consider v, let N be an LTL-node over v, and let us consider H. Let us assume that $H \in$ the new-component of N and H is either disjunctive or has *until* operator or *release* operator. The functor $\mathrm{SuccNode}_2(H, N)$ yields a strict LTL-node over v and is defined by the conditions (Def. 5).

(Def. 5)(i) The old-component of $\mathrm{SuccNode}_2(H, N)$ = (the old-component of $N) \cup \{H\}$,

(ii) the new-component of $\mathrm{SuccNode}_2(H, N)$ = ((the new-component of $N) \setminus \{H\}) \cup (\mathrm{LTLNew}_2\,H \setminus$ the old-component of N), and

(iii) the next-component of $\mathrm{SuccNode}_2(H, N)$ = the next-component of N.

Let us consider v, let N_1, N_2 be LTL-nodes over v, and let us consider H. We say that N_2 is a successor of N_1 and H if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) $H \in$ the new-component of N_1, and

(ii) $N_2 = \mathrm{SuccNode}_1(H, N_1)$ or H is either disjunctive or has *until* operator or *release* operator and $N_2 = \mathrm{SuccNode}_2(H, N_1)$.

Let us consider v and let N_1, N_2 be LTL-nodes over v. We say that N_2 is a 1st successor of N_1 if and only if:

(Def. 7) There exists H such that $H \in$ the new-component of N_1 and $N_2 = \mathrm{SuccNode}_1(H, N_1)$.

We say that N_2 is a 2nd successor of N_1 if and only if the condition (Def. 8) is satisfied.

(Def. 8) There exists H such that

(i) $H \in$ the new-component of N_1,

(ii) H is either disjunctive or has *until* operator or *release* operator, and

(iii) $N_2 = \mathrm{SuccNode}_2(H, N_1)$.

Let us consider v and let N_1, N_2 be LTL-nodes over v. We say that N_2 is a successor of N_1 if and only if:

(Def. 9) N_2 is a 1st successor of N_1 or a 2nd successor of N_1.

Let us consider v and let N be an LTL-node over v. We say that N is failure if and only if:

(Def. 10) There exist H, F such that H is atomic and $F = \neg H$ and $H \in$ the old-component of N and $F \in$ the old-component of N.

Let us consider v and let N be an LTL-node over v. We say that N is elementary if and only if:

(Def. 11) The new-component of $N = \emptyset$.

Let us consider v and let N be an LTL-node over v. We say that N is final if and only if:

(Def. 12) N is elementary and the next-component of $N = \emptyset$.

Let us consider v. The functor $\emptyset_v$ yielding a subset of $\mathrm{Subformulae}\, v$ is defined as follows:

(Def. 13) $\emptyset_v = \emptyset$.

Let us consider v. The functor $\mathrm{Seed}\, v$ yielding a subset of $\mathrm{Subformulae}\, v$ is defined by:

(Def. 14) $\mathrm{Seed}\, v = \{v\}$.

Let us consider v. Note that there exists an LTL-node over v which is elementary and strict.

Let us consider v. The functor $\mathrm{FinalNode}\, v$ yields an elementary strict LTL-node over v and is defined by:

(Def. 15) $\mathrm{FinalNode}\, v = \langle \emptyset_v, \emptyset_v, \emptyset_v \rangle$.

Let us consider x, v. The functor $\mathrm{CastNode}(x, v)$ yields a strict LTL-node over v and is defined by:

(Def. 16) $\mathrm{CastNode}(x, v) = \begin{cases} x, & \text{if } x \text{ is a strict LTL-node over } v, \\ \langle \emptyset_v, \emptyset_v, \emptyset_v \rangle, & \text{otherwise.} \end{cases}$

Let us consider v. The functor $\mathrm{init}\, v$ yields an elementary strict LTL-node over v and is defined by:

(Def. 17) $\mathrm{init}\, v = \langle \emptyset_v, \emptyset_v, \mathrm{Seed}\, v \rangle$.

Let us consider v and let N be an LTL-node over v. The functor $\mathcal{X}\, N$ yields a strict LTL-node over v and is defined as follows:

(Def. 18) $\mathcal{X}\, N = \langle \emptyset_v, \text{the next-component of } N, \emptyset_v \rangle$.

We follow the rules: N, N_1, N_2, M are strict LTL-nodes over v and w is an element of the infinite sequences of AtomicFamily.

Let us consider v, L. We say that L is a successor sequence for v if and only if:

(Def. 19) For every k such that $1 \leq k < \mathrm{len}\, L$ there exist N, M such that $N = L(k)$ and $M = L(k+1)$ and M is a successor of N.

Let us consider v, N_1, N_2. We say that N_2 is next to N_1 if and only if the conditions (Def. 20) are satisfied.

(Def. 20)(i) N_1 is elementary,

(ii) N_2 is elementary, and

(iii) there exists L such that $1 \leq \mathrm{len}\, L$ and L is a successor sequence for v and $L(1) = \mathcal{X}\, N_1$ and $L(\mathrm{len}\, L) = N_2$.

Let us consider v and let W be a subset of $\mathrm{Subformulae}\, v$. The functor $\mathrm{Cast}_{\mathrm{LTL}}\, W$ yielding a subset of $\mathrm{WFF}_{\mathrm{LTL}}$ is defined by:

(Def. 21) $\mathrm{Cast}_{\mathrm{LTL}}\, W = W$.

Let us consider v, N. The functor $\cdot N$ yields a subset of $\mathrm{WFF}_{\mathrm{LTL}}$ and is defined by:

(Def. 22) $\cdot N = (\text{the old-component of } N) \cup (\text{the new-component of } N) \cup \mathcal{X}\, \mathrm{Cast}_{\mathrm{LTL}}\, (\text{the next-component of } N)$.

We now state three propositions:

(1) Suppose $H \in$ the new-component of N and H is either atomic, or negative, or conjunctive, or has *next* operator. Then $w \models \cdot N$ if and only if $w \models \cdot\, \mathrm{SuccNode}_1(H, N)$.

(2) Suppose $H \in$ the new-component of N and H is either disjunctive or has *until* operator or *release* operator. Then $w \models \cdot N$ if and only if one of the following conditions is satisfied:

(i) $w \models \cdot\, \mathrm{SuccNode}_1(H, N)$, or

(ii) $w \models \cdot\, \mathrm{SuccNode}_2(H, N)$.

(3) There exists L such that $\mathrm{Subformulae}\, H = \mathrm{rng}\, L$.

Let us consider H. Observe that $\mathrm{Subformulae}\, H$ is finite.

Let us consider H, W, L, x. The length of L wrt W and x yields a natural number and is defined as follows:

(Def. 23) The length of L wrt W and $x = \begin{cases} \operatorname{len} \mathrm{Cast}_{\mathrm{LTL}} L(x), & \text{if } L(x) \in W, \\ 0, & \text{otherwise.} \end{cases}$

Let us consider H, W, L. The partial sequence of L wrt W yields a sequence of real numbers and is defined by the condition (Def. 24).

(Def. 24) Let given k. Then

(i) if $L(k) \in W$, then (the partial sequence of L wrt W)$(k) = \operatorname{len} \mathrm{Cast}_{\mathrm{LTL}} L(k)$, and

(ii) if $L(k) \notin W$, then (the partial sequence of L wrt W)$(k) = 0$.

Let us consider H, W, L. The functor $\operatorname{len}(L, W)$ yields a real number and is defined as follows:

(Def. 25) $\operatorname{len}(L, W) = \sum_{\kappa=0}^{\operatorname{len} L}$ (the partial sequence of L wrt W)(κ).

We now state several propositions:

(4) $\operatorname{len}(L, \emptyset_H) = 0$.

(5) If $F \notin W$, then $\operatorname{len}(L, W \setminus \{F\}) = \operatorname{len}(L, W)$.

(6) If $\operatorname{rng} L = \operatorname{Subformulae} H$ and L is one-to-one and $F \in W$, then $\operatorname{len}(L, W \setminus \{F\}) = \operatorname{len}(L, W) - \operatorname{len} F$.

(7) If $\operatorname{rng} L = \operatorname{Subformulae} H$ and L is one-to-one and $F \notin W$ and $W_1 = W \cup \{F\}$, then $\operatorname{len}(L, W_1) = \operatorname{len}(L, W) + \operatorname{len} F$.

(8) If $\operatorname{rng} L_1 = \operatorname{Subformulae} H$ and L_1 is one-to-one and $\operatorname{rng} L_2 = \operatorname{Subformulae} H$ and L_2 is one-to-one, then $\operatorname{len}(L_1, W) = \operatorname{len}(L_2, W)$.

Let us consider H, W. The functor $\operatorname{len} W$ yields a real number and is defined by:

(Def. 26) There exists L such that $\operatorname{rng} L = \operatorname{Subformulae} H$ and L is one-to-one and $\operatorname{len} W = \operatorname{len}(L, W)$.

The following propositions are true:

(9) If $F \notin W$, then $\operatorname{len}(W \setminus \{F\}) = \operatorname{len} W$.

(10) If $F \in W$, then $\operatorname{len}(W \setminus \{F\}) = \operatorname{len} W - \operatorname{len} F$.

(11) If $F \notin W$ and $W_1 = W \cup \{F\}$, then $\operatorname{len} W_1 = \operatorname{len} W + \operatorname{len} F$.

(12) $\operatorname{len}(W \cup \{F\}) \leq \operatorname{len} W + \operatorname{len} F$.

(13) $\operatorname{len}(\emptyset_H) = 0$.

(14) $\operatorname{len}(\{F\}) = \operatorname{len} F$.

(15) If $W \subseteq W_1$, then $\operatorname{len} W \leq \operatorname{len} W_1$.

(16) If $\operatorname{len} W < 1$, then $W = \emptyset_H$.

(17) $\operatorname{len} W \geq 0$.

(18) $\operatorname{len}(W_1 \cup W_2) \leq \operatorname{len} W_1 + \operatorname{len} W_2$.

Let us consider v, H. Let us assume that $H \in \mathrm{Subformulae}\, v$. The functor $\mathrm{LTLNew}_1(H, v)$ yielding a subset of Subformulae v is defined by:

(Def. 27) $\mathrm{LTLNew}_1(H, v) = \mathrm{LTLNew}_1\, H$.

The functor $\mathrm{LTLNew}_2(H, v)$ yields a subset of Subformulae v and is defined by:

(Def. 28) $\mathrm{LTLNew}_2(H, v) = \mathrm{LTLNew}_2\, H$.

The following propositions are true:

(19) If N_2 is a 1st successor of N_1, then len (the new-component of N_2) $\leq$ len (the new-component of N_1) $- 1$.

(20) If N_2 is a 2nd successor of N_1, then len (the new-component of N_2) $\leq$ len (the new-component of N_1) $- 1$.

Let us consider v, N. The functor $\mathrm{len}\, N$ yields a natural number and is defined by:

(Def. 29) $\mathrm{len}\, N = \lfloor$len (the new-component of N)$\rfloor$.

The following propositions are true:

(21) If N_2 is a successor of N_1, then $\mathrm{len}\, N_2 \leq \mathrm{len}\, N_1 - 1$.

(22) If $\mathrm{len}\, N \leq 0$, then the new-component of $N = \emptyset_v$.

(23) If $\mathrm{len}\, N > 0$, then the new-component of $N \neq \emptyset_v$.

(24) There exist n, L, M such that $1 \leq n$ and $\mathrm{len}\, L = n$ and $L(1) = N$ and $L(n) = M$ and the new-component of $M = \emptyset_v$ and L is a successor sequence for v.

(25) Suppose N_2 is a successor of N_1. Then

(i) the old-component of $N_1 \subseteq$ the old-component of N_2, and

(ii) the next-component of $N_1 \subseteq$ the next-component of N_2.

(26) If L is a successor sequence for v and $m \leq \mathrm{len}\, L$ and $L_1 = L \restriction \mathrm{Seg}\, m$, then L_1 is a successor sequence for v.

(27) Suppose that

(i) L is a successor sequence for v,

(ii) $F \notin$ the old-component of $\mathrm{CastNode}(L(1), v)$,

(iii) $1 < n$,

(iv) $n \leq \mathrm{len}\, L$, and

(v) $F \in$ the old-component of $\mathrm{CastNode}(L(n), v)$.

Then there exists m such that $1 \leq m < n$ and $F \notin$ the old-component of $\mathrm{CastNode}(L(m), v)$ and $F \in$ the old-component of $\mathrm{CastNode}(L(m+1), v)$.

(28) Suppose N_2 is a successor of N_1 and $F \notin$ the old-component of N_1 and $F \in$ the old-component of N_2. Then N_2 is a successor of N_1 and F.

(29) Suppose that

(i) L is a successor sequence for v,

(ii) $F \in$ the new-component of $\mathrm{CastNode}(L(1), v)$,

(iii) $1 < n$,

(iv) $n \leq \operatorname{len} L$, and

(v) $F \notin$ the new-component of CastNode$(L(n), v)$.

Then there exists m such that $1 \leq m < n$ and $F \in$ the new-component of CastNode$(L(m), v)$ and $F \notin$ the new-component of CastNode$(L(m+1), v)$.

(30) Suppose N_2 is a successor of N_1 and $F \in$ the new-component of N_1 and $F \notin$ the new-component of N_2. Then N_2 is a successor of N_1 and F.

(31) Suppose L is a successor sequence for v and $1 \leq m \leq n \leq \operatorname{len} L$. Then

(i) the old-component of CastNode$(L(m), v) \subseteq$ the old-component of CastNode$(L(n), v)$, and

(ii) the next-component of CastNode$(L(m), v) \subseteq$ the next-component of CastNode$(L(n), v)$.

(32) If N_2 is a successor of N_1 and F, then $F \in$ the old-component of N_2.

(33) Suppose L is a successor sequence for v and $1 \leq \operatorname{len} L$ and the new-component of CastNode$(L(\operatorname{len} L), v) = \emptyset_v$. Then the new-component of CastNode$(L(1), v) \subseteq$ the old-component of CastNode$(L(\operatorname{len} L), v)$.

(34) Suppose L is a successor sequence for v and $1 \leq m \leq \operatorname{len} L$ and the new-component of CastNode$(L(\operatorname{len} L), v) = \emptyset_v$. Then the new-component of CastNode$(L(m), v) \subseteq$ the old-component of CastNode$(L(\operatorname{len} L), v)$.

(35) If L is a successor sequence for v and $1 \leq k < \operatorname{len} L$, then CastNode$(L(k+1), v)$ is a successor of CastNode$(L(k), v)$.

(36) If L is a successor sequence for v and $1 \leq k \leq \operatorname{len} L$, then len CastNode$(L(k), v) \leq (\operatorname{len}$ CastNode$(L(1), v) - k) + 1$.

In the sequel s, s_0, s_1, s_2 denote elementary strict LTL-nodes over v.

The following propositions are true:

(37) If s_2 is next to s_1, then the next-component of $s_1 \subseteq$ the old-component of s_2.

(38) Suppose s_2 is next to s_1 and $F \in$ the old-component of s_2. Then there exist L, m such that

$1 \leq \operatorname{len} L$ and L is a successor sequence for v and $L(1) = \mathcal{X} s_1$ and $L(\operatorname{len} L) = s_2$ and $1 \leq m < \operatorname{len} L$ and CastNode$(L(m+1), v)$ is a successor of CastNode$(L(m), v)$ and F.

(39) Suppose s_2 is next to s_1 and H has *release* operator and $H \in$ the old-component of s_2 and LeftArg$(H) \notin$ the old-component of s_2. Then RightArg$(H) \in$ the old-component of s_2 and $H \in$ the next-component of s_2.

(40) Suppose s_2 is next to s_1 and H has *release* operator and $H \in$ the next-component of s_1. Then RightArg$(H) \in$ the old-component of s_2 and $H \in$ the old-component of s_2.

(41) Suppose s_1 is next to s_0 and $H \in$ the old-component of s_1. Then

(i) if H is conjunctive, then $\mathrm{LeftArg}(H) \in$ the old-component of s_1 and $\mathrm{RightArg}(H) \in$ the old-component of s_1,

(ii) if H is either disjunctive or has *until* operator, then $\mathrm{LeftArg}(H) \in$ the old-component of s_1 or $\mathrm{RightArg}(H) \in$ the old-component of s_1,

(iii) if H has *next* operator, then $\mathrm{Arg}(H) \in$ the next-component of s_1, and

(iv) if H has *release* operator, then $\mathrm{RightArg}(H) \in$ the old-component of s_1.

(42) Suppose s_1 is next to s_0 and s_2 is next to s_1 and $H \in$ the old-component of s_1 and H has *until* operator. Then $\mathrm{RightArg}(H) \in$ the old-component of s_1 or $\mathrm{LeftArg}(H) \in$ the old-component of s_1 and $H \in$ the old-component of s_2.

Let us consider v. The functor $\mathrm{Nodes}_{\mathrm{LTL}}\, v$ yields a non empty set and is defined as follows:

(Def. 30) $x \in \mathrm{Nodes}_{\mathrm{LTL}}\, v$ iff there exists a strict LTL-node N over v such that $x = N$.

Let us consider v. Note that $\mathrm{Nodes}_{\mathrm{LTL}}\, v$ is finite.

Let us consider v. The functor $\mathrm{States}_{\mathrm{LTL}}\, v$ yields a non empty set and is defined by:

(Def. 31) $\mathrm{States}_{\mathrm{LTL}}\, v = \{x \in \mathrm{Nodes}_{\mathrm{LTL}}\, v : x$ is an elementary strict LTL-node over $v\}$.

Let us consider v. Observe that $\mathrm{States}_{\mathrm{LTL}}\, v$ is finite.

The following propositions are true:

(43) $\mathrm{init}\, v$ is an element of $\mathrm{States}_{\mathrm{LTL}}\, v$.

(44) s is an element of $\mathrm{States}_{\mathrm{LTL}}\, v$.

(45) x is an element of $\mathrm{States}_{\mathrm{LTL}}\, v$ iff there exists s such that $s = x$.

Let us consider v, let us consider w, and let f be a function. We say that f is a successor homomorphism from v to w if and only if:

(Def. 32) For every x such that $x \in \mathrm{Nodes}_{\mathrm{LTL}}\, v$ and $\mathrm{CastNode}(x, v)$ is non elementary and $w \models \cdot \mathrm{CastNode}(x, v)$ holds $\mathrm{CastNode}(f(x), v)$ is a successor of $\mathrm{CastNode}(x, v)$ and $w \models \cdot \mathrm{CastNode}(f(x), v)$.

We say that f is a homomorphism of v into w if and only if:

(Def. 33) For every x such that $x \in \mathrm{Nodes}_{\mathrm{LTL}}\, v$ and $\mathrm{CastNode}(x, v)$ is non elementary and $w \models \cdot \mathrm{CastNode}(x, v)$ holds $w \models \cdot \mathrm{CastNode}(f(x), v)$.

The following propositions are true:

(46) Let f be a function from $\mathrm{Nodes}_{\mathrm{LTL}}\, v$ into $\mathrm{Nodes}_{\mathrm{LTL}}\, v$. Suppose f is a successor homomorphism from v to w. Then f is a homomorphism of v into w.

(47) Let f be a function from $\mathrm{Nodes}_{\mathrm{LTL}}\, v$ into $\mathrm{Nodes}_{\mathrm{LTL}}\, v$. Suppose f is a homomorphism of v into w. Let given x. Suppose $x \in \mathrm{Nodes}_{\mathrm{LTL}}\, v$ and

CastNode(x, v) is non elementary and $w \models \cdot\,$CastNode(x, v). Let given k. If for every i such that $i \leq k$ holds CastNode$(f^i(x), v)$ is non elementary, then $w \models \cdot\,$CastNode$(f^k(x), v)$.

(48) Let f be a function from $\mathrm{Nodes}_{\mathrm{LTL}}\, v$ into $\mathrm{Nodes}_{\mathrm{LTL}}\, v$. Suppose f is a successor homomorphism from v to w. Let given x. Suppose $x \in \mathrm{Nodes}_{\mathrm{LTL}}\, v$ and CastNode(x, v) is non elementary and $w \models \cdot\,$CastNode(x, v). Let given k. Suppose that for every i such that $i \leq k$ holds CastNode$(f^i(x), v)$ is non elementary. Then CastNode$(f^{k+1}(x), v)$ is a successor of CastNode$(f^k(x), v)$ and $w \models \cdot\,$CastNode$(f^k(x), v)$.

(49) Let f be a function from $\mathrm{Nodes}_{\mathrm{LTL}}\, v$ into $\mathrm{Nodes}_{\mathrm{LTL}}\, v$. Suppose f is a successor homomorphism from v to w. Let given x. Suppose $x \in \mathrm{Nodes}_{\mathrm{LTL}}\, v$ and CastNode(x, v) is non elementary and $w \models \cdot\,$CastNode(x, v). Then there exists n such that for every i such that $i < n$ holds CastNode$(f^i(x), v)$ is non elementary and CastNode$(f^n(x), v)$ is elementary.

