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BCI-algebras with Condition (S) and their Properties

Tao Sun
Qingdao University of Science and Technology
China

Junjie Zhao
Qingdao University of Science and Technology
China

Xiquan Liang
Qingdao University of Science and Technology
China

Summary. In this article we will first investigate the elementary properties of BCI-algebras with condition (S), see [8]. And then we will discuss the three classes of algebras: commutative, positive-implicative and implicative BCK-algebras with condition (S).

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The papers [5], [12], [3], [1], [6], [2], [10], [9], [4], [11], and [7] provide the notation and terminology for this paper.

We introduce BCI stuctures with complements which are extensions of BCI structure with 0 and zero structure and are systems

⟨ a carrier, an external complement, an internal complement, a zero ⟩,

where the carrier is a set, the external complement and the internal complement are binary operations on the carrier, and the zero is an element of the carrier.

Let us mention that there exists a BCI structure with complements which is non empty and strict.

Let A be a BCI structure with complements and let x, y be elements of A. The functor $x \cdot y$ yields an element of A and is defined as follows:

(Def. 1) $x \cdot y = (\text{the external complement of } A)(x, y)$.

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Let $\mathfrak{B}$ be a non empty BCI structure with complements. We say that $\mathfrak{B}$ satisfies condition (S) if and only if:

(Def. 2) For all elements x, y, z of $\mathfrak{B}$ holds $x \setminus y \setminus z = x \setminus y \cdot z$.

The BCI structure the BCI S-example with complements is defined by:

(Def. 3) The BCI S-example $= \langle 1, \mathrm{op}_2, \mathrm{op}_2, \mathrm{op}_0 \rangle$.

Let us observe that the BCI S-example is strict, non empty, and trivial.

Let us observe that the BCI S-example is B, C, I, BCI-4, and BCK-5 and satisfies condition (S).

Let us note that there exists a non empty BCI structure with complements which is strict, B, C, I, and BCI-4 and satisfies condition (S).

A BCI-algebra with condition (S) is B C I BCI-4 non empty BCI structure with complements satisfying condition (S).

In the sequel $\mathfrak{X}$ is a non empty BCI structure with complements, x, d are elements of $\mathfrak{X}$, and n is an element of $\mathbb{N}$.

Let $\mathfrak{X}$ be a BCI-algebra with condition (S) and let x, y be elements of $\mathfrak{X}$. The functor ConditionS(x, y) yields a non empty subset of $\mathfrak{X}$ and is defined as follows:

(Def. 4) $\mathrm{ConditionS}(x, y) = \{t \in \mathfrak{X} \colon t \setminus x \leq y\}$.

We now state four propositions:

(1) Let $\mathfrak{X}$ be a BCI-algebra with condition (S) and x, y, u, v be elements of $\mathfrak{X}$. If $u \in \mathrm{ConditionS}(x, y)$ and $v \leq u$, then $v \in \mathrm{ConditionS}(x, y)$.

(2) Let $\mathfrak{X}$ be a BCI-algebra with condition (S) and x, y be elements of $\mathfrak{X}$. Then there exists an element a of ConditionS(x, y) such that for every element z of ConditionS(x, y) holds $z \leq a$.

(3) $\mathfrak{X}$ is a BCI-algebra and for all elements x, y of $\mathfrak{X}$ holds $x \cdot y \setminus x \leq y$ and for every element t of $\mathfrak{X}$ such that $t \setminus x \leq y$ holds $t \leq x \cdot y$ if and only if $\mathfrak{X}$ is a BCI-algebra with condition (S).

(4) Let $\mathfrak{X}$ be a BCI-algebra with condition (S) and x, y be elements of $\mathfrak{X}$. Then there exists an element a of ConditionS(x, y) such that for every element z of ConditionS(x, y) holds $z \leq a$.

Let $\mathfrak{X}$ be a p-semisimple BCI-algebra. The adjoint p-group of $\mathfrak{X}$ yields a strict Abelian group and is defined by the conditions (Def. 5).

(Def. 5)(i) The carrier of the adjoint p-group of $\mathfrak{X}$ = the carrier of $\mathfrak{X}$,

(ii) for all elements x, y of $\mathfrak{X}$ holds (the addition of the adjoint p-group of $\mathfrak{X}$)$(x, y) = x \setminus (0_{\mathfrak{X}} \setminus y)$, and

(iii) $0_{\text{the adjoint p-group of } \mathfrak{X}} = 0_{\mathfrak{X}}$.

We now state a number of propositions:

(5) Let $\mathfrak{X}$ be a BCI-algebra. Then $\mathfrak{X}$ is p-semisimple if and only if for all elements x, y of $\mathfrak{X}$ such that $x \setminus y = 0_{\mathfrak{X}}$ holds $x = y$.

(6) Let $\mathfrak{X}$ be a BCI-algebra with condition (S). Suppose $\mathfrak{X}$ is p-semisimple. Let x, y be elements of $\mathfrak{X}$. Then $x \cdot y = x \setminus (0_{\mathfrak{X}} \setminus y)$.

(7) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements x, y of $\mathfrak{X}$ holds $x \cdot y = y \cdot x$.