(50) Let f be a function from $\mathrm{Nodes}_{\mathrm{LTL}}\, v$ into $\mathrm{Nodes}_{\mathrm{LTL}}\, v$. Suppose f is a homomorphism of v into w. Let given x. Suppose $x \in \mathrm{Nodes}_{\mathrm{LTL}}\, v$ and CastNode(x, v) is non elementary. Let given k. If CastNode$(f^k(x), v)$ is non elementary and $w \models \cdot\,$CastNode$(f^k(x), v)$, then $w \models \cdot\,$CastNode$(f^{k+1}(x), v)$.

(51) Let f be a function from $\mathrm{Nodes}_{\mathrm{LTL}}\, v$ into $\mathrm{Nodes}_{\mathrm{LTL}}\, v$. Suppose f is a successor homomorphism from v to w. Let given x. Suppose $x \in \mathrm{Nodes}_{\mathrm{LTL}}\, v$ and CastNode(x, v) is non elementary and $w \models \cdot\,$CastNode(x, v). Then there exists n such that
   (i) for every i such that $i < n$ holds CastNode$(f^i(x), v)$ is non elementary and CastNode$(f^{i+1}(x), v)$ is a successor of CastNode$(f^i(x), v)$,
   (ii) CastNode$(f^n(x), v)$ is elementary, and
   (iii) for every i such that $i \leq n$ holds $w \models \cdot\,$CastNode$(f^i(x), v)$.

In the sequel q denotes a sequence of $\mathrm{States}_{\mathrm{LTL}}\, v$.

One can prove the following propositions:

(52) There exists s such that $s =$ CastNode$(q(n), v)$.

(53) Suppose H has *until* operator and $H \in$ the old-component of CastNode$(q(1), v)$ and for every i holds CastNode$(q(i + 1), v)$ is next to CastNode$(q(i), v)$. Suppose that for every i such that $1 \leq i < n$ holds RightArg$(H) \notin$ the old-component of CastNode$(q(i), v)$. Let given i. Suppose $1 \leq i < n$. Then LeftArg$(H) \in$ the old-component of CastNode$(q(i), v)$ and $H \in$ the old-component of CastNode$(q(i), v)$.

(54) Suppose H has *until* operator and $H \in$ the old-component of CastNode$(q(1), v)$ and for every i holds CastNode$(q(i + 1), v)$ is next to CastNode$(q(i), v)$. Then

(i) for every i such that $i \geq 1$ holds $H \in$ the old-component of CastNode$(q(i), v)$ and LeftArg$(H) \in$ the old-component of CastNode$(q(i), v)$ and RightArg$(H) \notin$ the old-component of CastNode$(q(i), v)$, or

(ii) there exists j such that $j \geq 1$ and RightArg$(H) \in$ the old-component of CastNode$(q(j), v)$ and for every i such that $1 \leq i < j$ holds $H \in$ the old-component of CastNode$(q(i), v)$ and LeftArg$(H) \in$ the old-component of CastNode$(q(i), v)$.

(55) $\bigcup(2_+^X) = X$.

(56) If N is non elementary, then the new-component of $N \neq \emptyset$ and the new-component of $N \in 2_+^{\text{Subformulae}\, v}$.

Let us consider v. One can verify that $\bigcup(2_+^{\text{Subformulae}\, v})$ is non empty and $2_+^{\text{Subformulae}\, v}$ is non empty.

We now state the proposition

(57) There exists a choice function of $2_+^{\text{Subformulae}\, v}$ which is a function from $2_+^{\text{Subformulae}\, v}$ into Subformulae v.

In the sequel U denotes a choice function of $2_+^{\text{Subformulae}\, v}$.

Let us consider v, let us consider U, and let us consider N. Let us assume that N is non elementary. The U-chosen formula of N yielding an LTL-formula is defined as follows:

(Def. 34) The U-chosen formula of $N = U$(the new-component of N).

The following proposition is true

(58) If N is non elementary, then the U-chosen formula of $N \in$ the new-component of N.

Let us consider w, let us consider v, let us consider U, and let us consider N. The U-chosen successor of N w.r.t. w, v yields a strict LTL-node over v and is defined by:

(Def. 35) The U-chosen successor of N w.r.t. w, v

$$= \begin{cases} \text{SuccNode}_1(\text{the } U\text{-chosen formula of } N, N), \\ \quad \text{if the } U\text{-chosen formula of } N \text{does not have } \textit{until} \text{ operator and} \\ \quad w \models \cdot\, \text{SuccNode}_1(\text{the } U\text{-chosen formula of } N, N) \text{ or} \\ \quad \text{the } U\text{-chosen formula of } N \text{ has } \textit{until} \text{ operator and} \\ \quad w \not\models \text{RightArg}(\text{the } U\text{-chosen formula of } N), \\ \text{SuccNode}_2(\text{the } U\text{-chosen formula of } N,\ N), \text{ otherwise.} \end{cases}$$

One can prove the following propositions:

(59) Suppose $w \models \cdot N$ and N is non elementary. Then

(i) $w \models \cdot$(the U-chosen successor of N w.r.t. w, v), and

(ii) the U-chosen successor of N w.r.t. w, v is a successor of N.

(60) Suppose $w \models \cdot N$ and N is non elementary. Suppose the U-chosen formula of N has *until* operator and $w \models$ RightArg(the U-chosen formula of N).

Then

(i) RightArg(the U-chosen formula of N) $\in$ the new-component of the U-chosen successor of N w.r.t. w, v or RightArg(the U-chosen formula of N) $\in$ the old-component of N, and

(ii) the U-chosen formula of $N \in$ the old-component of the U-chosen successor of N w.r.t. w, v.

(61) Suppose $w \models \cdot N$ and N is non elementary. Then

(i) the old-component of $N \subseteq$ the old-component of the U-chosen successor of N w.r.t. w, v, and

(ii) the next-component of $N \subseteq$ the next-component of the U-chosen successor of N w.r.t. w, v.

Let us consider w, let us consider v, and let us consider U. The U-choice successor function w.r.t. w, v yielding a function from $\mathrm{Nodes}_{\mathrm{LTL}}\, v$ into $\mathrm{Nodes}_{\mathrm{LTL}}\, v$ is defined by the condition (Def. 36).

(Def. 36) Let given x. Suppose $x \in \mathrm{Nodes}_{\mathrm{LTL}}\, v$. Then (the U-choice successor function w.r.t. w, v)(x) = the U-chosen successor of CastNode(x, v) w.r.t. w, v.

We now state the proposition

(62) The U-choice successor function w.r.t. w, v is a successor homomorphism from v to w.

2. Negation Inner most LTL

Let us consider H. We say that H is negation-inner-most if and only if:

(Def. 37) For every LTL-formula G such that G is a subformula of H holds if G is negative, then Arg(G) is atomic.

Let us observe that there exists an LTL-formula which is negation-inner-most.

Let us consider H. We say that H is sub-atomic if and only if:

(Def. 38) H is atomic or there exists an LTL-formula G such that G is atomic and $H = \neg G$.

Next we state several propositions:

(63) If H is negation-inner-most and F is a subformula of H, then F is negation-inner-most.

(64) H is sub-atomic iff H is atomic or H is negative and Arg(H) is atomic.

(65) Suppose H is negation-inner-most. Then H is either sub-atomic, or conjunctive, or disjunctive, or has *next* operator, or *until* operator, or *release* operator.

(66) If H is negation-inner-most and has *next* operator, then Arg(H) is negation-inner-most.

(67) Suppose that
(i) H is conjunctive, or
(ii) H is disjunctive, or
(iii) H is negation-inner-most.
Then $\mathrm{LeftArg}(H)$ is negation-inner-most and $\mathrm{RightArg}(H)$ is negation-inner-most.

3. Definition of Buchi Automaton and Verification of the Main Theorem

Let W be a non empty set. We consider Buchi automatons over W as systems
$\langle$ a carrier, a transition, an initial state, final states $\rangle$,
where the carrier is a set, the transition is a relation between the carrier$\times W$ and the carrier, the initial state is an element of $2^{\text{the carrier}}$, and the final states constitute a subset of $2^{\text{the carrier}}$.

Let W be a non empty set, let B be a Buchi automaton over W, and let w be an element of the infinite sequences of W. We say that w is accepted by B if and only if the condition (Def. 39) is satisfied.

(Def. 39) There exists a sequence r_1 of the carrier of B such that
(i) $r_1(0) \in$ the initial state of B, and
(ii) for every natural number i holds $\langle\langle r_1(i), (\mathrm{CastSeq}(w, W))(i)\rangle, r_1(i+1)\rangle \in$ the transition of B and for every set F_1 such that $F_1 \in$ the final states of B holds $\{k \in \mathbb{N}: r_1(k) \in F_1\}$ is an infinite set.

For simplicity, we use the following convention: v denotes a negation-inner-most LTL-formula, U denotes a choice function of $2_{+}^{\mathrm{Subformulae}\, v}$, N denotes a strict LTL-node over v, and s, s_1 denote elementary strict LTL-nodes over v.

Let us consider v and let us consider N. The functor $\mathrm{atomic}_{\mathrm{LTL}}\, N$ yields a subset of $\mathrm{WFF}_{\mathrm{LTL}}$ and is defined by:

(Def. 40) $\mathrm{atomic}_{\mathrm{LTL}}\, N = \{x; x$ ranges over LTL-formulae: x is atomic $\wedge\ x \in$ the old-component of $N\}$.

The functor $\mathrm{NegAtomic}_{\mathrm{LTL}}\, N$ yields a subset of $\mathrm{WFF}_{\mathrm{LTL}}$ and is defined as follows:

(Def. 41) $\mathrm{NegAtomic}_{\mathrm{LTL}}\, N = \{x; x$ ranges over LTL-formulae: x is atomic $\wedge \neg x \in$ the old-component of $N\}$.

Let us consider v and let us consider N. The functor $\mathrm{Label}\, N$ yielding a set is defined by:

(Def. 42) $\mathrm{Label}\, N = \{x \subseteq \mathrm{atomic}_{\mathrm{LTL}}$: $\mathrm{atomic}_{\mathrm{LTL}}\, N \subseteq x \ \wedge\ \mathrm{NegAtomic}_{\mathrm{LTL}}\, N$ misses $x\}$.

Let us consider v. The functor $\mathrm{Tran}_{\mathrm{LTL}}\, v$ yields a relation between $\mathrm{States}_{\mathrm{LTL}}\, v \times \mathrm{AtomicFamily}$ and $\mathrm{States}_{\mathrm{LTL}}\, v$ and is defined as follows:

(Def. 43) $\mathrm{Tran}_{\mathrm{LTL}}\, v = \{y \in \mathrm{States}_{\mathrm{LTL}}\, v \times \mathrm{AtomicFamily} \times \mathrm{States}_{\mathrm{LTL}}\, v : \bigvee_{s,s_1,x}\ (y = \langle\langle s,\, x\rangle,\, s_1\rangle \ \wedge\ s_1$ is next to $s \ \wedge\ x \in \mathrm{Label}\, s_1)\}$.

The functor $\mathrm{InitS}_{\mathrm{LTL}}\, v$ yielding an element of $2^{\mathrm{States}_{\mathrm{LTL}}\, v}$ is defined as follows:

(Def. 44) $\mathrm{InitS}_{\mathrm{LTL}}\, v = \{\mathrm{init}\, v\}$.

Let us consider v and let us consider F. The functor $\mathrm{FinalS}_{\mathrm{LTL}}(F, v)$ yields an element of $2^{\mathrm{States}_{\mathrm{LTL}}\, v}$ and is defined as follows:

(Def. 45) $\mathrm{FinalS}_{\mathrm{LTL}}(F, v) = \{x \in \mathrm{States}_{\mathrm{LTL}}\, v : F \notin$ the old-component of $\mathrm{CastNode}(x, v) \ \vee\ \mathrm{RightArg}(F) \in$ the old-component of $\mathrm{CastNode}(x, v)\}$.

Let us consider v. The functor $\mathrm{FinalS}_{\mathrm{LTL}}\, v$ yields a subset of $2^{\mathrm{States}_{\mathrm{LTL}}\, v}$ and is defined by:

(Def. 46) $\mathrm{FinalS}_{\mathrm{LTL}}\, v = \{x \in 2^{\mathrm{States}_{\mathrm{LTL}}\, v} : \bigvee_F\ (F$ is a subformula of $v \ \wedge\ F$ has *until* operator $\wedge\ x = \mathrm{FinalS}_{\mathrm{LTL}}(F, v))\}$.

Let us consider v. The functor $\mathrm{BAutomaton}\, v$ yields a Buchi automaton over AtomicFamily and is defined as follows:

(Def. 47) $\mathrm{BAutomaton}\, v = \langle \mathrm{States}_{\mathrm{LTL}}\, v, \mathrm{Tran}_{\mathrm{LTL}}\, v, \mathrm{InitS}_{\mathrm{LTL}}\, v, \mathrm{FinalS}_{\mathrm{LTL}}\, v\rangle$.

The following proposition is true

(68) If w is accepted by $\mathrm{BAutomaton}\, v$, then $w \models v$.

Let us consider w, let us consider v, let us consider U, and let us consider N. Let us assume that N is non elementary and $w \models \cdot N$. The U-chosen successor end number of N w.r.t. w, v yields an element of $\mathbb{N}$ and is defined by the conditions (Def. 48).

(Def. 48)(i) For every i such that $i <$ the U-chosen successor end number of N w.r.t. w, v holds CastNode((the U-choice successor function w.r.t. w, $v)^i(N), v)$ is non elementary and CastNode((the U-choice successor function w.r.t. w, $v)^{i+1}(N), v)$ is a successor of CastNode((the U-choice successor function w.r.t. w, $v)^i(N), v)$,

(ii) CastNode((the U-choice successor function w.r.t. w, $v)^{\text{the } U\text{-chosen successor end number of } N \text{ w.r.t. } w,\ v}(N), v)$ is elementary, and

(iii) for every i such that $i \leq$ the U-chosen successor end number of N w.r.t. w, v holds $w \models \cdot$ CastNode((the U-choice successor function w.r.t. w, $v)^i(N), v)$.

Let us consider w, let us consider v, let us consider U, and let us consider N. Let us assume that $w \models \cdot \mathcal{X} N$. The U-chosen next node to N w.r.t. w, v yielding an elementary strict LTL-node over v is defined by:

(Def. 49) The U-chosen next node to N w.r.t. w, v

$$= \begin{cases} \text{CastNode((the } U\text{-choice successor function w.r.t. } w, \\ \quad v)^{\text{the } U\text{-chosen successor end number of } \mathcal{X} N \text{ w.r.t. } w,\ v}(\mathcal{X} N), v), \\ \quad \text{if } \mathcal{X} N \text{ is non elementary}, \\ \mathrm{FinalNode}\, v, \ \text{otherwise}. \end{cases}$$

One can prove the following proposition

(69) Suppose $w \models \cdot \mathcal{X}\, s$. Then the U-chosen next node to s w.r.t. w, v is next to s and $w \models \cdot$(the U-chosen next node to s w.r.t. w, v).

Let us consider w, let us consider v, and let us consider U. The U-chosen run w.r.t. w, v yields a sequence of $\mathrm{States}_{\mathrm{LTL}}\, v$ and is defined by the conditions (Def. 50).

(Def. 50)(i) (The U-chosen run w.r.t. w, v)$(0) = \mathrm{init}\, v$, and

(ii) for every n holds (the U-chosen run w.r.t. w, v)$(n+1)$ = the U-chosen next node to CastNode((the U-chosen run w.r.t. w, v)$(n), v)$ w.r.t. $\mathrm{Shift}(w, n)$, v.

The following propositions are true:

(70) If $w \models \cdot N$, then $\mathrm{Shift}(w, 1) \models \cdot \mathcal{X}\, N$.

(71) If $w \models \mathcal{X}\, v$, then $w \models \cdot\, \mathrm{init}\, v$.

(72) $w \models v$ iff $w \models \cdot \mathcal{X}\, \mathrm{init}\, v$.

(73) Suppose $w \models v$. Let given n. Then

(i) CastNode((the U-chosen run w.r.t. w, v)$(n+1), v)$ is next to CastNode((the U-chosen run w.r.t. w, v)$(n), v)$, and

(ii) $\mathrm{Shift}(w, n) \models \cdot \mathcal{X}$ CastNode((the U-chosen run w.r.t. w, v)$(n), v)$.

(74) Suppose $w \models v$. Let given i. Suppose $H \in$ the old-component of CastNode((the U-chosen run w.r.t. w, v)$(i+1), v)$ and H has *until* operator and $\mathrm{Shift}(w, i) \models \mathrm{RightArg}(H)$. Then $\mathrm{RightArg}(H) \in$ the old-component of CastNode((the U-chosen run w.r.t. w, v)$(i+1), v)$.

(75) w is accepted by BAutomaton v iff $w \models v$.

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Basic Properties of Circulant Matrices and Anti-Circular Matrices

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Summary. This article introduces definitions of circulant matrices, line- and column-circulant matrices as well as anti-circular matrices and describes their main properties.

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The articles [6], [9], [4], [10], [1], [14], [13], [2], [5], [8], [12], [11], [3], and [7] provide the notation and terminology for this paper.

1. Some Properties of Circulant Matrices

For simplicity, we adopt the following convention: i, j, k, n, l denote elements of $\mathbb{N}$, K denotes a field, a, b, c denote elements of K, p, q denote finite sequences of elements of K, and M_1, M_2, M_3 denote square matrices over K of dimension n.

Next we state two propositions:

(1) $\mathbf{1}_K \cdot p = p$.

(2) $(-\mathbf{1}_K) \cdot p = -p$.

Let K be a set, let M be a matrix over K, and let p be a finite sequence. We say that M is line circulant about p if and only if:

(Def. 1) $\operatorname{len} p = \operatorname{width} M$ and for all natural numbers i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} = p(((j - i) \bmod \operatorname{len} p) + 1)$.

Let K be a set and let M be a matrix over K. We say that M is line circulant if and only if:

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(Def. 2) There exists a finite sequence p of elements of K such that $\operatorname{len} p = \operatorname{width} M$ and M is line circulant about p.

Let K be a non empty set and let p be a finite sequence of elements of K. We say that p is first-line-of-circulant if and only if:

(Def. 3) There exists a square matrix over K of dimension $\operatorname{len} p$ which is line circulant about p.

Let K be a set, let M be a matrix over K, and let p be a finite sequence. We say that M is column circulant about p if and only if:

(Def. 4) $\operatorname{len} p = \operatorname{len} M$ and for all natural numbers i, j such that $\langle i, j\rangle \in$ the indices of M holds $M_{i,j} = p(((i - j) \bmod \operatorname{len} p) + 1)$.

Let K be a set and let M be a matrix over K. We say that M is column circulant if and only if:

(Def. 5) There exists a finite sequence p of elements of K such that $\operatorname{len} p = \operatorname{len} M$ and M is column circulant about p.

Let K be a non empty set and let p be a finite sequence of elements of K. We say that p is first-column-of-circulant if and only if:

(Def. 6) There exists a square matrix over K of dimension $\operatorname{len} p$ which is column circulant about p.

Let K be a non empty set and let p be a finite sequence of elements of K. Let us assume that p is first-line-of-circulant. The functor $\operatorname{LCirc} p$ yields a square matrix over K of dimension $\operatorname{len} p$ and is defined by:

(Def. 7) $\operatorname{LCirc} p$ is line circulant about p.

Let K be a non empty set and let p be a finite sequence of elements of K. Let us assume that p is first-column-of-circulant. The functor $\operatorname{CCirc} p$ yielding a square matrix over K of dimension $\operatorname{len} p$ is defined by:

(Def. 8) $\operatorname{CCirc} p$ is column circulant about p.

Let K be a field. One can verify that there exists a finite sequence of elements of K which is first-line-of-circulant and first-column-of-circulant.

Let us consider K, n. Observe that $0_K^{n \times n}$ is line circulant and column circulant.

Let us consider K, let us consider n, and let a be an element of K. Observe that $(a)^{n \times n}$ is line circulant and $(a)^{n \times n}$ is column circulant.

Let us consider K. Note that there exists a matrix over K which is line circulant and column circulant.

In the sequel D denotes a non empty set, t denotes a finite sequence of elements of D, and A denotes a square matrix over D of dimension n.

We now state a number of propositions:

(3) If A is line circulant and $n > 0$, then A^{T} is column circulant.

(4) If A is line circulant about t and $n > 0$, then $t = \operatorname{Line}(A, 1)$.

(5) If A is line circulant and $\langle i, j\rangle \in \operatorname{Seg} n \times \operatorname{Seg} n$ and $k = i+1$ and $l = j+1$ and $i < n$ and $j < n$, then $A_{i,j} = A_{k,l}$.

(6) If M_1 is line circulant, then $a \cdot M_1$ is line circulant.

(7) If M_1 is line circulant and M_2 is line circulant, then $M_1 + M_2$ is line circulant.

(8) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $M_1 + M_2 + M_3$ is line circulant.

(9) If M_1 is line circulant and M_2 is line circulant, then $a \cdot M_1 + b \cdot M_2$ is line circulant.

(10) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $a \cdot M_1 + b \cdot M_2 + c \cdot M_3$ is line circulant.

(11) If M_1 is line circulant, then $-M_1$ is line circulant.

(12) If M_1 is line circulant and M_2 is line circulant, then $M_1 - M_2$ is line circulant.

(13) If M_1 is line circulant and M_2 is line circulant, then $a \cdot M_1 - b \cdot M_2$ is line circulant.

(14) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $(a \cdot M_1 + b \cdot M_2) - c \cdot M_3$ is line circulant.

(15) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $a \cdot M_1 - b \cdot M_2 - c \cdot M_3$ is line circulant.

(16) If M_1 is line circulant and M_2 is line circulant and M_3 is line circulant, then $(a \cdot M_1 - b \cdot M_2) + c \cdot M_3$ is line circulant.

(17) If A is column circulant and $n > 0$, then A^{T} is line circulant.

(18) If A is column circulant about t and $n > 0$, then $t = A_{\square,1}$.

(19) If A is column circulant and $\langle i, j\rangle \in \operatorname{Seg} n \times \operatorname{Seg} n$ and $k = i + 1$ and $l = j + 1$ and $i < n$ and $j < n$, then $A_{i,j} = A_{k,l}$.

(20) If M_1 is column circulant, then $a \cdot M_1$ is column circulant.

(21) If M_1 is column circulant and M_2 is column circulant, then $M_1 + M_2$ is column circulant.

(22) If M_1 is column circulant and M_2 is column circulant and M_3 is column circulant, then $M_1 + M_2 + M_3$ is column circulant.

(23) If M_1 is column circulant and M_2 is column circulant, then $a \cdot M_1 + b \cdot M_2$ is column circulant.

(24) Suppose M_1 is column circulant and M_2 is column circulant and M_3 is column circulant. Then $a \cdot M_1 + b \cdot M_2 + c \cdot M_3$ is column circulant.

(25) If M_1 is column circulant, then $-M_1$ is column circulant.

(26) If M_1 is column circulant and M_2 is column circulant, then $M_1 - M_2$ is column circulant.

(27) If M_1 is column circulant and M_2 is column circulant, then $a \cdot M_1 - b \cdot M_2$ is column circulant.

(28) Suppose M_1 is column circulant and M_2 is column circulant and M_3 is column circulant. Then $(a \cdot M_1 + b \cdot M_2) - c \cdot M_3$ is column circulant.

(29) Suppose M_1 is column circulant and M_2 is column circulant and M_3 is column circulant. Then $a \cdot M_1 - b \cdot M_2 - c \cdot M_3$ is column circulant.

(30) Suppose M_1 is column circulant and M_2 is column circulant and M_3 is column circulant. Then $(a \cdot M_1 - b \cdot M_2) + c \cdot M_3$ is column circulant.

(31) If p is first-line-of-circulant, then $-p$ is first-line-of-circulant.

(32) If p is first-line-of-circulant, then $\mathrm{LCirc}(-p) = -\mathrm{LCirc}\, p$.

(33) Suppose p is first-line-of-circulant and q is first-line-of-circulant and $\mathrm{len}\, p = \mathrm{len}\, q$. Then $p + q$ is first-line-of-circulant.

(34) If $\mathrm{len}\, p = \mathrm{len}\, q$ and p is first-line-of-circulant and q is first-line-of-circulant, then $\mathrm{LCirc}(p + q) = \mathrm{LCirc}\, p + \mathrm{LCirc}\, q$.

(35) If p is first-column-of-circulant, then $-p$ is first-column-of-circulant.

(36) For every finite sequence p of elements of K such that p is first-column-of-circulant holds $\mathrm{CCirc}(-p) = -\mathrm{CCirc}\, p$.

(37) Suppose p is first-column-of-circulant and q is first-column-of-circulant and $\mathrm{len}\, p = \mathrm{len}\, q$. Then $p + q$ is first-column-of-circulant.

(38) If $\mathrm{len}\, p = \mathrm{len}\, q$ and p is first-column-of-circulant and q is first-column-of-circulant, then $\mathrm{CCirc}(p + q) = \mathrm{CCirc}\, p + \mathrm{CCirc}\, q$.

(39) If $n > 0$, then $I_K^{n \times n}$ is column circulant.

(40) If $n > 0$, then $I_K^{n \times n}$ is line circulant.

(41) If p is first-line-of-circulant, then $a \cdot p$ is first-line-of-circulant.

(42) If p is first-line-of-circulant, then $\mathrm{LCirc}(a \cdot p) = a \cdot \mathrm{LCirc}\, p$.

(43) If p is first-line-of-circulant, then $a \cdot \mathrm{LCirc}\, p + b \cdot \mathrm{LCirc}\, p = \mathrm{LCirc}((a+b) \cdot p)$.

(44) If p is first-line-of-circulant and q is first-line-of-circulant and $\mathrm{len}\, p = \mathrm{len}\, q$ and $\mathrm{len}\, p > 0$, then $a \cdot \mathrm{LCirc}\, p + a \cdot \mathrm{LCirc}\, q = \mathrm{LCirc}(a \cdot (p + q))$.

(45) If p is first-line-of-circulant and q is first-line-of-circulant and $\mathrm{len}\, p = \mathrm{len}\, q$, then $a \cdot \mathrm{LCirc}\, p + b \cdot \mathrm{LCirc}\, q = \mathrm{LCirc}(a \cdot p + b \cdot q)$.

(46) If p is first-column-of-circulant, then $a \cdot p$ is first-column-of-circulant.

(47) If p is first-column-of-circulant, then $\mathrm{CCirc}(a \cdot p) = a \cdot \mathrm{CCirc}\, p$.

(48) If p is first-column-of-circulant, then $a \cdot \mathrm{CCirc}\, p + b \cdot \mathrm{CCirc}\, p = \mathrm{CCirc}((a + b) \cdot p)$.

(49) Suppose p is first-column-of-circulant and q is first-column-of-circulant and $\mathrm{len}\, p = \mathrm{len}\, q$ and $\mathrm{len}\, p > 0$. Then $a \cdot \mathrm{CCirc}\, p + a \cdot \mathrm{CCirc}\, q = \mathrm{CCirc}(a \cdot (p + q))$.

(50) If p is first-column-of-circulant and q is first-column-of-circulant and $\operatorname{len} p = \operatorname{len} q$, then $a \cdot \mathrm{CCirc}\, p + b \cdot \mathrm{CCirc}\, q = \mathrm{CCirc}(a \cdot p + b \cdot q)$.

Let K be a set and let M be a matrix over K. We introduce M is circulant as a synonym of M is line circulant.

2. Some Properties of Anti-circular Matrices

Let K be a field, let M_1 be a matrix over K, and let p be a finite sequence of elements of K. We say that M_1 is anti-circular about p if and only if the conditions (Def. 9) are satisfied.

(Def. 9)(i) $\operatorname{len} p = \operatorname{width} M_1$,

(ii) for all natural numbers i, j such that $\langle i, j\rangle \in$ the indices of M_1 and $i \leq j$ holds $(M_1)_{i,j} = p(((j-i) \bmod \operatorname{len} p) + 1)$, and

(iii) for all natural numbers i, j such that $\langle i, j\rangle \in$ the indices of M_1 and $i \geq j$ holds $(M_1)_{i,j} = (-p)(((j-i) \bmod \operatorname{len} p) + 1)$.

Let K be a field and let M be a matrix over K. We say that M is anti-circular if and only if:

(Def. 10) There exists a finite sequence p of elements of K such that $\operatorname{len} p = \operatorname{width} M$ and M is anti-circular about p.

Let K be a field and let p be a finite sequence of elements of K. We say that p is first-line-of-anti-circular if and only if:

(Def. 11) There exists a square matrix over K of dimension $\operatorname{len} p$ which is anti-circular about p.

Let K be a field and let p be a finite sequence of elements of K. Let us assume that p is first-line-of-anti-circular. The functor $\mathrm{ACirc}\, p$ yields a square matrix over K of dimension $\operatorname{len} p$ and is defined by:

(Def. 12) $\mathrm{ACirc}\, p$ is anti-circular about p.

One can prove the following propositions:

(51) If M_1 is anti-circular, then $a \cdot M_1$ is anti-circular.

(52) If M_1 is anti-circular and M_2 is anti-circular, then $M_1 + M_2$ is anti-circular.

(53) Let K be a Fanoian field, n, i, j be natural numbers, and M_1 be a square matrix over K of dimension n. Suppose $\langle i, j\rangle \in$ the indices of M_1 and $i = j$ and M_1 is anti-circular. Then $(M_1)_{i,j} = 0_K$.

(54) If M_1 is anti-circular and $\langle i, j\rangle \in \operatorname{Seg} n \times \operatorname{Seg} n$ and $k = i+1$ and $l = j+1$ and $i < n$ and $j < n$, then $(M_1)_{k,l} = (M_1)_{i,j}$.

(55) If M_1 is anti-circular, then $-M_1$ is anti-circular.

(56) If M_1 is anti-circular and M_2 is anti-circular, then $M_1 - M_2$ is anti-circular.

(57) If M_1 is anti-circular about p and $n > 0$, then $p = \mathrm{Line}(M_1, 1)$.

(58) If p is first-line-of-anti-circular, then $-p$ is first-line-of-anti-circular.

(59) If p is first-line-of-anti-circular, then $\mathrm{ACirc}(-p) = -\mathrm{ACirc}\, p$.

(60) Suppose p is first-line-of-anti-circular and q is first-line-of-anti-circular and $\mathrm{len}\, p = \mathrm{len}\, q$. Then $p + q$ is first-line-of-anti-circular.

(61) If p is first-line-of-anti-circular and q is first-line-of-anti-circular and $\mathrm{len}\, p = \mathrm{len}\, q$, then $\mathrm{ACirc}(p + q) = \mathrm{ACirc}\, p + \mathrm{ACirc}\, q$.

(62) If p is first-line-of-anti-circular, then $a \cdot p$ is first-line-of-anti-circular.

(63) If p is first-line-of-anti-circular, then $\mathrm{ACirc}(a \cdot p) = a \cdot \mathrm{ACirc}\, p$.

(64) If p is first-line-of-anti-circular, then $a \cdot \mathrm{ACirc}\, p + b \cdot \mathrm{ACirc}\, p = \mathrm{ACirc}((a + b) \cdot p)$.

(65) Suppose p is first-line-of-anti-circular and q is first-line-of-anti-circular and $\mathrm{len}\, p = \mathrm{len}\, q$ and $\mathrm{len}\, p > 0$. Then $a \cdot \mathrm{ACirc}\, p + a \cdot \mathrm{ACirc}\, q = \mathrm{ACirc}(a \cdot (p + q))$.

(66) Suppose p is first-line-of-anti-circular and q is first-line-of-anti-circular and $\mathrm{len}\, p = \mathrm{len}\, q$. Then $a \cdot \mathrm{ACirc}\, p + b \cdot \mathrm{ACirc}\, q = \mathrm{ACirc}(a \cdot p + b \cdot q)$.

Let us consider K, n. Observe that $0_K^{n \times n}$ is anti-circular.

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On L^1 Space Formed by Real-Valued Partial Functions

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Summary. This article contains some definitions and properties refering to function spaces formed by partial functions defined over a measurable space. We formalized a function space, the so-called L^1 space and proved that the space turns out to be a normed space. The formalization of a real function space was given in [16]. The set of all function forms additive group. Here addition is defined by point-wise addition of two functions. However it is not true for partial functions. The set of partial functions does not form an additive group due to lack of right zeroed condition. Therefore, firstly we introduced a kind of a quasi-linear space, then, we introduced the definition of an equivalent relation of two functions which are almost everywhere equal ($=_{\rm a.e.}$), thirdly we formalized a linear space by taking the quotient of a quasi-linear space by the relation ($=_{\rm a.e.}$).

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The papers [11], [24], [4], [5], [3], [8], [25], [10], [9], [14], [7], [20], [13], [23], [22], [1], [17], [21], [18], [15], [6], [12], [19], and [2] provide the notation and terminology for this paper.

1. Preliminaries of Real Linear Space

Let V be a non empty RLS structure and let V_1 be a subset of V. We say that V_1 is multiplicatively-closed if and only if:

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(Def. 1) For every real number a and for every vector v of V such that $v \in V_1$ holds $a \cdot v \in V_1$.

The following proposition is true

(1) Let V be a real linear space and V_1 be a subset of V. Then V_1 is linearly closed if and only if V_1 is add closed and multiplicatively-closed.

Let V be a non empty RLS structure. Observe that there exists a subset of V which is add closed, multiplicatively-closed, and non empty.

Let X be a non empty RLS structure and let X_1 be a multiplicatively-closed non empty subset of X. The functor $\cdot_{(X_1)}$ yields a function from $\mathbb{R} \times X_1$ into X_1 and is defined by:

(Def. 2) $\cdot_{(X_1)} =$ (the external multiplication of X)$\restriction(\mathbb{R} \times X_1)$.

In the sequel a, b, r denote real numbers.

Next we state four propositions:

(2) Let V be an Abelian add-associative right zeroed real linear space-like non empty RLS structure, V_1 be a non empty subset of V, d_1 be an element of V_1, A be a binary operation on V_1, and M be a function from $\mathbb{R} \times V_1$ into V_1. Suppose $d_1 = 0_V$ and $A =$ (the addition of V) $\restriction$ (V_1) and $M =$ (the external multiplication of V)$\restriction(\mathbb{R} \times V_1)$. Then $\langle V_1, d_1, A, M\rangle$ is Abelian, add-associative, right zeroed, and real linear space-like.

(3) Let V be an Abelian add-associative right zeroed real linear space-like non empty RLS structure and V_1 be an add closed multiplicatively-closed non empty subset of V. Suppose $0_V \in V_1$. Then $\langle V_1, 0_V(\in V_1), \mathrm{add}\,|(V_1, V), \cdot_{(V_1)}\rangle$ is Abelian, add-associative, right zeroed, and real linear space-like.

(4) Let V be a non empty RLS structure, V_1 be an add closed multiplicatively-closed non empty subset of V, v, u be vectors of V, and w_1, w_2 be vectors of $\langle V_1, 0_V(\in V_1), \mathrm{add}\,|(V_1, V), \cdot_{(V_1)}\rangle$. If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(5) Let V be a non empty RLS structure, V_1 be an add closed multiplicatively-closed non empty subset of V, a be a real number, v be a vector of V, and w be a vector of $\langle V_1, 0_V(\in V_1), \mathrm{add}\,|(V_1, V), \cdot_{(V_1)}\rangle$. If $w = v$, then $a \cdot w = a \cdot v$.

2. Quasi-Real Linear Space of Partial Functions

We adopt the following convention: A, B denote non empty sets and f, g, h denote elements of $A \dot{\to} \mathbb{R}$.

Let us consider A, B, let F be a binary operation on $A \dot{\to} B$, and let f, g be elements of $A \dot{\to} B$. Then $F(f, g)$ is an element of $A \dot{\to} B$.

Let us consider A. The functor $\cdot_{A \dot{\to} \mathbb{R}}$ yielding a binary operation on $A \dot{\to} \mathbb{R}$ is defined as follows:

(Def. 3) For all elements f, g of $A \dot{\to} \mathbb{R}$ holds $\cdot_{A \dot{\to} \mathbb{R}}(f, g) = f\, g$.

Let us consider A. The functor $\cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}$ yielding a function from $\mathbb{R} \times (A \dot{\to} \mathbb{R})$ into $A \dot{\to} \mathbb{R}$ is defined as follows:

(Def. 4) For every real number a and for every element f of $A \dot{\to} \mathbb{R}$ holds $\cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a, f) = a\, f$.

Let us consider A. The functor $0_{A \dot{\to} \mathbb{R}}$ yielding an element of $A \dot{\to} \mathbb{R}$ is defined as follows:

(Def. 5) $0_{A \dot{\to} \mathbb{R}} = A \longmapsto 0$.

Let us consider A. The functor $1_{A \dot{\to} \mathbb{R}}$ yields an element of $A \dot{\to} \mathbb{R}$ and is defined as follows:

(Def. 6) $1_{A \dot{\to} \mathbb{R}} = A \longmapsto 1$.

The following propositions are true:

(6) $h = +_{A \dot{\to} \mathbb{R}}(f, g)$ iff $\mathrm{dom}\, h = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and for every element x of A such that $x \in \mathrm{dom}\, h$ holds $h(x) = f(x) + g(x)$.

(7) $h = \cdot_{A \dot{\to} \mathbb{R}}(f, g)$ iff $\mathrm{dom}\, h = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and for every element x of A such that $x \in \mathrm{dom}\, h$ holds $h(x) = f(x) \cdot g(x)$.

(8) $0_{A \dot{\to} \mathbb{R}} \neq 1_{A \dot{\to} \mathbb{R}}$.

(9) $h = \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a, f)$ iff $\mathrm{dom}\, h = \mathrm{dom}\, f$ and for every element x of A such that $x \in \mathrm{dom}\, f$ holds $h(x) = a \cdot f(x)$.

(10) $+_{A \dot{\to} \mathbb{R}}(f, g) = +_{A \dot{\to} \mathbb{R}}(g, f)$.

(11) $+_{A \dot{\to} \mathbb{R}}(f, +_{A \dot{\to} \mathbb{R}}(g, h)) = +_{A \dot{\to} \mathbb{R}}(+_{A \dot{\to} \mathbb{R}}(f, g), h)$.

(12) $\cdot_{A \dot{\to} \mathbb{R}}(f, g) = \cdot_{A \dot{\to} \mathbb{R}}(g, f)$.

(13) $\cdot_{A \dot{\to} \mathbb{R}}(f, \cdot_{A \dot{\to} \mathbb{R}}(g, h)) = \cdot_{A \dot{\to} \mathbb{R}}(\cdot_{A \dot{\to} \mathbb{R}}(f, g), h)$.

(14) $\cdot_{A \dot{\to} \mathbb{R}}(1_{A \dot{\to} \mathbb{R}}, f) = f$.

(15) $+_{A \dot{\to} \mathbb{R}}(0_{A \dot{\to} \mathbb{R}}, f) = f$.

(16) $+_{A \dot{\to} \mathbb{R}}(f, \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(-1, f)) = 0_{A \dot{\to} \mathbb{R}} \restriction \mathrm{dom}\, f$.

(17) $\cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(1, f) = f$.

(18) $\cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a, \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(b, f)) = \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a \cdot b, f)$.

(19) $+_{A \dot{\to} \mathbb{R}}(\cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a, f), \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(b, f)) = \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a + b, f)$.

(20) $\cdot_{A \dot{\to} \mathbb{R}}(f, +_{A \dot{\to} \mathbb{R}}(g, h)) = +_{A \dot{\to} \mathbb{R}}(\cdot_{A \dot{\to} \mathbb{R}}(f, g), \cdot_{A \dot{\to} \mathbb{R}}(f, h))$.

(21) $\cdot_{A \dot{\to} \mathbb{R}}(\cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a, f), g) = \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}}(a, \cdot_{A \dot{\to} \mathbb{R}}(f, g))$.

Let us consider A. The functor $\mathrm{PFunct}_{\mathrm{RLS}}\, A$ yields a non empty RLS structure and is defined by:

(Def. 7) $\mathrm{PFunct}_{\mathrm{RLS}}\, A = \langle A \dot{\to} \mathbb{R}, 0_{A \dot{\to} \mathbb{R}}, +_{A \dot{\to} \mathbb{R}}, \cdot^{\mathbb{R}}_{A \dot{\to} \mathbb{R}} \rangle$.

Let us consider A. One can verify that $\mathrm{PFunct}_{\mathrm{RLS}}\, A$ is strict, Abelian, add-associative, right zeroed, and real linear space-like.

3. Quasi-Real Linear Space of Integrable Functions

For simplicity, we use the following convention: X is a non empty set, x is an element of X, S is a σ-field of subsets of X, M is a σ-measure on S, E is an element of S, and f, g, h, f_1, g_1 are partial functions from X to $\mathbb{R}$.

Next we state the proposition

(22) Let given X, S, M and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose there exists E such that $E = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $0 = f(x)$. Then f is integrable on M and $\int f\, \mathrm{d}M = 0$.

Let X be a non empty set and let r be a real number. Then $X \longmapsto r$ is a partial function from X to $\mathbb{R}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The L^1 functions of M yielding a non empty subset of $\mathrm{PFunct}_{\mathrm{RLS}}\, X$ is defined by the condition (Def. 8).

(Def. 8) The L^1 functions of $M = \{f; f$ ranges over partial functions from X to $\mathbb{R}$: $\bigvee_{N_1 : \text{element of } S} (M(N_1) = 0 \land \mathrm{dom}\, f = {N_1}^{\mathrm{c}} \land f$ is integrable on $M)\}$.

We now state two propositions:

(23) Suppose $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M. Then $f + g \in$ the L^1 functions of M.

(24) If $f \in$ the L^1 functions of M, then $a\, f \in$ the L^1 functions of M.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. Observe that the L^1 functions of M is multiplicatively-closed and add closed.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor $L^1\text{-}\mathrm{Funct}_{\mathrm{RLS}} M$ yielding a non empty RLS structure is defined by the condition (Def. 9).

(Def. 9) $L^1\text{-}\mathrm{Funct}_{\mathrm{RLS}} M = \langle$the L^1 functions of M, $0_{\mathrm{PFunct}_{\mathrm{RLS}}\, X}(\in$ the L^1 functions of $M)$, add $|$(the L^1 functions of M, $\mathrm{PFunct}_{\mathrm{RLS}}\, X)$, $\cdot_{\text{the } L^1 \text{ functions of } M}\rangle$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. Observe that $L^1\text{-}\mathrm{Funct}_{\mathrm{RLS}} M$ is strict, Abelian, add-associative, right zeroed, and real linear space-like.

4. Quotient Space of Quasi-Real Linear Space of Integrable Functions

In the sequel v, u are vectors of $L^1\text{-}\mathrm{Funct}_{\mathrm{RLS}} M$.

Next we state two propositions:

(25) $(v) + (u) = v + u$.

(26) $a\,(u) = a \cdot u$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f, g be partial functions from X to $\mathbb{R}$. The predicate $f =^{M}_{\text{a.e.}} g$ is defined by:

(Def. 10) There exists an element E of S such that $M(E) = 0$ and $f{\restriction}E^{\rm c} = g{\restriction}E^{\rm c}$.

We now state several propositions:

(27) Suppose $f = u$. Then

(i) $u + (-1) \cdot u = (X \longmapsto 0){\restriction}\operatorname{dom} f$, and

(ii) there exist partial functions v, g from X to $\mathbb{R}$ such that $v \in$ the L^1 functions of M and $g \in$ the L^1 functions of M and $v = u + (-1) \cdot u$ and $g = X \longmapsto 0$ and $v =^{M}_{\text{a.e.}} g$.

(28) $f =^{M}_{\text{a.e.}} f$.

(29) If $f =^{M}_{\text{a.e.}} g$, then $g =^{M}_{\text{a.e.}} f$.

(30) If $f =^{M}_{\text{a.e.}} g$ and $g =^{M}_{\text{a.e.}} h$, then $f =^{M}_{\text{a.e.}} h$.

(31) If $f =^{M}_{\text{a.e.}} f_1$ and $g =^{M}_{\text{a.e.}} g_1$, then $f + g =^{M}_{\text{a.e.}} f_1 + g_1$.

(32) If $f =^{M}_{\text{a.e.}} g$, then $a\,f =^{M}_{\text{a.e.}} a\,g$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor AlmostZeroFunctions M yielding a non empty subset of L^1-Funct$_{\rm RLS}M$ is defined as follows:

(Def. 11) AlmostZeroFunctions $M = \{f; f$ ranges over partial functions from X to $\mathbb{R}$: $f \in$ the L^1 functions of $M \ \wedge\ f =^{M}_{\text{a.e.}} X \longmapsto 0\}$.

The following proposition is true

(33) $(X \longmapsto 0) + (X \longmapsto 0) = X \longmapsto 0$ and $a\,(X \longmapsto 0) = X \longmapsto 0$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. One can check that AlmostZeroFunctions M is add closed and multiplicatively-closed.

Next we state the proposition

(34) $0_{L^1\text{-Funct}_{\rm RLS}M} = X \longmapsto 0$ and $0_{L^1\text{-Funct}_{\rm RLS}M} \in$ AlmostZeroFunctions M.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor AlmostZeroFunct$_{\rm RLS}$ M yielding a non empty RLS structure is defined as follows:

(Def. 12) AlmostZeroFunct$_{\rm RLS}$ M = $\langle$AlmostZeroFunctions $M, 0_{L^1\text{-Funct}_{\rm RLS}M}(\in$ AlmostZeroFunctions $M)$, add $|($AlmostZeroFunctions M, L^1-Funct$_{\rm RLS}M)$, $\cdot_{\text{AlmostZeroFunctions}\,M}\rangle$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. Note that L^1-Funct$_{\rm RLS}M$ is strict, strict, Abelian, add-associative, right zeroed, and real linear space-like.

In the sequel v, u are vectors of AlmostZeroFunct$_{\rm RLS}$ M.

Next we state two propositions:

(35) $(v) + (u) = v + u$.

(36) $a\,(u) = a \cdot u$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\mathbb{R}$. The functor $[f]^{M}_{\text{a.e.}}$ yielding a subset of the L^1 functions of M is defined by the condition (Def. 13).

(Def. 13) $[f]^{M}_{\text{a.e.}} = \{g; g$ ranges over partial functions from X to $\mathbb{R}$: $g \in$ the L^1 functions of $M \ \wedge\ f \in$ the L^1 functions of $M \ \wedge\ f =^{M}_{\text{a.e.}} g\}$.

The following propositions are true:

(37) If $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M, then $g =^{M}_{\text{a.e.}} f$ iff $g \in [f]^{M}_{\text{a.e.}}$.

(38) If $f \in$ the L^1 functions of M, then $f \in [f]^{M}_{\text{a.e.}}$.

(39) If $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M, then $[f]^{M}_{\text{a.e.}} = [g]^{M}_{\text{a.e.}}$ iff $f =^{M}_{\text{a.e.}} g$.

(40) Suppose $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M. Then $[f]^{M}_{\text{a.e.}} = [g]^{M}_{\text{a.e.}}$ if and only if $g \in [f]^{M}_{\text{a.e.}}$.

(41) Suppose that

(i) $f \in$ the L^1 functions of M,

(ii) $f_1 \in$ the L^1 functions of M,

(iii) $g \in$ the L^1 functions of M,

(iv) $g_1 \in$ the L^1 functions of M,

(v) $[f]^{M}_{\text{a.e.}} = [f_1]^{M}_{\text{a.e.}}$, and

(vi) $[g]^{M}_{\text{a.e.}} = [g_1]^{M}_{\text{a.e.}}$.

Then $[f + g]^{M}_{\text{a.e.}} = [f_1 + g_1]^{M}_{\text{a.e.}}$.

(42) If $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M and $[f]^{M}_{\text{a.e.}} = [g]^{M}_{\text{a.e.}}$, then $[a\,f]^{M}_{\text{a.e.}} = [a\,g]^{M}_{\text{a.e.}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor CosetSet M yields a non empty family of subsets of the L^1 functions of M and is defined by:

(Def. 14) CosetSet $M = \{[f]^{M}_{\text{a.e.}}; f$ ranges over partial functions from X to $\mathbb{R}$: $f \in$ the L^1 functions of $M\}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor addCoset M yields a binary operation on CosetSet M and is defined by the condition (Def. 15).

(Def. 15) Let A, B be elements of CosetSet M and a, b be partial functions from X to $\mathbb{R}$. If $a \in A$ and $b \in B$, then $(\text{addCoset}\,M)(A, B) = [a + b]^{M}_{\text{a.e.}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor zeroCoset M yielding an element of CosetSet M is defined by:

(Def. 16) There exists a partial function f from X to $\mathbb{R}$ such that $f = X \longmapsto 0$ and $f \in$ the L^1 functions of M and zeroCoset $M = [f]^{M}_{\text{a.e.}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor lmultCoset M yields a function from $\mathbb{R} \times$ CosetSet M into CosetSet M and is defined by the condition (Def. 17).

(Def. 17) Let z be an element of $\mathbb{R}$, A be an element of CosetSet M, and f be a partial function from X to $\mathbb{R}$. If $f \in A$, then (lmultCoset M)$(z, A) = [z\,f]^{M}_{\text{a.e.}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor pre-L-Space M yields a strict Abelian add-associative right zeroed right complementable real linear space-like non empty RLS structure and is defined by the conditions (Def. 18).

(Def. 18)(i) The carrier of pre-L-Space M = CosetSet M,

(ii) the addition of pre-L-Space M = addCoset M,

(iii) $0_{\text{pre-}L\text{-Space}\,M}$ = zeroCoset M, and

(iv) the external multiplication of pre-L-Space M = lmultCoset M.

5. Real Normed Space of Integrable Functions

One can prove the following propositions:

(43) If $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M and $f =^{M}_{\text{a.e.}} g$, then $\int f \,\mathrm{d}M = \int g \,\mathrm{d}M$.

(44) If f is integrable on M, then $\int f \,\mathrm{d}M$, $\int |f| \,\mathrm{d}M \in \mathbb{R}$ and $|f|$ is integrable on M.

(45) Suppose $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M and $f =^{M}_{\text{a.e.}} g$. Then $|f| =^{M}_{\text{a.e.}} |g|$ and $\int |f| \,\mathrm{d}M = \int |g| \,\mathrm{d}M$.

(46) Given a vector x of pre-L-Space M such that f, $g \in x$. Then $f =^{M}_{\text{a.e.}} g$ and $f \in$ the L^1 functions of M and $g \in$ the L^1 functions of M.

(47) There exists a function N_2 from the carrier of pre-L-Space M into $\mathbb{R}$ such that for every point x of pre-L-Space M holds there exists a partial function f from X to $\mathbb{R}$ such that $f \in x$ and $N_2(x) = \int |f| \,\mathrm{d}M$.

In the sequel x is a point of pre-L-Space M.

The following two propositions are true:

(48) If $f \in x$, then f is integrable on M and $f \in$ the L^1 functions of M and $|f|$ is integrable on M.

(49) If $f, g \in x$, then $f =^{M}_{\text{a.e.}} g$ and $\int f \,\mathrm{d}M = \int g \,\mathrm{d}M$ and $\int |f| \,\mathrm{d}M = \int |g| \,\mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor L^1-Norm(M) yields a function from the carrier of pre-L-Space M into $\mathbb{R}$ and is defined by:

(Def. 19) For every point x of pre-L-Space M there exists a partial function f from X to $\mathbb{R}$ such that $f \in x$ and $(L^1\text{-Norm}(M))(x) = \int |f| \,\mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. The functor L^1-Space(M) yielding a non empty strict normed structure is defined by:

(Def. 20) The RLS structure of L^1-Space(M) = pre-L-Space M and the norm of L^1-Space$(M) = L^1$-Norm(N).

In the sequel x, y are points of L^1-Space(M).

Next we state several propositions:

(50)(i) There exists a partial function f from X to $\mathbb{R}$ such that $f \in$ the L^1 functions of M and $x = [f]^M_{\mathrm{a.e.}}$ and $\|x\| = \int|f|\,\mathrm{d}M$, and

(ii) for every partial function f from X to $\mathbb{R}$ such that $f \in x$ holds $\int|f|\,\mathrm{d}M = \|x\|$.

(51) If $f \in x$, then $x = [f]^M_{\mathrm{a.e.}}$ and $\|x\| = \int|f|\,\mathrm{d}M$.

(52) If $f \in x$ and $g \in y$, then $f + g \in x + y$ and if $f \in x$, then $a\,f \in a \cdot x$.

(53) If $E = \mathrm{dom}\, f$ and for every set x such that $x \in \mathrm{dom}\, f$ holds $f(x) = r$, then f is measurable on E.

(54) If $f \in$ the L^1 functions of M and $\int|f|\,\mathrm{d}M = 0$, then $f =^M_{\mathrm{a.e.}} X \longmapsto 0$.

(55) $\int|X \longmapsto 0|\,\mathrm{d}M = 0$.

(56) If f is integrable on M and g is integrable on M, then $\int|f + g|\,\mathrm{d}M \leq \int|f|\,\mathrm{d}M + \int|g|\,\mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. One can check that L^1-Space(M) is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

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BCI-homomorphisms

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Summary. In this article the notion of the power of an element of BCI-algebra and its period in the book [11], sections 1.4 to 1.5 are firstly given. Then the definition of BCI-homomorphism is defined and the fundamental theorem of homomorphism, the first isomorphism theorem and the second isomorphism theorem are proved following the book [9], section 1.6.

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The notation and terminology used in this paper have been introduced in the following articles: [6], [14], [3], [15], [5], [4], [2], [7], [10], [1], [13], [8], and [12].

1. The Power of an Element of BCI-algebras

In this paper X is a BCI-algebra and n is an element of $\mathbb{N}$.

Let D be a set, let f be a function from $\mathbb{N}$ into D, and let n be a natural number. Then $f(n)$ is an element of D.

Let G be a non empty BCI structure with 0. The functor BCI-power G yielding a function from (the carrier of G) $\times$ $\mathbb{N}$ into the carrier of G is defined as follows:

(Def. 1) For every element x of G holds (BCI-power G)$(x, 0) = 0_G$ and for every n holds (BCI-power G)$(x, n+1) = x \setminus$ (BCI-power G)$(x, n)^{\rm c}$.

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For simplicity, we adopt the following convention: x, y are elements of X, a, b are elements of AtomSet X, m, n are natural numbers, and i, j are integers.

Let us consider X, i, x. The functor x^i yielding an element of X is defined by:

(Def. 2) $x^i = \begin{cases} (\text{BCI-power}\,X)(x,\,|i|), & \text{if } 0 \leq i, \\ (\text{BCI-power}\,X)(x^{\mathrm{c}},\,|i|), & \text{otherwise.} \end{cases}$

Let us consider X, n, x. Then x^n can be characterized by the condition:

(Def. 3) $x^n = (\text{BCI-power}\,X)(x,\,n)$.

One can prove the following propositions:

(1) $a \setminus (x \setminus b) = b \setminus (x \setminus a)$.

(2) $x^{n+1} = x \setminus (x^n)^{\mathrm{c}}$.

(3) $x^0 = 0_X$.

(4) $x^1 = x$.

(5) $x^{-1} = x^{\mathrm{c}}$.

(6) $x^2 = x \setminus x^{\mathrm{c}}$.

(7) $(0_X)^n = 0_X$.

(8) $(a^{-1})^{-1} = a$.

(9) $x^{-n} = ((x^{\mathrm{c}})^{\mathrm{c}})^{-n}$.

(10) $(a^{\mathrm{c}})^n = a^{-n}$.

(11) If $x \in$ BCK-part X and $n \geq 1$, then $x^n = x$.

(12) If $x \in$ BCK-part X, then $x^{-n} = 0_X$.

(13) $a^i \in$ AtomSet X.

(14) $(a^{n+1})^{\mathrm{c}} = (a^n)^{\mathrm{c}} \setminus a$.

(15) $(a \setminus b)^n = a^n \setminus b^n$.

(16) $(a \setminus b)^{-n} = a^{-n} \setminus b^{-n}$.

(17) $(a^{\mathrm{c}})^n = (a^n)^{\mathrm{c}}$.

(18) $(x^{\mathrm{c}})^n = (x^n)^{\mathrm{c}}$.

(19) $(a^{\mathrm{c}})^{-n} = (a^{-n})^{\mathrm{c}}$.

(20) $x^n \in \text{BranchV}(((x^{\mathrm{c}})^{\mathrm{c}})^n)$.

(21) $(x^n)^{\mathrm{c}} = (((x^{\mathrm{c}})^{\mathrm{c}})^n)^{\mathrm{c}}$.

(22) $a^i \setminus a^j = a^{i-j}$.

(23) $(a^i)^j = a^{i \cdot j}$.

(24) $a^{i+j} = a^i \setminus (a^j)^{\mathrm{c}}$.

Let us consider X, x. We say that x is finite-period if and only if:

(Def. 4) There exists an element n of $\mathbb{N}$ such that $n \neq 0$ and $x^n \in$ BCK-part X.

One can prove the following proposition

(25) If x is finite-period, then $(x^{\mathrm{c}})^{\mathrm{c}}$ is finite-period.

Let us consider X, x. Let us assume that x is finite-period. The functor $\mathrm{ord}(x)$ yielding an element of $\mathbb{N}$ is defined as follows:

(Def. 5) $x^{\mathrm{ord}(x)} \in$ BCK-part X and $\mathrm{ord}(x) \neq 0$ and for every element m of $\mathbb{N}$ such that $x^m \in$ BCK-part X and $m \neq 0$ holds $\mathrm{ord}(x) \leq m$.

One can prove the following propositions:

(26) If a is finite-period and $\mathrm{ord}(a) = n$, then $a^n = 0_X$.

(27) X is a BCK-algebra iff for every x holds x is finite-period and $\mathrm{ord}(x) = 1$.

(28) If x is finite-period and a is finite-period and $x \in$ BranchV a, then $\mathrm{ord}(x) = \mathrm{ord}(a)$.

(29) If x is finite-period and $\mathrm{ord}(x) = n$, then $x^m \in$ BCK-part X iff $n \mid m$.

(30) If x is finite-period and x^m is finite-period and $\mathrm{ord}(x) = n$ and $m > 0$, then $\mathrm{ord}(x^m) = n \div (m \gcd n)$.

(31) If x is finite-period and x^{c} is finite-period, then $\mathrm{ord}(x) = \mathrm{ord}(x^{\mathrm{c}})$.

(32) If $x \setminus y$ is finite-period and x, $y \in$ BranchV a, then $\mathrm{ord}(x \setminus y) = 1$.

(33) Suppose that $x \setminus y$ is finite-period and $a \setminus b$ is finite-period and x is finite-period and y is finite-period and a is finite-period and b is finite-period and $a \neq b$ and $x \in$ BranchV a and $y \in$ BranchV b. Then $\mathrm{ord}(a \setminus b) \mid \mathrm{lcm}(\mathrm{ord}(x), \mathrm{ord}(y))$.

2. Definition of BCI-homomorphisms

For simplicity, we follow the rules: X, X', Y, Z, W are BCI-algebras, H' denotes a subalgebra of X', G denotes a subalgebra of X, A' denotes a non empty subset of X', I denotes an ideal of X, C_1, K are closed ideals of X, x, y are elements of X, R_1 denotes an I-congruence of X by I, and R_2 denotes an I-congruence of X by K.

One can prove the following proposition

(34) Let X be a BCI-algebra, Y be a subalgebra of X, x, y be elements of X, and x', y' be elements of Y. If $x = x'$ and $y = y'$, then $x \setminus y = x' \setminus y'$.

Let X, X' be non empty BCI structures with 0 and let f be a function from X into X'. We say that f is multiplicative if and only if:

(Def. 6) For all elements a, b of X holds $f(a \setminus b) = f(a) \setminus f(b)$.

Let X, X' be BCI-algebras. Note that there exists a function from X into X' which is multiplicative.

Let X, X' be BCI-algebras. A BCI-homomorphism from X to X' is a multiplicative function from X into X'.

In the sequel f denotes a BCI-homomorphism from X to X', g denotes a BCI-homomorphism from X' to X, and h denotes a BCI-homomorphism from X' to Y.

Let us consider X, X', f. We say that f is isotonic if and only if:

(Def. 7) For all x, y such that $x \leq y$ holds $f(x) \leq f(y)$.

Let us consider X. An endomorphism of X is a BCI-homomorphism from X to X.

Let us consider X, X', f. The functor $\operatorname{Ker} f$ is defined by:

(Def. 8) $\operatorname{Ker} f = \{x \in X \colon f(x) = 0_{X'}\}$.

The following proposition is true

(35) $f(0_X) = 0_{X'}$.

Let us consider X, X', f. Observe that $\operatorname{Ker} f$ is non empty.

We now state several propositions:

(36) If $x \leq y$, then $f(x) \leq f(y)$.

(37) f is one-to-one iff $\operatorname{Ker} f = \{0_X\}$.

(38) If f is bijective and $g = f^{-1}$, then g is bijective.

(39) $h \cdot f$ is a BCI-homomorphism from X to Y.

(40) Let f be a BCI-homomorphism from X to Y, g be a BCI-homomorphism from Y to Z, and h be a BCI-homomorphism from Z to W. Then $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

(41) For every subalgebra Z of X' such that the carrier of $Z = \operatorname{rng} f$ holds f is a BCI-homomorphism from X to Z.

(42) $\operatorname{Ker} f$ is a closed ideal of X.

Let us consider X, X', f. Observe that $\operatorname{Ker} f$ is closed.

Next we state several propositions:

(43) If f is onto, then for every element c of X' there exists x such that $c = f(x)$.

(44) For every element a of X such that a is minimal holds $f(a)$ is minimal.

(45) For every element a of AtomSet X and for every element b of AtomSet X' such that $b = f(a)$ holds $f^\circ \operatorname{BranchV} a \subseteq \operatorname{BranchV} b$.

(46) If A' is an ideal of X', then $f^{-1}(A')$ is an ideal of X.

(47) If A' is a closed ideal of X', then $f^{-1}(A')$ is a closed ideal of X.

(48) If f is onto, then $f^\circ I$ is an ideal of X'.

(49) If f is onto, then $f^\circ C_1$ is a closed ideal of X'.

Let X, X' be BCI-algebras. We say that X and X' are isomorphic if and only if:

(Def. 9) There exists a BCI-homomorphism from X to X' which is bijective.

Let us consider X, let I be an ideal of X, and let R_1 be an I-congruence of X by I. Note that ${}^{X}/_{R_1}$ is strict, B, C, I, and BCI-4.

Let us consider X, let I be an ideal of X, and let R_1 be an I-congruence of X by I. The canonical homomorphism onto cosets of R_1 yielding a BCI-homomorphism from X to ${}^{X}/_{R_1}$ is defined as follows:

(Def. 10) For every x holds (the canonical homomorphism onto cosets of R_1)(x) = $[x]_{(R_1)}$.

3. Fundamental Theorem of Homomorphisms

The following four propositions are true:

(50) The canonical homomorphism onto cosets of R_1 is onto.

(51) Suppose $I = \mathrm{Ker}\, f$. Then there exists a BCI-homomorphism h from ${}^{X}/_{R_1}$ to X' such that $f = h \cdot$ the canonical homomorphism onto cosets of R_1 and h is one-to-one.

(52) Let given X, X', I, R_1, f. Suppose $I = \mathrm{Ker}\, f$. Then there exists a BCI-homomorphism h from ${}^{X}/_{R_1}$ to X' such that $f = h \cdot$ the canonical homomorphism onto cosets of R_1 and h is one-to-one.

(53) Ker (the canonical homomorphism onto cosets of R_2) = K.

4. First Isomorphism Theorem

One can prove the following propositions:

(54) If $I = \mathrm{Ker}\, f$ and the carrier of $H' = \mathrm{rng}\, f$, then ${}^{X}/_{R_1}$ and H' are isomorphic.

(55) If $I = \mathrm{Ker}\, f$ and f is onto, then ${}^{X}/_{R_1}$ and X' are isomorphic.

5. Second Isomorphism Theorem

Let us consider X, G, K, R_2. The functor Union(G, R_2) yielding a non empty subset of X is defined by:

(Def. 11) $\mathrm{Union}(G, R_2) = \bigcup\{[a]_{(R_2)}; a$ ranges over elements of G: $[a]_{(R_2)} \in$ the carrier of ${}^{X}/_{R_2}\}$.

Let us consider X, G, K, R_2. The functor HKOp(G, R_2) yielding a binary operation on Union(G, R_2) is defined as follows:

(Def. 12) For all elements w_1, w_2 of Union(G, R_2) and for all elements x, y of X such that $w_1 = x$ and $w_2 = y$ holds $(\mathrm{HKOp}(G, R_2))(w_1, w_2) = x \setminus y$.

Let us consider X, G, K, R_2. The functor zeroHK(G, R_2) yields an element of Union(G, R_2) and is defined as follows:

(Def. 13) $\mathrm{zeroHK}(G, R_2) = 0_X$.

Let us consider X, G, K, R_2. The functor HK(G, R_2) yielding a BCI structure with 0 is defined as follows:

(Def. 14) $\mathrm{HK}(G, R_2) = \langle \mathrm{Union}(G, R_2), \mathrm{HKOp}(G, R_2), \mathrm{zeroHK}(G, R_2) \rangle$.

Let us consider X, G, K, R_2. Observe that $\mathrm{HK}(G, R_2)$ is non empty.

Let us consider X, G, K, R_2 and let w_1, w_2 be elements of $\mathrm{Union}(G, R_2)$. The functor $w_1 \setminus w_2$ yielding an element of $\mathrm{Union}(G, R_2)$ is defined by:

(Def. 15) $w_1 \setminus w_2 = (\mathrm{HKOp}(G, R_2))(w_1, w_2)$.

We now state the proposition

(56) $\mathrm{HK}(G, R_2)$ is a BCI-algebra.

Let us consider X, G, K, R_2. Observe that $\mathrm{HK}(G, R_2)$ is strict, B, C, I, and BCI-4.

We now state three propositions:

(57) $\mathrm{HK}(G, R_2)$ is a subalgebra of X.

(58) (The carrier of G) $\cap K$ is a closed ideal of G.

(59) Let K_1 be an ideal of $\mathrm{HK}(G, R_2)$, R_3 be an I-congruence of $\mathrm{HK}(G, R_2)$ by K_1, I be an ideal of G, and R_1 be an I-congruence of G by I. Suppose $K_1 = K$ and $R_3 = R_2$ and $I =$ (the carrier of G) $\cap K$. Then ${}^{G}/_{R_1}$ and ${}^{\mathrm{HK}(G,R_2)}/_{R_3}$ are isomorphic.

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Stability of the 4-2 Binary Addition Circuit Cells. Part I

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Summary. To evaluate our formal verification method on a real-size calculation circuit, in this article, we continue to formalize the concept of the 4-2 Binary Addition Cell primitives (FTAs) to define the structures of calculation units for a very fast multiplication algorithm for VLSI implementation [11]. We define the circuit structure of four-types FTAs, TYPE-0 to TYPE-3, using the series constructions of the Generalized Full Adder Circuits (GFAs) that generalized adder to have for each positive and negative weights to inputs and outputs [15]. We then successfully prove its circuit stability of the calculation outputs after four-steps. The motivation for this research is to establish a technique based on formalized mathematics and its applications for calculation circuits with high reliability.

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The terminology and notation used in this paper are introduced in the following papers: [8], [10], [14], [3], [13], [1], [7], [9], [6], [5], [4], [2], [12], and [15]. For simplicity the following abbreviations are introduced

$$\text{BitGFA}i\text{Str} \longmapsto \Sigma_i$$
$$\text{BitGFA}i\text{Circ} \longmapsto \mathfrak{C}_i$$
$$\text{GFA}i\text{AdderOutput} \longmapsto \mathfrak{a}_i$$
$$\text{GFA}i\text{CarryOutput} \longmapsto \mathfrak{c}_i$$
$$\text{InnerVertices} \longmapsto \mathcal{IV}$$

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1. Stability of 4-2 Binary Addition Circuit Cell (TYPE-0)

Let a_1, b_1, c_1, d_1, c_2 be sets. The functor BitFTA0Str(a_1, b_1, c_1, d_1, c_2) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 1) $\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2) = \Sigma_0(a_1, b_1, c_1) +\cdot \Sigma_0(\mathfrak{a}_0(a_1, b_1, c_1), c_2, d_1)$.

Let a_1, b_1, c_1, d_1, c_2 be sets. The functor BitFTA0Circ(a_1, b_1, c_1, d_1, c_2) yields a strict Boolean circuit of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) with denotation held in gates and is defined as follows:

(Def. 2) $\mathrm{BitFTA0Circ}(a_1, b_1, c_1, d_1, c_2) = \mathfrak{C}_0(a_1, b_1, c_1) +\cdot \mathfrak{C}_0(\mathfrak{a}_0(a_1, b_1, c_1), c_2, d_1)$.

One can prove the following propositions:

(1) Let a_1, b_1, c_1, d_1, c_2 be sets. Then $\mathcal{IV}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)) = \{\langle\langle a_1, b_1\rangle, \mathrm{xor}_2\rangle, \mathfrak{a}_0(a_1, b_1, c_1)\} \cup \{\langle\langle a_1, b_1\rangle, \mathrm{and}_2\rangle, \langle\langle b_1, c_1\rangle, \mathrm{and}_2\rangle, \langle\langle c_1, a_1\rangle, \mathrm{and}_2\rangle, \mathfrak{c}_0(a_1, b_1, c_1)\} \cup \{\langle\langle \mathfrak{a}_0(a_1, b_1, c_1), c_2\rangle, \mathrm{xor}_2\rangle, \mathfrak{a}_0(\mathfrak{a}_0(a_1, b_1, c_1), c_2, d_1)\} \cup \{\langle\langle \mathfrak{a}_0(a_1, b_1, c_1), c_2\rangle, \mathrm{and}_2\rangle, \langle\langle c_2, d_1\rangle, \mathrm{and}_2\rangle, \langle\langle d_1, \mathfrak{a}_0(a_1, b_1, c_1)\rangle, \mathrm{and}_2\rangle, \mathfrak{c}_0(\mathfrak{a}_0(a_1, b_1, c_1), c_2, d_1)\}$.

(2) For all sets a_1, b_1, c_1, d_1, c_2 holds $\mathcal{IV}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ is a binary relation.

(3) For all non pair sets a_1, b_1, c_1, d_1 and for every set c_2 such that $c_2 \neq \langle\langle d_1, \mathfrak{a}_0(a_1, b_1, c_1)\rangle, \mathrm{and}_2\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_0(a_1, b_1, c_1))$ holds $\mathrm{InputVertices}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)) = \{a_1, b_1, c_1, d_1, c_2\}$.

(4) Let a_1, b_1, c_1, d_1, c_2 be sets. Then $a_1 \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $b_1 \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $c_1 \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $d_1 \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $c_2 \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $\langle\langle a_1, b_1\rangle, \mathrm{xor}_2\rangle \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $\mathfrak{a}_0(a_1, b_1, c_1) \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $\langle\langle a_1, b_1\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $\langle\langle b_1, c_1\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $\langle\langle c_1, a_1\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $\mathfrak{c}_0(a_1, b_1, c_1) \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $\langle\langle \mathfrak{a}_0(a_1, b_1, c_1), c_2\rangle, \mathrm{xor}_2\rangle \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $\mathfrak{a}_0(\mathfrak{a}_0(a_1, b_1, c_1), c_2, d_1) \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $\langle\langle \mathfrak{a}_0(a_1, b_1, c_1), c_2\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $\langle\langle c_2, d_1\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $\langle\langle d_1, \mathfrak{a}_0(a_1, b_1, c_1)\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2) and $\mathfrak{c}_0(\mathfrak{a}_0(a_1, b_1, c_1), c_2, d_1) \in$ the carrier of BitFTA0Str(a_1, b_1, c_1, d_1, c_2).

(5) Let a_1, b_1, c_1, d_1, c_2 be sets. Then $\langle\langle a_1, b_1\rangle, \mathrm{xor}_2\rangle \in \mathcal{IV}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and $\mathfrak{a}_0(a_1, b_1, c_1) \in \mathcal{IV}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and

$\langle\langle a_1, b_1\rangle, \mathrm{and}_2\rangle$, $\langle\langle b_1, c_1\rangle, \mathrm{and}_2\rangle$, $\langle\langle c_1, a_1\rangle, \mathrm{and}_2\rangle \in \mathcal{IV}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and $\mathfrak{c}_0(a_1, b_1, c_1) \in \mathcal{IV}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and $\langle\langle \mathfrak{a}_0(a_1, b_1, c_1), c_2\rangle, \mathrm{xor}_2\rangle$, $\mathfrak{a}_0(\mathfrak{a}_0(a_1, b_1, c_1), c_2, d_1)$, $\langle\langle \mathfrak{a}_0(a_1, b_1, c_1), c_2\rangle, \mathrm{and}_2\rangle$, $\langle\langle c_2, d_1\rangle, \mathrm{and}_2\rangle$, $\langle\langle d_1, \mathfrak{a}_0(a_1, b_1, c_1)\rangle, \mathrm{and}_2\rangle$, $\mathfrak{c}_0(\mathfrak{a}_0(a_1, b_1, c_1), c_2, d_1) \in \mathcal{IV}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$.

(6) Let a_1, b_1, c_1, d_1 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_1, \mathfrak{a}_0(a_1, b_1, c_1)\rangle, \mathrm{and}_2\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_0(a_1, b_1, c_1))$. Then a_1, b_1, c_1, d_1, $c_2 \in \mathrm{InputVertices}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$.

Let a_1, b_1, c_1, d_1, c_2 be sets. The functor BitFTA0CarryOutput$(a_1, b_1, c_1, d_1, c_2)$ yields an element of $\mathcal{IV}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and is defined as follows:

(Def. 3) $\mathrm{BitFTA0CarryOutput}(a_1, b_1, c_1, d_1, c_2) = \mathfrak{c}_0(a_1, b_1, c_1)$.

The functor BitFTA0AdderOutputI$(a_1, b_1, c_1, d_1, c_2)$ yields an element of $\mathcal{IV}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and is defined as follows:

(Def. 4) $\mathrm{BitFTA0AdderOutputI}(a_1, b_1, c_1, d_1, c_2) = \mathfrak{a}_0(a_1, b_1, c_1)$.

The functor BitFTA0AdderOutputP$(a_1, b_1, c_1, d_1, c_2)$ yielding an element of $\mathcal{IV}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ is defined by:

(Def. 5) $\mathrm{BitFTA0AdderOutputP}(a_1, b_1, c_1, d_1, c_2) = \mathfrak{c}_0(\mathfrak{a}_0(a_1, b_1, c_1), c_2, d_1)$.

The functor BitFTA0AdderOutputQ$(a_1, b_1, c_1, d_1, c_2)$ yields an element of $\mathcal{IV}(\mathrm{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and is defined by:

(Def. 6) $\mathrm{BitFTA0AdderOutputQ}(a_1, b_1, c_1, d_1, c_2) = \mathfrak{a}_0(\mathfrak{a}_0(a_1, b_1, c_1), c_2, d_1)$.

The following propositions are true:

(7) Let a_1, b_1, c_1 be non pair sets, d_1, c_2 be sets, s be a state of BitFTA0Circ$(a_1, b_1, c_1, d_1, c_2)$, and a_2, a_3, a_4 be elements of *Boolean*. Suppose $a_2 = s(a_1)$ and $a_3 = s(b_1)$ and $a_4 = s(c_1)$. Then $(\mathrm{Following}(s, 2))(\mathrm{BitFTA0CarryOutput}(a_1, b_1, c_1, d_1, c_2)) = a_2 \wedge a_3 \vee a_3 \wedge a_4 \vee a_4 \wedge a_2$ and $(\mathrm{Following}(s, 2))(\mathrm{BitFTA0AdderOutputI}(a_1, b_1, c_1, d_1, c_2)) = a_2 \oplus a_3 \oplus a_4$.

(8) Let a_1, b_1, c_1, d_1 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_1, \mathfrak{a}_0(a_1, b_1, c_1)\rangle, \mathrm{and}_2\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_0(a_1, b_1, c_1))$. Let s be a state of BitFTA0Circ$(a_1, b_1, c_1, d_1, c_2)$ and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_1)$ and $a_3 = s(b_1)$ and $a_4 = s(c_1)$ and $a_5 = s(d_1)$ and $a_6 = s(c_2)$. Then $(\mathrm{Following}(s, 2))(\mathfrak{a}_0(a_1, b_1, c_1)) = a_2 \oplus a_3 \oplus a_4$ and $(\mathrm{Following}(s, 2))(a_1) = a_2$ and $(\mathrm{Following}(s, 2))(b_1) = a_3$ and $(\mathrm{Following}(s, 2))(c_1) = a_4$ and $(\mathrm{Following}(s, 2))(d_1) = a_5$ and $(\mathrm{Following}(s, 2))(c_2) = a_6$.

(9) Let a_1, b_1, c_1, d_1 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_1, \mathfrak{a}_0(a_1, b_1, c_1)\rangle, \mathrm{and}_2\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_0(a_1, b_1, c_1))$. Let s be a state of BitFTA0Circ$(a_1, b_1, c_1, d_1, c_2)$ and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_1)$ and $a_3 = s(b_1)$ and $a_4 = s(c_1)$ and $a_5 = s(d_1)$

and $a_6 = s(c_2)$. Then (Following$(s, 4)$)(BitFTA0AdderOutputP$(a_1, b_1, c_1, d_1, c_2)$) $= (a_2 \oplus a_3 \oplus a_4) \wedge a_6 \vee a_6 \wedge a_5 \vee a_5 \wedge (a_2 \oplus a_3 \oplus a_4)$ and (Following$(s, 4)$)(BitFTA0AdderOutputQ$(a_1, b_1, c_1, d_1, c_2)$) $= a_2 \oplus a_3 \oplus a_4 \oplus a_5 \oplus a_6$.

(10) Let a_1, b_1, c_1, d_1 be non pair sets and c_2 be a set. If $c_2 \neq \langle\langle d_1, \mathfrak{a}_0(a_1, b_1, c_1)\rangle, \text{and}_2\rangle$, then for every state s of BitFTA0Circ$(a_1, b_1, c_1, d_1, c_2)$ holds Following$(s, 4)$ is stable.

2. Stability of 4-2 Binary Addition Circuit Cell (TYPE-1)

Let a_1, b_2, c_1, d_2, c_2 be sets. The functor BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$ yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 7) BitFTA1Str$(a_1, b_2, c_1, d_2, c_2) = \Sigma_1(a_1, b_2, c_1) +\cdot \Sigma_2(\mathfrak{a}_1(a_1, b_2, c_1), c_2, d_2)$.

Let a_1, b_2, c_1, d_2, c_2 be sets. The functor BitFTA1Circ$(a_1, b_2, c_1, d_2, c_2)$ yields a strict Boolean circuit of BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$ with denotation held in gates and is defined by:

(Def. 8) BitFTA1Circ$(a_1, b_2, c_1, d_2, c_2) = \mathfrak{C}_1(a_1, b_2, c_1) +\cdot \mathfrak{C}_2(\mathfrak{a}_1(a_1, b_2, c_1), c_2, d_2)$.

Next we state several propositions:

(11) Let a_1, b_2, c_1, d_2, c_2 be sets. Then $\mathcal{IV}$(BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$) $= \{\langle\langle a_1, b_2\rangle, \text{xor2c}\rangle, \mathfrak{a}_1(a_1, b_2, c_1)\} \cup \{\langle\langle a_1, b_2\rangle, \text{and2c}\rangle, \langle\langle b_2, c_1\rangle, \text{and}_{2a}\rangle, \langle\langle c_1, a_1\rangle, \text{and}_2\rangle, \mathfrak{c}_1(a_1, b_2, c_1)\} \cup \{\langle\langle \mathfrak{a}_1(a_1, b_2, c_1), c_2\rangle, \text{xor2c}\rangle, \mathfrak{a}_2(\mathfrak{a}_1(a_1, b_2, c_1), c_2, d_2)\} \cup \{\langle\langle \mathfrak{a}_1(a_1, b_2, c_1), c_2\rangle, \text{and}_{2a}\rangle, \langle\langle c_2, d_2\rangle, \text{and2c}\rangle, \langle\langle d_2, \mathfrak{a}_1(a_1, b_2, c_1)\rangle, \text{and}_{2b}\rangle, \mathfrak{c}_2(\mathfrak{a}_1(a_1, b_2, c_1), c_2, d_2)\}$.

(12) For all sets a_1, b_2, c_1, d_2, c_2 holds $\mathcal{IV}$(BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$) is a binary relation.

(13) For all non pair sets a_1, b_2, c_1, d_2 and for every set c_2 such that $c_2 \neq \langle\langle d_2, \mathfrak{a}_1(a_1, b_2, c_1)\rangle, \text{and}_{2b}\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_1(a_1, b_2, c_1))$ holds InputVertices(BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$) $= \{a_1, b_2, c_1, d_2, c_2\}$.

(14) Let a_1, b_2, c_1, d_2, c_2 be sets. Then $a_1 \in$ the carrier of BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$ and $b_2 \in$ the carrier of BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$ and $c_1 \in$ the carrier of BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$ and $d_2 \in$ the carrier of BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$ and $c_2 \in$ the carrier of BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$ and $\langle\langle a_1, b_2\rangle, \text{xor2c}\rangle \in$ the carrier of BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$ and $\mathfrak{a}_1(a_1, b_2, c_1) \in$ the carrier of BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$ and $\langle\langle a_1, b_2\rangle, \text{and2c}\rangle \in$ the carrier of BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$ and $\langle\langle b_2, c_1\rangle, \text{and}_{2a}\rangle \in$ the carrier of BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$ and $\langle\langle c_1, a_1\rangle, \text{and}_2\rangle \in$ the carrier of BitFTA1Str$(a_1, b_2, c_1, d_2, c_2)$ and $\mathfrak{c}_1(a_1, b_2,$

$c_1) \in$ the carrier of BitFTA1Str(a_1, b_2, c_1, d_2, c_2) and $\langle\langle \mathfrak{a}_1(a_1, b_2, c_1), c_2\rangle$, xor2c $\rangle \in$ the carrier of BitFTA1Str(a_1, b_2, c_1, d_2, c_2) and $\mathfrak{a}_2(\mathfrak{a}_1(a_1, b_2, c_1), c_2, d_2) \in$ the carrier of BitFTA1Str(a_1, b_2, c_1, d_2, c_2) and $\langle\langle \mathfrak{a}_1(a_1, b_2, c_1), c_2\rangle$, and$_{2a}$ $\rangle \in$ the carrier of BitFTA1Str(a_1, b_2, c_1, d_2, c_2) and $\langle\langle c_2, d_2\rangle$, and2c $\rangle \in$ the carrier of BitFTA1Str(a_1, b_2, c_1, d_2, c_2) and $\langle\langle d_2, \mathfrak{a}_1(a_1, b_2, c_1)\rangle$, and$_{2b}$ $\rangle \in$ the carrier of BitFTA1Str(a_1, b_2, c_1, d_2, c_2) and $\mathfrak{c}_2(\mathfrak{a}_1(a_1, b_2, c_1), c_2, d_2) \in$ the carrier of BitFTA1Str(a_1, b_2, c_1, d_2, c_2).

(15) Let a_1, b_2, c_1, d_2, c_2 be sets. Then $\langle\langle a_1, b_2\rangle$, xor2c $\rangle \in \mathcal{IV}$(BitFTA1Str(a_1, b_2, c_1, d_2, c_2)) and $\mathfrak{a}_1(a_1, b_2, c_1) \in \mathcal{IV}$(BitFTA1Str($a_1$, b_2, c_1, d_2, c_2)) and $\langle\langle a_1, b_2\rangle$, and2c $\rangle$, $\langle\langle b_2, c_1\rangle$, and$_{2a}$ $\rangle$, $\langle\langle c_1, a_1\rangle$, and$_2$ $\rangle \in \mathcal{IV}$(BitFTA1Str(a_1, b_2, c_1, d_2, c_2)) and $\mathfrak{c}_1(a_1, b_2, c_1) \in \mathcal{IV}$(BitFTA1Str($a_1$, b_2, c_1, d_2, c_2)) and $\langle\langle \mathfrak{a}_1(a_1, b_2, c_1), c_2\rangle$, xor2c $\rangle$, $\mathfrak{a}_2(\mathfrak{a}_1(a_1, b_2, c_1), c_2, d_2)$, $\langle\langle \mathfrak{a}_1(a_1, b_2, c_1), c_2\rangle$, and$_{2a}$ $\rangle$, $\langle\langle c_2, d_2\rangle$, and2c $\rangle$, $\langle\langle d_2, \mathfrak{a}_1(a_1, b_2, c_1)\rangle$, and$_{2b}$ $\rangle$, $\mathfrak{c}_2(\mathfrak{a}_1(a_1, b_2, c_1), c_2, d_2) \in \mathcal{IV}$(BitFTA1Str($a_1$, b_2, c_1, d_2, c_2)).

(16) Let a_1, b_2, c_1, d_2 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_2, \mathfrak{a}_1(a_1, b_2, c_1)\rangle$, and$_{2b}$ $\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_1(a_1, b_2, c_1))$. Then a_1, b_2, c_1, d_2, $c_2 \in$ InputVertices(BitFTA1Str(a_1, b_2, c_1, d_2, c_2)).

Let a_1, b_2, c_1, d_2, c_2 be sets. The functor BitFTA1CarryOutput(a_1, b_2, c_1, d_2, c_2) yielding an element of $\mathcal{IV}$(BitFTA1Str(a_1, b_2, c_1, d_2, c_2)) is defined as follows:

(Def. 9) BitFTA1CarryOutput(a_1, b_2, c_1, d_2, c_2) $= \mathfrak{c}_1(a_1, b_2, c_1)$.

The functor BitFTA1AdderOutputI(a_1, b_2, c_1, d_2, c_2) yields an element of $\mathcal{IV}$(BitFTA1Str(a_1, b_2, c_1, d_2, c_2)) and is defined by:

(Def. 10) BitFTA1AdderOutputI(a_1, b_2, c_1, d_2, c_2) $= \mathfrak{a}_1(a_1, b_2, c_1)$.

The functor BitFTA1AdderOutputP(a_1, b_2, c_1, d_2, c_2) yields an element of $\mathcal{IV}$(BitFTA1Str(a_1, b_2, c_1, d_2, c_2)) and is defined as follows:

(Def. 11) BitFTA1AdderOutputP(a_1, b_2, c_1, d_2, c_2) $= \mathfrak{c}_2(\mathfrak{a}_1(a_1, b_2, c_1), c_2, d_2)$.

The functor BitFTA1AdderOutputQ(a_1, b_2, c_1, d_2, c_2) yielding an element of $\mathcal{IV}$(BitFTA1Str(a_1, b_2, c_1, d_2, c_2)) is defined as follows:

(Def. 12) BitFTA1AdderOutputQ(a_1, b_2, c_1, d_2, c_2) $= \mathfrak{a}_2(\mathfrak{a}_1(a_1, b_2, c_1), c_2, d_2)$.

The following four propositions are true:

(17) Let a_1, b_2, c_1 be non pair sets, d_2, c_2 be sets, s be a state of BitFTA1Circ(a_1, b_2, c_1, d_2, c_2), and a_2, a_3, a_4 be elements of *Boolean*. Suppose $a_2 = s(a_1)$ and $a_3 = s(b_2)$ and $a_4 = s(c_1)$. Then (Following($s, 2$))(BitFTA1CarryOutput(a_1, b_2, c_1, d_2, c_2)) $= a_2 \wedge \neg a_3 \vee \neg a_3 \wedge a_4 \vee a_4 \wedge a_2$ and (Following($s, 2$))(BitFTA1AdderOutputI(a_1, b_2, c_1, d_2, c_2)) $= \neg(a_2 \oplus \neg a_3 \oplus a_4)$.

(18) Let a_1, b_2, c_1, d_2 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_2, \mathfrak{a}_1(a_1, b_2, c_1)\rangle$, and$_{2b}$ $\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_1(a_1, b_2, c_1))$. Let s be a state of BitFTA1Circ(a_1, b_2, c_1, d_2, c_2) and a_2, a_3, a_4, a_5, a_6 be elements of

Boolean. Suppose $a_2 = s(a_1)$ and $a_3 = s(b_2)$ and $a_4 = s(c_1)$ and $a_5 = s(d_2)$ and $a_6 = s(c_2)$. Then $(\mathrm{Following}(s,2))(\mathfrak{a}_1(a_1, b_2, c_1)) = \neg(a_2 \oplus \neg a_3 \oplus a_4)$ and $(\mathrm{Following}(s,2))(a_1) = a_2$ and $(\mathrm{Following}(s,2))(b_2) = a_3$ and $(\mathrm{Following}(s,2))(c_1) = a_4$ and $(\mathrm{Following}(s,2))(d_2) = a_5$ and $(\mathrm{Following}(s,2))(c_2) = a_6$.

(19) Let a_1, b_2, c_1, d_2 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_2, \mathfrak{a}_1(a_1, b_2, c_1)\rangle, \mathrm{and}_{2b}\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_1(a_1, b_2, c_1))$. Let s be a state of BitFTA1Circ(a_1, b_2, c_1, d_2, c_2) and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_1)$ and $a_3 = s(b_2)$ and $a_4 = s(c_1)$ and $a_5 = s(d_2)$ and $a_6 = s(c_2)$. Then $(\mathrm{Following}(s,4))(\mathrm{BitFTA1AdderOutputP}(a_1, b_2, c_1, d_2, c_2)) = \neg((a_2 \oplus \neg a_3 \oplus a_4) \wedge a_6 \vee a_6 \wedge \neg a_5 \vee \neg a_5 \wedge (a_2 \oplus \neg a_3 \oplus a_4))$ and $(\mathrm{Following}(s,4))(\mathrm{BitFTA1AdderOutputQ}(a_1, b_2, c_1, d_2, c_2)) = a_2 \oplus \neg a_3 \oplus a_4 \oplus \neg a_5 \oplus a_6$.

(20) Let a_1, b_2, c_1, d_2 be non pair sets and c_2 be a set. If $c_2 \neq \langle\langle d_2, \mathfrak{a}_1(a_1, b_2, c_1)\rangle, \mathrm{and}_{2b}\rangle$, then for every state s of BitFTA1Circ(a_1, b_2, c_1, d_2, c_2) holds Following$(s, 4)$ is stable.

3. Stability of 4-2 Binary Addition Circuit Cell (TYPE-2)

Let a_7, b_1, c_3, d_1, c_2 be sets. The functor BitFTA2Str(a_7, b_1, c_3, d_1, c_2) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 13) $\mathrm{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2) = \Sigma_2(a_7, b_1, c_3) +\cdot \Sigma_1(\mathfrak{a}_2(a_7, b_1, c_3), c_2, d_1)$.

Let a_7, b_1, c_3, d_1, c_2 be sets. The functor BitFTA2Circ(a_7, b_1, c_3, d_1, c_2) yielding a strict Boolean circuit of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) with denotation held in gates is defined by:

(Def. 14) $\mathrm{BitFTA2Circ}(a_7, b_1, c_3, d_1, c_2) = \mathfrak{C}_2(a_7, b_1, c_3) +\cdot \mathfrak{C}_1(\mathfrak{a}_2(a_7, b_1, c_3), c_2, d_1)$.

Next we state several propositions:

(21) Let a_7, b_1, c_3, d_1, c_2 be sets. Then $\mathcal{IV}(\mathrm{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)) = \{\langle\langle a_7, b_1\rangle, \mathrm{xor2c}\rangle, \mathfrak{a}_2(a_7, b_1, c_3)\} \cup \{\langle\langle a_7, b_1\rangle, \mathrm{and}_{2a}\rangle, \langle\langle b_1, c_3\rangle, \mathrm{and2c}\rangle, \langle\langle c_3, a_7\rangle, \mathrm{and}_{2b}\rangle, \mathfrak{c}_2(a_7, b_1, c_3)\} \cup \{\langle\langle \mathfrak{a}_2(a_7, b_1, c_3), c_2\rangle, \mathrm{xor2c}\rangle, \mathfrak{a}_1(\mathfrak{a}_2(a_7, b_1, c_3), c_2, d_1)\} \cup \{\langle\langle \mathfrak{a}_2(a_7, b_1, c_3), c_2\rangle, \mathrm{and2c}\rangle, \langle\langle c_2, d_1\rangle, \mathrm{and}_{2a}\rangle, \langle\langle d_1, \mathfrak{a}_2(a_7, b_1, c_3)\rangle, \mathrm{and}_2\rangle, \mathfrak{c}_1(\mathfrak{a}_2(a_7, b_1, c_3), c_2, d_1)\}$.

(22) For all sets a_7, b_1, c_3, d_1, c_2 holds $\mathcal{IV}(\mathrm{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2))$ is a binary relation.

(23) For all non pair sets a_7, b_1, c_3, d_1 and for every set c_2 such that $c_2 \neq \langle\langle d_1, \mathfrak{a}_2(a_7, b_1, c_3)\rangle, \mathrm{and}_2\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_2(a_7, b_1, c_3))$ holds $\mathrm{InputVertices}(\mathrm{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)) = \{a_7, b_1, c_3, d_1, c_2\}$.

(24) Let a_7, b_1, c_3, d_1, c_2 be sets. Then $a_7 \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $b_1 \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $c_3 \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $d_1 \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $c_2 \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $\langle\langle a_7, b_1\rangle, \mathrm{xor2c}\rangle \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $\mathfrak{a}_2(a_7, b_1, c_3) \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $\langle\langle a_7, b_1\rangle, \mathrm{and}_{2a}\rangle \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $\langle\langle b_1, c_3\rangle, \mathrm{and2c}\rangle \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $\langle\langle c_3, a_7\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $\mathfrak{c}_2(a_7, b_1, c_3) \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $\langle\langle \mathfrak{a}_2(a_7, b_1, c_3), c_2\rangle, \mathrm{xor2c}\rangle \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $\mathfrak{a}_1(\mathfrak{a}_2(a_7, b_1, c_3), c_2, d_1) \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $\langle\langle \mathfrak{a}_2(a_7, b_1, c_3), c_2\rangle, \mathrm{and2c}\rangle \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $\langle\langle c_2, d_1\rangle, \mathrm{and}_{2a}\rangle \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $\langle\langle d_1, \mathfrak{a}_2(a_7, b_1, c_3)\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2) and $\mathfrak{c}_1(\mathfrak{a}_2(a_7, b_1, c_3), c_2, d_1) \in$ the carrier of BitFTA2Str(a_7, b_1, c_3, d_1, c_2).

(25) Let a_7, b_1, c_3, d_1, c_2 be sets. Then $\langle\langle a_7, b_1\rangle, \mathrm{xor2c}\rangle \in \mathcal{IV}$(BitFTA2Str($a_7$, b_1, c_3, d_1, c_2)) and $\mathfrak{a}_2(a_7, b_1, c_3) \in \mathcal{IV}$(BitFTA2Str($a_7$, b_1, c_3, d_1, c_2)) and $\langle\langle a_7, b_1\rangle, \mathrm{and}_{2a}\rangle$, $\langle\langle b_1, c_3\rangle, \mathrm{and2c}\rangle$, $\langle\langle c_3, a_7\rangle, \mathrm{and}_{2b}\rangle \in \mathcal{IV}$(BitFTA2Str($a_7$, b_1, c_3, d_1, c_2)) and $\mathfrak{c}_2(a_7, b_1, c_3) \in \mathcal{IV}$(BitFTA2Str($a_7$, b_1, c_3, d_1, c_2)) and $\langle\langle \mathfrak{a}_2(a_7, b_1, c_3), c_2\rangle, \mathrm{xor2c}\rangle$, $\mathfrak{a}_1(\mathfrak{a}_2(a_7, b_1, c_3), c_2, d_1)$, $\langle\langle \mathfrak{a}_2(a_7, b_1, c_3), c_2\rangle, \mathrm{and2c}\rangle$, $\langle\langle c_2, d_1\rangle, \mathrm{and}_{2a}\rangle$, $\langle\langle d_1, \mathfrak{a}_2(a_7, b_1, c_3)\rangle, \mathrm{and}_2\rangle$, $\mathfrak{c}_1(\mathfrak{a}_2(a_7, b_1, c_3), c_2, d_1) \in \mathcal{IV}$(BitFTA2Str($a_7$, b_1, c_3, d_1, c_2)).

(26) Let a_7, b_1, c_3, d_1 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_1, \mathfrak{a}_2(a_7, b_1, c_3)\rangle, \mathrm{and}_2\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_2(a_7, b_1, c_3))$. Then a_7, b_1, c_3, d_1, $c_2 \in$ InputVertices(BitFTA2Str(a_7, b_1, c_3, d_1, c_2)).

Let a_7, b_1, c_3, d_1, c_2 be sets. The functor BitFTA2CarryOutput(a_7, b_1, c_3, d_1, c_2) yields an element of $\mathcal{IV}$(BitFTA2Str(a_7, b_1, c_3, d_1, c_2)) and is defined as follows:

(Def. 15) BitFTA2CarryOutput(a_7, b_1, c_3, d_1, c_2) $= \mathfrak{c}_2(a_7, b_1, c_3)$.

The functor BitFTA2AdderOutputI(a_7, b_1, c_3, d_1, c_2) yields an element of $\mathcal{IV}$(BitFTA2Str(a_7, b_1, c_3, d_1, c_2)) and is defined as follows:

(Def. 16) BitFTA2AdderOutputI(a_7, b_1, c_3, d_1, c_2) $= \mathfrak{a}_2(a_7, b_1, c_3)$.

The functor BitFTA2AdderOutputP(a_7, b_1, c_3, d_1, c_2) yields an element of $\mathcal{IV}$(BitFTA2Str(a_7, b_1, c_3, d_1, c_2)) and is defined by:

(Def. 17) BitFTA2AdderOutputP(a_7, b_1, c_3, d_1, c_2) $= \mathfrak{c}_1(\mathfrak{a}_2(a_7, b_1, c_3), c_2, d_1)$.

The functor BitFTA2AdderOutputQ(a_7, b_1, c_3, d_1, c_2) yielding an element of $\mathcal{IV}$(BitFTA2Str(a_7, b_1, c_3, d_1, c_2)) is defined as follows:

(Def. 18) BitFTA2AdderOutputQ(a_7, b_1, c_3, d_1, c_2) $= \mathfrak{a}_1(\mathfrak{a}_2(a_7, b_1, c_3), c_2, d_1)$.

One can prove the following propositions:

(27) Let a_7, b_1, c_3 be non pair sets, d_1, c_2 be sets, s be a state of BitFTA2Circ(a_7, b_1, c_3, d_1, c_2), and a_2, a_3, a_4 be elements of *Boolean*. Suppose $a_2 = s(a_7)$ and $a_3 = s(b_1)$ and $a_4 = s(c_3)$. Then (Following$(s, 2)$)(BitFTA2CarryOutput(a_7, b_1, c_3, d_1, c_2)) $= \neg(\neg a_2 \wedge a_3 \vee a_3 \wedge \neg a_4 \vee \neg a_4 \wedge \neg a_2)$ and (Following$(s, 2)$)(BitFTA2AdderOutputI(a_7, b_1, c_3, d_1, c_2)) $= \neg a_2 \oplus a_3 \oplus \neg a_4$.

(28) Let a_7, b_1, c_3, d_1 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_1, \mathfrak{a}_2(a_7, b_1, c_3)\rangle, \mathrm{and}_2\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_2(a_7, b_1, c_3))$. Let s be a state of BitFTA2Circ(a_7, b_1, c_3, d_1, c_2) and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_7)$ and $a_3 = s(b_1)$ and $a_4 = s(c_3)$ and $a_5 = s(d_1)$ and $a_6 = s(c_2)$. Then (Following$(s, 2)$)($\mathfrak{a}_2(a_7, b_1, c_3)$) $= \neg a_2 \oplus a_3 \oplus \neg a_4$ and (Following$(s, 2)$)$(a_7) = a_2$ and (Following$(s, 2)$)$(b_1) = a_3$ and (Following$(s, 2)$)$(c_3) = a_4$ and (Following$(s, 2)$)$(d_1) = a_5$ and (Following$(s, 2)$)$(c_2) = a_6$.

(29) Let a_7, b_1, c_3, d_1 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_1, \mathfrak{a}_2(a_7, b_1, c_3)\rangle, \mathrm{and}_2\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_2(a_7, b_1, c_3))$. Let s be a state of BitFTA2Circ(a_7, b_1, c_3, d_1, c_2) and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_7)$ and $a_3 = s(b_1)$ and $a_4 = s(c_3)$ and $a_5 = s(d_1)$ and $a_6 = s(c_2)$. Then (Following$(s, 4)$)(BitFTA2AdderOutputP(a_7, b_1, c_3, d_1, c_2)) $= (\neg a_2 \oplus a_3 \oplus \neg a_4) \wedge \neg a_6 \vee \neg a_6 \wedge a_5 \vee a_5 \wedge (\neg a_2 \oplus a_3 \oplus \neg a_4)$ and (Following$(s, 4)$)(BitFTA2AdderOutputQ(a_7, b_1, c_3, d_1, c_2)) $= \neg(\neg a_2 \oplus a_3 \oplus \neg a_4 \oplus a_5 \oplus \neg a_6)$.

(30) Let a_7, b_1, c_3, d_1 be non pair sets and c_2 be a set. If $c_2 \neq \langle\langle d_1, \mathfrak{a}_2(a_7, b_1, c_3)\rangle, \mathrm{and}_2\rangle$, then for every state s of BitFTA2Circ(a_7, b_1, c_3, d_1, c_2) holds Following$(s, 4)$ is stable.

4. Stability of 4-2 Binary Addition Circuit Cell (TYPE-3)

Let a_7, b_2, c_3, d_2, c_2 be sets. The functor BitFTA3Str(a_7, b_2, c_3, d_2, c_2) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by:

(Def. 19) BitFTA3Str(a_7, b_2, c_3, d_2, c_2) $= \Sigma_3(a_7, b_2, c_3) +\cdot \Sigma_3(\mathfrak{a}_3(a_7, b_2, c_3), c_2, d_2)$.

Let a_7, b_2, c_3, d_2, c_2 be sets. The functor BitFTA3Circ(a_7, b_2, c_3, d_2, c_2) yielding a strict Boolean circuit of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) with denotation held in gates is defined by:

(Def. 20) BitFTA3Circ(a_7, b_2, c_3, d_2, c_2) $= \mathfrak{C}_3(a_7, b_2, c_3) +\cdot \mathfrak{C}_3(\mathfrak{a}_3(a_7, b_2, c_3), c_2, d_2)$.

We now state several propositions:

(31) Let a_7, b_2, c_3, d_2, c_2 be sets. Then $\mathcal{IV}$(BitFTA3Str(a_7, b_2, c_3, d_2, c_2)) = $\{\langle\langle a_7, b_2\rangle, \mathrm{xor}_2\rangle, \mathfrak{a}_3(a_7, b_2, c_3)\} \cup \{\langle\langle a_7, b_2\rangle, \mathrm{and}_{2b}\rangle, \langle\langle b_2, c_3\rangle, \mathrm{and}_{2b}\rangle, \langle\langle c_3, a_7\rangle, \mathrm{and}_{2b}\rangle, \mathfrak{c}_3(a_7, b_2, c_3)\} \cup \{\langle\langle \mathfrak{a}_3(a_7, b_2, c_3), c_2\rangle, \mathrm{xor}_2\rangle, \mathfrak{a}_3(\mathfrak{a}_3(a_7, b_2, c_3), c_2, d_2)\} \cup \{\langle\langle \mathfrak{a}_3(a_7, b_2, c_3), c_2\rangle, \mathrm{and}_{2b}\rangle, \langle\langle c_2, d_2\rangle, \mathrm{and}_{2b}\rangle, \langle\langle d_2, \mathfrak{a}_3(a_7, b_2, c_3)\rangle, \mathrm{and}_{2b}\rangle, \mathfrak{c}_3(\mathfrak{a}_3(a_7, b_2, c_3), c_2, d_2)\}$.

(32) For all sets a_7, b_2, c_3, d_2, c_2 holds $\mathcal{IV}$(BitFTA3Str(a_7, b_2, c_3, d_2, c_2)) is a binary relation.

(33) For all non pair sets a_7, b_2, c_3, d_2 and for every set c_2 such that $c_2 \neq \langle\langle d_2, \mathfrak{a}_3(a_7, b_2, c_3)\rangle, \mathrm{and}_{2b}\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_3(a_7, b_2, c_3))$ holds InputVertices(BitFTA3Str(a_7, b_2, c_3, d_2, c_2)) = $\{a_7, b_2, c_3, d_2, c_2\}$.

(34) Let a_7, b_2, c_3, d_2, c_2 be sets. Then $a_7 \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $b_2 \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $c_3 \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $d_2 \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $c_2 \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $\langle\langle a_7, b_2\rangle, \mathrm{xor}_2\rangle \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $\mathfrak{a}_3(a_7, b_2, c_3) \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $\langle\langle a_7, b_2\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $\langle\langle b_2, c_3\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $\langle\langle c_3, a_7\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $\mathfrak{c}_3(a_7, b_2, c_3) \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $\langle\langle \mathfrak{a}_3(a_7, b_2, c_3), c_2\rangle, \mathrm{xor}_2\rangle \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $\mathfrak{a}_3(\mathfrak{a}_3(a_7, b_2, c_3), c_2, d_2) \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $\langle\langle \mathfrak{a}_3(a_7, b_2, c_3), c_2\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $\langle\langle c_2, d_2\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $\langle\langle d_2, \mathfrak{a}_3(a_7, b_2, c_3)\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2) and $\mathfrak{c}_3(\mathfrak{a}_3(a_7, b_2, c_3), c_2, d_2) \in$ the carrier of BitFTA3Str(a_7, b_2, c_3, d_2, c_2).

(35) Let a_7, b_2, c_3, d_2, c_2 be sets. Then $\langle\langle a_7, b_2\rangle, \mathrm{xor}_2\rangle \in \mathcal{IV}$(BitFTA3Str($a_7$, b_2, c_3, d_2, c_2)) and $\mathfrak{a}_3(a_7, b_2, c_3) \in \mathcal{IV}$(BitFTA3Str($a_7$, b_2, c_3, d_2, c_2)) and $\langle\langle a_7, b_2\rangle, \mathrm{and}_{2b}\rangle, \langle\langle b_2, c_3\rangle, \mathrm{and}_{2b}\rangle, \langle\langle c_3, a_7\rangle, \mathrm{and}_{2b}\rangle \in \mathcal{IV}$(BitFTA3Str($a_7$, b_2, c_3, d_2, c_2)) and $\mathfrak{c}_3(a_7, b_2, c_3) \in \mathcal{IV}$(BitFTA3Str($a_7$, b_2, c_3, d_2, c_2)) and $\langle\langle \mathfrak{a}_3(a_7, b_2, c_3), c_2\rangle, \mathrm{xor}_2\rangle$, $\mathfrak{a}_3(\mathfrak{a}_3(a_7, b_2, c_3), c_2, d_2)$, $\langle\langle \mathfrak{a}_3(a_7, b_2, c_3), c_2\rangle, \mathrm{and}_{2b}\rangle$, $\langle\langle c_2, d_2\rangle, \mathrm{and}_{2b}\rangle$, $\langle\langle d_2, \mathfrak{a}_3(a_7, b_2, c_3)\rangle, \mathrm{and}_{2b}\rangle$, $\mathfrak{c}_3(\mathfrak{a}_3(a_7, b_2, c_3), c_2, d_2) \in \mathcal{IV}$(BitFTA3Str($a_7$, b_2, c_3, d_2, c_2)).

(36) Let a_7, b_2, c_3, d_2 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_2, \mathfrak{a}_3(a_7, b_2, c_3)\rangle, \mathrm{and}_{2b}\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_3(a_7, b_2, c_3))$. Then a_7, b_2, c_3, d_2, $c_2 \in$ InputVertices(BitFTA3Str(a_7, b_2, c_3, d_2, c_2)).

Let a_7, b_2, c_3, d_2, c_2 be sets. The functor BitFTA3CarryOutput(a_7, b_2, c_3, d_2, c_2) yields an element of $\mathcal{IV}$(BitFTA3Str(a_7, b_2, c_3, d_2, c_2)) and is defined by:

(Def. 21) BitFTA3CarryOutput(a_7, b_2, c_3, d_2, c_2) = $\mathfrak{c}_3(a_7, b_2, c_3)$.

The functor BitFTA3AdderOutputI(a_7, b_2, c_3, d_2, c_2) yields an element of

$\mathcal{IV}$(BitFTA3Str(a_7, b_2, c_3, d_2, c_2)) and is defined by:

(Def. 22) BitFTA3AdderOutputI(a_7, b_2, c_3, d_2, c_2) $= \mathfrak{a}_3(a_7, b_2, c_3)$.

The functor BitFTA3AdderOutputP(a_7, b_2, c_3, d_2, c_2) yields an element of $\mathcal{IV}$(BitFTA3Str(a_7, b_2, c_3, d_2, c_2)) and is defined by:

(Def. 23) BitFTA3AdderOutputP(a_7, b_2, c_3, d_2, c_2) $= \mathfrak{c}_3(\mathfrak{a}_3(a_7, b_2, c_3), c_2, d_2)$.

The functor BitFTA3AdderOutputQ(a_7, b_2, c_3, d_2, c_2) yielding an element of $\mathcal{IV}$(BitFTA3Str(a_7, b_2, c_3, d_2, c_2)) is defined by:

(Def. 24) BitFTA3AdderOutputQ(a_7, b_2, c_3, d_2, c_2) $= \mathfrak{a}_3(\mathfrak{a}_3(a_7, b_2, c_3), c_2, d_2)$.

One can prove the following propositions:

(37) Let a_7, b_2, c_3 be non pair sets, d_2, c_2 be sets, s be a state of BitFTA3Circ(a_7, b_2, c_3, d_2, c_2), and a_2, a_3, a_4 be elements of *Boolean*. Suppose $a_2 = s(a_7)$ and $a_3 = s(b_2)$ and $a_4 = s(c_3)$. Then (Following$(s, 2)$)(BitFTA3CarryOutput(a_7, b_2, c_3, d_2, c_2)) $= \neg(\neg a_2 \wedge \neg a_3 \vee \neg a_3 \wedge \neg a_4 \vee \neg a_4 \wedge \neg a_2)$ and (Following$(s, 2)$)(BitFTA3AdderOutputI(a_7, b_2, c_3, d_2, c_2)) $= \neg(\neg a_2 \oplus \neg a_3 \oplus \neg a_4)$.

(38) Let a_7, b_2, c_3, d_2 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_2, \mathfrak{a}_3(a_7, b_2, c_3)\rangle, \mathrm{and}_{2b}\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_3(a_7, b_2, c_3))$. Let s be a state of BitFTA3Circ(a_7, b_2, c_3, d_2, c_2) and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_7)$ and $a_3 = s(b_2)$ and $a_4 = s(c_3)$ and $a_5 = s(d_2)$ and $a_6 = s(c_2)$. Then (Following$(s, 2)$)$(\mathfrak{a}_3(a_7, b_2, c_3)) = \neg(\neg a_2 \oplus \neg a_3 \oplus \neg a_4)$ and (Following$(s, 2)$)$(a_7) = a_2$ and (Following$(s, 2)$)$(b_2) = a_3$ and (Following$(s, 2)$)$(c_3) = a_4$ and (Following$(s, 2)$)$(d_2) = a_5$ and (Following$(s, 2)$)$(c_2) = a_6$.

(39) Let a_7, b_2, c_3, d_2 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_2, \mathfrak{a}_3(a_7, b_2, c_3)\rangle, \mathrm{and}_{2b}\rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_3(a_7, b_2, c_3))$. Let s be a state of BitFTA3Circ(a_7, b_2, c_3, d_2, c_2) and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_7)$ and $a_3 = s(b_2)$ and $a_4 = s(c_3)$ and $a_5 = s(d_2)$ and $a_6 = s(c_2)$. Then (Following$(s, 4)$)(BitFTA3AdderOutputP(a_7, b_2, c_3, d_2, c_2)) $= \neg((\neg a_2 \oplus \neg a_3 \oplus \neg a_4) \wedge \neg a_6 \vee \neg a_6 \wedge \neg a_5 \vee \neg a_5 \wedge (\neg a_2 \oplus \neg a_3 \oplus \neg a_4))$ and (Following$(s, 4)$)(BitFTA3AdderOutputQ(a_7, b_2, c_3, d_2, c_2)) $= \neg(\neg a_2 \oplus \neg a_3 \oplus \neg a_4 \oplus \neg a_5 \oplus \neg a_6)$.

(40) Let a_7, b_2, c_3, d_2 be non pair sets and c_2 be a set. If $c_2 \neq \langle\langle d_2, \mathfrak{a}_3(a_7, b_2, c_3)\rangle, \mathrm{and}_{2b}\rangle$, then for every state s of BitFTA3Circ(a_7, b_2, c_3, d_2, c_2) holds Following$(s, 4)$ is stable.

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Several Differentiation Formulas of Special Functions. Part VII

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Summary. In this article, we prove a series of differentiation identities [2] involving the arctan and arccot functions and specific combinations of special functions including trigonometric and exponential functions.

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The papers [13], [15], [1], [10], [16], [5], [12], [3], [6], [9], [4], [11], [8], [14], and [7] provide the terminology and notation for this paper.

For simplicity, we adopt the following rules: x denotes a real number, n denotes an element of $\mathbb{N}$, Z denotes an open subset of $\mathbb{R}$, and f, g denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Next we state a number of propositions:

(1) Suppose $Z \subseteq \mathrm{dom}((\text{the function arctan}) \cdot (\text{the function sin}))$ and for every x such that $x \in Z$ holds $-1 < \sin x < 1$. Then

(i) (the function arctan) $\cdot$(the function sin) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function arctan) $\cdot$(the function sin))$'_{\restriction Z}(x) = \frac{\cos x}{1+(\sin x)^2}$.

(2) Suppose $Z \subseteq \mathrm{dom}((\text{the function arccot}) \cdot (\text{the function sin}))$ and for every x such that $x \in Z$ holds $-1 < \sin x < 1$. Then

(i) (the function arccot) $\cdot$(the function sin) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function arccot) $\cdot$(the function sin))$'_{\restriction Z}(x) = -\frac{\cos x}{1+(\sin x)^2}$.

(3) Suppose $Z \subseteq \mathrm{dom}((\text{the function arctan}) \cdot (\text{the function cos}))$ and for every x such that $x \in Z$ holds $-1 < \cos x < 1$. Then

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(i) (the function arctan) $\cdot$(the function cos) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function arctan) $\cdot$(the function cos))$'_{\restriction Z}(x) = -\frac{\sin x}{1+(\cos x)^{\mathbf{2}}}$.

(4) Suppose $Z \subseteq$ dom((the function arccot) $\cdot$(the function cos)) and for every x such that $x \in Z$ holds $-1 < \cos x < 1$. Then
(i) (the function arccot) $\cdot$(the function cos) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function arccot) $\cdot$(the function cos))$'_{\restriction Z}(x) = \frac{\sin x}{1+(\cos x)^{\mathbf{2}}}$.

(5) Suppose $Z \subseteq$ dom((the function arctan) $\cdot$(the function tan)) and for every x such that $x \in Z$ holds $-1 < \tan x < 1$. Then
(i) (the function arctan) $\cdot$(the function tan) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function arctan) $\cdot$(the function tan))$'_{\restriction Z}(x) = 1$.

(6) Suppose $Z \subseteq$ dom((the function arccot) $\cdot$(the function tan)) and for every x such that $x \in Z$ holds $-1 < \tan x < 1$. Then
(i) (the function arccot) $\cdot$(the function tan) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function arccot) $\cdot$(the function tan))$'_{\restriction Z}(x) = -1$.

(7) Suppose $Z \subseteq$ dom((the function arctan) $\cdot$(the function cot)) and for every x such that $x \in Z$ holds $-1 < \cot x < 1$. Then
(i) (the function arctan) $\cdot$(the function cot) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function arctan) $\cdot$(the function cot))$'_{\restriction Z}(x) = -1$.

(8) Suppose $Z \subseteq$ dom((the function arccot) $\cdot$(the function cot)) and for every x such that $x \in Z$ holds $-1 < \cot x < 1$. Then
(i) (the function arccot) $\cdot$(the function cot) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function arccot) $\cdot$(the function cot))$'_{\restriction Z}(x) = 1$.

(9) Suppose $Z \subseteq$ dom((the function arctan) $\cdot$(the function arctan)) and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $-1 < \arctan x < 1$. Then
(i) (the function arctan) $\cdot$(the function arctan) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function arctan) $\cdot$(the function arctan))$'_{\restriction Z}(x) = \frac{1}{(1+x^{\mathbf{2}})\cdot(1+(\arctan x)^{\mathbf{2}})}$.

(10) Suppose $Z \subseteq$ dom((the function arccot) $\cdot$(the function arctan)) and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $-1 < \arctan x < 1$. Then
(i) (the function arccot) $\cdot$(the function arctan) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function arccot) $\cdot$(the function arctan))$'_{\restriction Z}(x) = -\frac{1}{(1+x^{\mathbf{2}})\cdot(1+(\arctan x)^{\mathbf{2}})}$.

(11) Suppose $Z \subseteq \mathrm{dom}((\text{the function arctan}) \cdot (\text{the function arccot}))$ and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $-1 < \operatorname{arccot} x < 1$. Then

(i) (the function arctan) $\cdot$(the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function arctan) $\cdot$(the function arccot))$'_{\restriction Z}(x) = -\frac{1}{(1+x^2)\cdot(1+(\operatorname{arccot} x)^2)}$.

(12) Suppose $Z \subseteq \mathrm{dom}((\text{the function arccot}) \cdot (\text{the function arccot}))$ and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $-1 < \operatorname{arccot} x < 1$. Then

(i) (the function arccot) $\cdot$(the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function arccot) $\cdot$(the function arccot))$'_{\restriction Z}(x) = \frac{1}{(1+x^2)\cdot(1+(\operatorname{arccot} x)^2)}$.

(13) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin}) \cdot (\text{the function arctan}))$ and $Z \subseteq]-1, 1[$. Then

(i) (the function sin) $\cdot$(the function arctan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) $\cdot$(the function arctan))$'_{\restriction Z}(x) = \frac{\cos \arctan x}{1+x^2}$.

(14) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin}) \cdot (\text{the function arccot}))$ and $Z \subseteq]-1, 1[$. Then

(i) (the function sin) $\cdot$(the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) $\cdot$(the function arccot))$'_{\restriction Z}(x) = -\frac{\cos \operatorname{arccot} x}{1+x^2}$.

(15) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos}) \cdot (\text{the function arctan}))$ and $Z \subseteq]-1, 1[$. Then

(i) (the function cos) $\cdot$(the function arctan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) $\cdot$(the function arctan))$'_{\restriction Z}(x) = -\frac{\sin \arctan x}{1+x^2}$.

(16) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos}) \cdot (\text{the function arccot}))$ and $Z \subseteq]-1, 1[$. Then

(i) (the function cos) $\cdot$(the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) $\cdot$(the function arccot))$'_{\restriction Z}(x) = \frac{\sin \operatorname{arccot} x}{1+x^2}$.

(17) Suppose $Z \subseteq \mathrm{dom}((\text{the function tan}) \cdot (\text{the function arctan}))$ and $Z \subseteq]-1, 1[$. Then

(i) (the function tan) $\cdot$(the function arctan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function tan) $\cdot$(the function arctan))$'_{\restriction Z}(x) = \frac{1}{(\cos \arctan x)^2\cdot(1+x^2)}$.

(18) Suppose $Z \subseteq \mathrm{dom}((\text{the function tan}) \cdot (\text{the function arccot}))$ and $Z \subseteq]-1, 1[$. Then

(i) (the function tan) $\cdot$(the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function tan) $\cdot$(the function arccot))$'_{\restriction Z}(x) = -\frac{1}{(\cos \operatorname{arccot} x)^2\cdot(1+x^2)}$.

(19) Suppose $Z \subseteq$ dom((the function cot) $\cdot$(the function arctan)) and $Z \subseteq]-1, 1[$. Then
(i) (the function cot) $\cdot$(the function arctan) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function cot) $\cdot$(the function arctan))$'_{\restriction Z}(x) = -\frac{1}{(\sin \arctan x)^{2} \cdot (1+x^{2})}$.

(20) Suppose $Z \subseteq$ dom((the function cot) $\cdot$(the function arccot)) and $Z \subseteq]-1, 1[$. Then
(i) (the function cot) $\cdot$(the function arccot) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function cot) $\cdot$(the function arccot))$'_{\restriction Z}(x) = \frac{1}{(\sin \operatorname{arccot} x)^{2} \cdot (1+x^{2})}$.

(21) Suppose $Z \subseteq$ dom((the function sec) $\cdot$(the function arctan)) and $Z \subseteq]-1, 1[$. Then
(i) (the function sec) $\cdot$(the function arctan) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function sec) $\cdot$(the function arctan))$'_{\restriction Z}(x) = \frac{\sin \arctan x}{(\cos \arctan x)^{2} \cdot (1+x^{2})}$.

(22) Suppose $Z \subseteq$ dom((the function sec) $\cdot$(the function arccot)) and $Z \subseteq]-1, 1[$. Then
(i) (the function sec) $\cdot$(the function arccot) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function sec) $\cdot$(the function arccot))$'_{\restriction Z}(x) = -\frac{\sin \operatorname{arccot} x}{(\cos \operatorname{arccot} x)^{2} \cdot (1+x^{2})}$.

(23) Suppose $Z \subseteq$ dom((the function cosec) $\cdot$(the function arctan)) and $Z \subseteq]-1, 1[$. Then
(i) (the function cosec) $\cdot$(the function arctan) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function cosec) $\cdot$(the function arctan))$'_{\restriction Z}(x) = -\frac{\cos \arctan x}{(\sin \arctan x)^{2} \cdot (1+x^{2})}$.

(24) Suppose $Z \subseteq$ dom((the function cosec) $\cdot$(the function arccot)) and $Z \subseteq]-1, 1[$. Then
(i) (the function cosec) $\cdot$(the function arccot) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function cosec) $\cdot$(the function arccot))$'_{\restriction Z}(x) = \frac{\cos \operatorname{arccot} x}{(\sin \operatorname{arccot} x)^{2} \cdot (1+x^{2})}$.

(25) Suppose $Z \subseteq$ dom((the function sin) (the function arctan)) and $Z \subseteq]-1, 1[$. Then
(i) (the function sin) (the function arctan) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function sin) (the function arctan))$'_{\restriction Z}(x) = \cos x \cdot \arctan x + \frac{\sin x}{1+x^{2}}$.

(26) Suppose $Z \subseteq$ dom((the function sin) (the function arccot)) and $Z \subseteq]-1, 1[$. Then
(i) (the function sin) (the function arccot) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function sin) (the function arccot))$'_{\restriction Z}(x) = \cos x \cdot \operatorname{arccot} x - \frac{\sin x}{1+x^{2}}$.

(27) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos})\ (\text{the function arctan}))$ and $Z \subseteq]-1,1[$. Then

(i) (the function cos) (the function arctan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function cos})\ (\text{the function arctan}))'_{\restriction Z}(x) = -\sin x \cdot \arctan x + \frac{\cos x}{1+x^2}$.

(28) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos})\ (\text{the function arccot}))$ and $Z \subseteq]-1,1[$. Then

(i) (the function cos) (the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function cos})\ (\text{the function arccot}))'_{\restriction Z}(x) = -\sin x \cdot \mathrm{arccot}\, x - \frac{\cos x}{1+x^2}$.

(29) Suppose $Z \subseteq \mathrm{dom}((\text{the function tan})\ (\text{the function arctan}))$ and $Z \subseteq]-1,1[$. Then

(i) (the function tan) (the function arctan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function tan})\ (\text{the function arctan}))'_{\restriction Z}(x) = \frac{\arctan x}{(\cos x)^2} + \frac{\tan x}{1+x^2}$.

(30) Suppose $Z \subseteq \mathrm{dom}((\text{the function tan})\ (\text{the function arccot}))$ and $Z \subseteq]-1,1[$. Then

(i) (the function tan) (the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function tan})\ (\text{the function arccot}))'_{\restriction Z}(x) = \frac{\mathrm{arccot}\, x}{(\cos x)^2} - \frac{\tan x}{1+x^2}$.

(31) Suppose $Z \subseteq \mathrm{dom}((\text{the function cot})\ (\text{the function arctan}))$ and $Z \subseteq]-1,1[$. Then

(i) (the function cot) (the function arctan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function cot})\ (\text{the function arctan}))'_{\restriction Z}(x) = -\frac{\arctan x}{(\sin x)^2} + \frac{\cot x}{1+x^2}$.

(32) Suppose $Z \subseteq \mathrm{dom}((\text{the function cot})\ (\text{the function arccot}))$ and $Z \subseteq]-1,1[$. Then

(i) (the function cot) (the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function cot})\ (\text{the function arccot}))'_{\restriction Z}(x) = -\frac{\mathrm{arccot}\, x}{(\sin x)^2} - \frac{\cot x}{1+x^2}$.

(33) Suppose $Z \subseteq \mathrm{dom}((\text{the function sec})\ (\text{the function arctan}))$ and $Z \subseteq]-1,1[$. Then

(i) (the function sec) (the function arctan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function sec})\ (\text{the function arctan}))'_{\restriction Z}(x) = \frac{\sin x \cdot \arctan x}{(\cos x)^2} + \frac{1}{\cos x \cdot (1+x^2)}$.

(34) Suppose $Z \subseteq \mathrm{dom}((\text{the function sec})\ (\text{the function arccot}))$ and $Z \subseteq]-1,1[$. Then

(i) (the function sec) (the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function sec})\ (\text{the function arccot}))'_{\restriction Z}(x) = \frac{\sin x \cdot \mathrm{arccot}\, x}{(\cos x)^2} - \frac{1}{\cos x \cdot (1+x^2)}$.

(35) Suppose $Z \subseteq$ dom((the function cosec) (the function arctan)) and $Z \subseteq]-1,1[$. Then

(i) (the function cosec) (the function arctan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cosec) (the function arctan))$'_{\restriction Z}(x) = -\frac{\cos x \cdot \arctan x}{(\sin x)^{\mathbf{2}}} + \frac{1}{\sin x \cdot (1+x^{\mathbf{2}})}$.

(36) Suppose $Z \subseteq$ dom((the function cosec) (the function arccot)) and $Z \subseteq]-1,1[$. Then

(i) (the function cosec) (the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cosec) (the function arccot))$'_{\restriction Z}(x) = -\frac{\cos x \cdot \operatorname{arccot} x}{(\sin x)^{\mathbf{2}}} - \frac{1}{\sin x \cdot (1+x^{\mathbf{2}})}$.

(37) Suppose $Z \subseteq]-1,1[$. Then

(i) (the function arctan)+(the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function arctan)+(the function arccot))$'_{\restriction Z}(x) = 0$.

(38) Suppose $Z \subseteq]-1,1[$. Then

(i) (the function arctan)−(the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function arctan)−(the function arccot))$'_{\restriction Z}(x) = \frac{2}{1+x^{\mathbf{2}}}$.

(39) Suppose $Z \subseteq]-1,1[$. Then

(i) (the function sin) ((the function arctan)+(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) ((the function arctan)+(the function arccot)))$'_{\restriction Z}(x) = \cos x \cdot (\arctan x + \operatorname{arccot} x)$.

(40) Suppose $Z \subseteq]-1,1[$. Then

(i) (the function sin) ((the function arctan)−(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) ((the function arctan)−(the function arccot)))$'_{\restriction Z}(x) = \cos x \cdot (\arctan x - \operatorname{arccot} x) + \frac{2 \cdot \sin x}{1+x^{\mathbf{2}}}$.

(41) Suppose $Z \subseteq]-1,1[$. Then

(i) (the function cos) ((the function arctan)+(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) ((the function arctan)+(the function arccot)))$'_{\restriction Z}(x) = -\sin x \cdot (\arctan x + \operatorname{arccot} x)$.

(42) Suppose $Z \subseteq]-1,1[$. Then

(i) (the function cos) ((the function arctan)−(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) ((the function arctan)−(the function arccot)))$'_{\restriction Z}(x) = -\sin x \cdot (\arctan x - \operatorname{arccot} x) + \frac{2 \cdot \cos x}{1+x^{\mathbf{2}}}$.

(43) Suppose $Z \subseteq$ dom (the function tan) and $Z \subseteq]-1,1[$. Then

(i) (the function tan) ((the function arctan)+(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function tan) ((the function arctan)+(the function arccot)))$'_{\restriction Z}(x) = \frac{\arctan x + \operatorname{arccot} x}{(\cos x)^{2}}$.

(44) Suppose $Z \subseteq \operatorname{dom}$ (the function tan) and $Z \subseteq]-1, 1[$. Then

(i) (the function tan) ((the function arctan)−(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function tan) ((the function arctan)−(the function arccot)))$'_{\restriction Z}(x) = \frac{\arctan x - \operatorname{arccot} x}{(\cos x)^{2}} + \frac{2 \cdot \tan x}{1+x^{2}}$.

(45) Suppose $Z \subseteq \operatorname{dom}$ (the function cot) and $Z \subseteq]-1, 1[$. Then

(i) (the function cot) ((the function arctan)+(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cot) ((the function arctan)+(the function arccot)))$'_{\restriction Z}(x) = -\frac{\arctan x + \operatorname{arccot} x}{(\sin x)^{2}}$.

(46) Suppose $Z \subseteq \operatorname{dom}$ (the function cot) and $Z \subseteq]-1, 1[$. Then

(i) (the function cot) ((the function arctan)−(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cot) ((the function arctan)−(the function arccot)))$'_{\restriction Z}(x) = -\frac{\arctan x - \operatorname{arccot} x}{(\sin x)^{2}} + \frac{2 \cdot \cot x}{1+x^{2}}$.

(47) Suppose $Z \subseteq \operatorname{dom}$ (the function sec) and $Z \subseteq]-1, 1[$. Then

(i) (the function sec) ((the function arctan)+(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sec) ((the function arctan)+(the function arccot)))$'_{\restriction Z}(x) = \frac{(\arctan x + \operatorname{arccot} x) \cdot \sin x}{(\cos x)^{2}}$.

(48) Suppose $Z \subseteq \operatorname{dom}$ (the function sec) and $Z \subseteq]-1, 1[$. Then

(i) (the function sec) ((the function arctan)−(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sec) ((the function arctan)−(the function arccot)))$'_{\restriction Z}(x) = \frac{(\arctan x - \operatorname{arccot} x) \cdot \sin x}{(\cos x)^{2}} + \frac{2 \cdot \sec x}{1+x^{2}}$.

(49) Suppose $Z \subseteq \operatorname{dom}$ (the function cosec) and $Z \subseteq]-1, 1[$. Then

(i) (the function cosec) ((the function arctan)+(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cosec) ((the function arctan)+(the function arccot)))$'_{\restriction Z}(x) = -\frac{(\arctan x + \operatorname{arccot} x) \cdot \cos x}{(\sin x)^{2}}$.

(50) Suppose $Z \subseteq \operatorname{dom}$ (the function cosec) and $Z \subseteq]-1, 1[$. Then

(i) (the function cosec) ((the function arctan)−(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cosec) ((the function arctan)−(the function arccot)))$'_{\restriction Z}(x) = -\frac{(\arctan x - \operatorname{arccot} x) \cdot \cos x}{(\sin x)^{2}} + \frac{2 \cdot \operatorname{cosec} x}{1+x^{2}}$.

(51) Suppose $Z \subseteq]-1, 1[$. Then

(i) (the function exp) ((the function arctan)+(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) ((the function arctan)+(the function arccot)))$'_{\restriction Z}(x) = \exp x \cdot (\arctan x + \operatorname{arccot} x)$.

(52) Suppose $Z \subseteq]-1, 1[$. Then

(i) (the function exp) ((the function arctan)−(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) ((the function arctan)−(the function arccot)))$'_{\restriction Z}(x) = \exp x \cdot (\arctan x - \operatorname{arccot} x) + \frac{2 \cdot \exp x}{1+x^2}$.

(53) Suppose $Z \subseteq]-1, 1[$. Then

(i) $\frac{\text{(the function arctan)+(the function arccot)}}{\text{the function exp}}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{\text{(the function arctan)+(the function arccot)}}{\text{the function exp}})'_{\restriction Z}(x) = -\frac{\arctan x + \operatorname{arccot} x}{\exp x}$.

(54) Suppose $Z \subseteq]-1, 1[$. Then

(i) $\frac{\text{(the function arctan)−(the function arccot)}}{\text{the function exp}}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{\text{(the function arctan)−(the function arccot)}}{\text{the function exp}})'_{\restriction Z}(x) = \frac{(\frac{2}{1+x^2} - \arctan x) + \operatorname{arccot} x}{\exp x}$.

(55) Suppose $Z \subseteq$ dom((the function exp) $\cdot$((the function arctan)+(the function arccot))) and $Z \subseteq]-1, 1[$. Then

(i) (the function exp) $\cdot$((the function arctan)+(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) $\cdot$((the function arctan)+(the function arccot)))$'_{\restriction Z}(x) = 0$.

(56) Suppose $Z \subseteq$ dom((the function exp) $\cdot$((the function arctan)−(the function arccot))) and $Z \subseteq]-1, 1[$. Then

(i) (the function exp) $\cdot$((the function arctan)−(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function exp) $\cdot$((the function arctan)−(the function arccot)))$'_{\restriction Z}(x) = \frac{2 \cdot \exp(\arctan x - \operatorname{arccot} x)}{1+x^2}$.

(57) Suppose $Z \subseteq$ dom((the function sin) $\cdot$((the function arctan)+(the function arccot))) and $Z \subseteq]-1, 1[$. Then

(i) (the function sin) $\cdot$((the function arctan)+(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) $\cdot$((the function arctan)+(the function arccot)))$'_{\restriction Z}(x) = 0$.

(58) Suppose $Z \subseteq$ dom((the function sin) $\cdot$((the function arctan)−(the function arccot))) and $Z \subseteq]-1, 1[$. Then

(i) (the function sin) $\cdot$((the function arctan)−(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function sin) $\cdot$((the function arctan)$-$(the function arccot)))$'_{\restriction Z}(x) = \frac{2 \cdot \cos(\arctan x - \operatorname{arccot} x)}{1+x^2}$.

(59) Suppose $Z \subseteq$ dom((the function cos) $\cdot$((the function arctan)+(the function arccot))) and $Z \subseteq]-1, 1[$. Then

(i) (the function cos) $\cdot$((the function arctan)+(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) $\cdot$((the function arctan)+(the function arccot)))$'_{\restriction Z}(x) = 0$.

(60) Suppose $Z \subseteq$ dom((the function cos) $\cdot$((the function arctan)$-$(the function arccot))) and $Z \subseteq]-1, 1[$. Then

(i) (the function cos) $\cdot$((the function arctan)$-$(the function arccot)) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function cos) $\cdot$((the function arctan)$-$(the function arccot)))$'_{\restriction Z}(x) = -\frac{2 \cdot \sin(\arctan x - \operatorname{arccot} x)}{1+x^2}$.

(61) Suppose $Z \subseteq]-1, 1[$. Then

(i) (the function arctan) (the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function arctan) (the function arccot))$'_{\restriction Z}(x) = \frac{\operatorname{arccot} x - \arctan x}{1+x^2}$.

(62) Suppose that

(i) $Z \subseteq$ dom(((the function arctan) $\cdot \frac{1}{f}$) ((the function arccot) $\cdot \frac{1}{f}$)), and

(ii) for every x such that $x \in Z$ holds $f(x) = x$ and $-1 < (\frac{1}{f})(x) < 1$.

Then

(iii) ((the function arctan) $\cdot \frac{1}{f}$) ((the function arccot) $\cdot \frac{1}{f}$) is differentiable on Z, and

(iv) for every x such that $x \in Z$ holds (((the function arctan) $\cdot \frac{1}{f}$) ((the function arccot) $\cdot \frac{1}{f}$))$'_{\restriction Z}(x) = \frac{\arctan(\frac{1}{x}) - \operatorname{arccot}(\frac{1}{x})}{1+x^2}$.

(63) Suppose $Z \subseteq$ dom(id_Z ((the function arctan) $\cdot \frac{1}{f}$)) and for every x such that $x \in Z$ holds $f(x) = x$ and $-1 < (\frac{1}{f})(x) < 1$. Then

(i) id_Z ((the function arctan) $\cdot \frac{1}{f}$) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds (id_Z ((the function arctan) $\cdot \frac{1}{f}$))$'_{\restriction Z}(x) = \arctan(\frac{1}{x}) - \frac{x}{1+x^2}$.

(64) Suppose $Z \subseteq$ dom(id_Z ((the function arccot) $\cdot \frac{1}{f}$)) and for every x such that $x \in Z$ holds $f(x) = x$ and $-1 < (\frac{1}{f})(x) < 1$. Then

(i) id_Z ((the function arccot) $\cdot \frac{1}{f}$) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds (id_Z ((the function arccot) $\cdot \frac{1}{f}$))$'_{\restriction Z}(x) = \operatorname{arccot}(\frac{1}{x}) + \frac{x}{1+x^2}$.

(65) Suppose $Z \subseteq$ dom(g ((the function arctan) $\cdot \frac{1}{f}$)) and $g = \square^2$ and for every x such that $x \in Z$ holds $f(x) = x$ and $-1 < (\frac{1}{f})(x) < 1$. Then

(i) g ((the function arctan) $\cdot \frac{1}{f}$) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(g\,((\text{the function arctan}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = 2 \cdot x \cdot \operatorname{arctan}(\frac{1}{x}) - \frac{x^2}{1+x^2}$.

(66) Suppose $Z \subseteq \operatorname{dom}(g\,((\text{the function arccot}) \cdot \frac{1}{f}))$ and $g = \square^2$ and for every x such that $x \in Z$ holds $f(x) = x$ and $-1 < (\frac{1}{f})(x) < 1$. Then

(i) $g\,((\text{the function arccot}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(g\,((\text{the function arccot}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = 2 \cdot x \cdot \operatorname{arccot}(\frac{1}{x}) + \frac{x^2}{1+x^2}$.

(67) Suppose $Z \subseteq \,]-1,1[$ and for every x such that $x \in Z$ holds (the function arctan)$(x) \neq 0$. Then

(i) $\frac{1}{\text{the function arctan}}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{\text{the function arctan}})'_{\restriction Z}(x) = -\frac{1}{(\arctan x)^2 \cdot (1+x^2)}$.

(68) Suppose $Z \subseteq \,]-1,1[$. Then

(i) $\frac{1}{\text{the function arccot}}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{\text{the function arccot}})'_{\restriction Z}(x) = \frac{1}{(\operatorname{arccot} x)^2 \cdot (1+x^2)}$.

One can prove the following propositions:

(69) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{n\,(\text{the function arctan})^n})$ and $Z \subseteq \,]-1,1[$ and $n > 0$ and for every x such that $x \in Z$ holds $\arctan x \neq 0$. Then

(i) $\frac{1}{n\,(\text{the function arctan})^n}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{n\,(\text{the function arctan})^n})'_{\restriction Z}(x) = -\frac{1}{((\arctan x)^{n+1}) \cdot (1+x^2)}$.

(70) Suppose $Z \subseteq \operatorname{dom}(\frac{1}{n\,(\text{the function arccot})^n})$ and $Z \subseteq \,]-1,1[$ and $n > 0$. Then

(i) $\frac{1}{n\,(\text{the function arccot})^n}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{n\,(\text{the function arccot})^n})'_{\restriction Z}(x) = \frac{1}{((\operatorname{arccot} x)^{n+1}) \cdot (1+x^2)}$.

(71) Suppose $Z \subseteq \operatorname{dom}(2\,(\text{the function arctan})^{\frac{1}{2}})$ and $Z \subseteq \,]-1,1[$ and for every x such that $x \in Z$ holds $\arctan x > 0$. Then

(i) $2\,(\text{the function arctan})^{\frac{1}{2}}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(2\,(\text{the function arctan})^{\frac{1}{2}})'_{\restriction Z}(x) = \frac{(\arctan x)^{-\frac{1}{2}}}{1+x^2}$.

(72) Suppose $Z \subseteq \operatorname{dom}(2\,(\text{the function arccot})^{\frac{1}{2}})$ and $Z \subseteq \,]-1,1[$. Then

(i) $2\,(\text{the function arccot})^{\frac{1}{2}}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(2\,(\text{the function arccot})^{\frac{1}{2}})'_{\restriction Z}(x) = -\frac{(\operatorname{arccot} x)^{-\frac{1}{2}}}{1+x^2}$.

(73) Suppose $Z \subseteq \mathrm{dom}(\frac{2}{3}\,(\text{the function arctan})^{\frac{3}{2}})$ and $Z \subseteq]-1,1[$ and for every x such that $x \in Z$ holds $\arctan x > 0$. Then

(i) $\frac{2}{3}\,(\text{the function arctan})^{\frac{3}{2}}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{2}{3}\,(\text{the function arctan})^{\frac{3}{2}})'_{\restriction Z}(x) = \frac{(\arctan x)^{\frac{1}{2}}}{1+x^{\mathbf{2}}}$.

(74) Suppose $Z \subseteq \mathrm{dom}(\frac{2}{3}\,(\text{the function arccot})^{\frac{3}{2}})$ and $Z \subseteq]-1,1[$. Then

(i) $\frac{2}{3}\,(\text{the function arccot})^{\frac{3}{2}}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{2}{3}\,(\text{the function arccot})^{\frac{3}{2}})'_{\restriction Z}(x) = -\frac{(\mathrm{arccot}\, x)^{\frac{1}{2}}}{1+x^{\mathbf{2}}}$.

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Open Mapping Theorem

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Summary. In this article we formalize one of the most important theorems of linear operator theory the Open Mapping Theorem commonly used in a standard book such as [8] in chapter 2.4.2. It states that a surjective continuous linear operator between Banach spaces is an open map.

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The notation and terminology used here are introduced in the following papers: [13], [14], [3], [9], [2], [7], [1], [4], [5], [10], [6], [12], [11], and [15].

The following proposition is true

(1) For all real numbers x, y such that $0 \leq x < y$ there exists a real number s_0 such that $0 < s_0$ and $x < \frac{y}{1+s_0} < y$.

The scheme *RecExD3* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, an element $\mathcal{C}$ of $\mathcal{A}$, and a 4-ary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and $f(1) = \mathcal{C}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), f(n+1), f(n+2)]$

provided the parameters meet the following requirement:

- For every element n of $\mathbb{N}$ and for all elements x, y of $\mathcal{A}$ there exists an element z of $\mathcal{A}$ such that $\mathcal{P}[n, x, y, z]$.

In the sequel X, Y denote real normed spaces.

The following propositions are true:

(2) For every point y_1 of X and for every real number r holds $\mathrm{Ball}(y_1, r) = y_1 + \mathrm{Ball}(0_X, r)$.

(3) For every real number r and for every real number a such that $0 < a$ holds $\mathrm{Ball}(0_X, a \cdot r) = a \cdot \mathrm{Ball}(0_X, r)$.

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(4) For every linear operator T from X into Y and for all subsets B_0, B_1 of X holds $T^\circ(B_0 + B_1) = T^\circ B_0 + T^\circ B_1$.

(5) Let T be a linear operator from X into Y, B_0 be a subset of X, and a be a real number. Then $T^\circ(a \cdot B_0) = a \cdot T^\circ B_0$.

(6) Let T be a linear operator from X into Y, B_0 be a subset of X, and x_1 be a point of X. Then $T^\circ(x_1 + B_0) = T(x_1) + T^\circ B_0$.

(7) For all subsets V, W of X and for all subsets V_1, W_1 of LinearTopSpaceNorm X such that $V = V_1$ and $W = W_1$ holds $V + W = V_1 + W_1$.

(8) Let V be a subset of X, x be a point of X, V_1 be a subset of LinearTopSpaceNorm X, and x_1 be a point of LinearTopSpaceNorm X. If $V = V_1$ and $x = x_1$, then $x + V = x_1 + V_1$.

(9) For every subset V of X and for every real number a and for every subset V_1 of LinearTopSpaceNorm X such that $V = V_1$ holds $a \cdot V = a \cdot V_1$.

(10) For every subset V of TopSpaceNorm X and for every subset V_1 of LinearTopSpaceNorm X such that $V = V_1$ holds $\overline{V} = \overline{V_1}$.

(11) For every point x of X and for every real number r holds $\mathrm{Ball}(0_X, r) = (-1) \cdot \mathrm{Ball}(0_X, r)$.

(12) For every point x of X and for every real number r and for every subset V of LinearTopSpaceNorm X such that $V = \mathrm{Ball}(x, r)$ holds V is convex.

(13) Let x be a point of X, r be a real number, T be a linear operator from X into Y, and V be a subset of LinearTopSpaceNorm Y. If $V = T^\circ\,\mathrm{Ball}(x, r)$, then V is convex.

(14) For every point x of X and for all real numbers r, s such that $r \leq s$ holds $\mathrm{Ball}(x, r) \subseteq \mathrm{Ball}(x, s)$.

(15) Let X be a real Banach space, Y be a real normed space, T be a bounded linear operator from X into Y, r be a real number, B_2 be a subset of LinearTopSpaceNorm X, and T_1, B_3 be subsets of LinearTopSpaceNorm Y. If $r > 0$ and $B_2 = \mathrm{Ball}(0_X, 1)$ and $B_3 = \mathrm{Ball}(0_Y, r)$ and $T_1 = T^\circ\,\mathrm{Ball}(0_X, 1)$ and $B_3 \subseteq \overline{T_1}$, then $B_3 \subseteq T_1$.

(16) Let X, Y be real Banach spaces, T be a bounded linear operator from X into Y, and T_2 be a function from LinearTopSpaceNorm X into LinearTopSpaceNorm Y. If $T_2 = T$ and T_2 is onto, then T_2 is open.

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