(8) Let $\mathfrak{X}$ be a BCI-algebra with condition (S) and x, y, z be elements of $\mathfrak{X}$. If $x \leq y$, then $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$.

(9) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for every element x of $\mathfrak{X}$ holds $0_{\mathfrak{X}} \cdot x = x$ and $x \cdot 0_{\mathfrak{X}} = x$.

(10) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements x, y, z of $\mathfrak{X}$ holds $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(11) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements x, y, z of $\mathfrak{X}$ holds $x \cdot y \cdot z = x \cdot z \cdot y$.

(12) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements x, y, z of $\mathfrak{X}$ holds $x \setminus y \setminus z = x \setminus y \cdot z$.

(13) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements x, y of $\mathfrak{X}$ holds $y \leq x \cdot (y \setminus x)$.

(14) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements x, y, z of $\mathfrak{X}$ holds $x \cdot z \setminus y \cdot z \leq x \setminus y$.

(15) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements x, y, z of $\mathfrak{X}$ holds $x \setminus y \leq z$ iff $x \leq y \cdot z$.

(16) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements x, y, z of $\mathfrak{X}$ holds $x \setminus y \leq (x \setminus z) \cdot (z \setminus y)$.

Let $\mathfrak{X}$ be a BCI-algebra with condition (S). One can check that the external complement of $\mathfrak{X}$ is commutative and associative.

Next we state three propositions:

(17) For every BCI-algebra $\mathfrak{X}$ with condition (S) holds $0_{\mathfrak{X}}$ is a unity w.r.t. the external complement of $\mathfrak{X}$.

(18) For every BCI-algebra $\mathfrak{X}$ with condition (S) holds $\mathbf{1}_{\text{the external complement of } \mathfrak{X}} = 0_{\mathfrak{X}}$.

(19) For every BCI-algebra $\mathfrak{X}$ with condition (S) holds the external complement of $\mathfrak{X}$ has a unity.

Let $\mathfrak{X}$ be a BCI-algebra with condition (S). The functor $\mathrm{power}_{\mathfrak{X}}$ yielding a function from (the carrier of $\mathfrak{X}$) $\times$ $\mathbb{N}$ into the carrier of $\mathfrak{X}$ is defined as follows:

(Def. 6) For every element h of $\mathfrak{X}$ holds $\mathrm{power}_{\mathfrak{X}}(h, 0) = 0_{\mathfrak{X}}$ and for every n holds $\mathrm{power}_{\mathfrak{X}}(h, n+1) = \mathrm{power}_{\mathfrak{X}}(h, n) \cdot h$.

Let $\mathfrak{X}$ be a BCI-algebra with condition (S), let x be an element of $\mathfrak{X}$, and let us consider n. The functor x^n yields an element of $\mathfrak{X}$ and is defined by:

(Def. 7) $x^n = \mathrm{power}_{\mathfrak{X}}(x, n)$.

The following propositions are true:

(20) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for every element x of $\mathfrak{X}$ holds $x^0 = 0_{\mathfrak{X}}$.

(21) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for every element x of $\mathfrak{X}$ holds $x^{n+1} = x^n \cdot x$.

(22) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for every element x of $\mathfrak{X}$ holds $x^1 = x$.

(23) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for every element x of $\mathfrak{X}$ holds $x^2 = x \cdot x$.

(24) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for every element x of $\mathfrak{X}$ holds $x^3 = x \cdot x \cdot x$.

(25) For every BCI-algebra $\mathfrak{X}$ with condition (S) holds $(0_{\mathfrak{X}})^2 = 0_{\mathfrak{X}}$.

(26) For every BCI-algebra $\mathfrak{X}$ with condition (S) holds $(0_{\mathfrak{X}})^n = 0_{\mathfrak{X}}$.

(27) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements x, a of $\mathfrak{X}$ holds $x \setminus a \setminus a \setminus a = x \setminus a^3$.

(28) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements x, a of $\mathfrak{X}$ holds $(x \setminus a)^n = x \setminus a^n$.

Let $\mathfrak{X}$ be a non empty BCI structure with complements and let F be a finite sequence of elements of the carrier of $\mathfrak{X}$. The functor ProductS(F) yielding an element of $\mathfrak{X}$ is defined by:

(Def. 8) ProductS(F) = the external complement of $\mathfrak{X} \odot F$.

One can prove the following propositions:

(29) The external complement of $\mathfrak{X} \odot \langle d \rangle = d$.

(30) Let $\mathfrak{X}$ be a BCI-algebra with condition (S) and F_1, F_2 be finite sequences of elements of the carrier of $\mathfrak{X}$. Then ProductS($F_1 \frown F_2$) = ProductS(F_1) $\cdot$ ProductS(F_2).

(31) Let $\mathfrak{X}$ be a BCI-algebra with condition (S), F be a finite sequence of elements of the carrier of $\mathfrak{X}$, and a be an element of $\mathfrak{X}$. Then ProductS($F \frown \langle a \rangle$) = ProductS($F$) $\cdot\, a$.

(32) Let $\mathfrak{X}$ be a BCI-algebra with condition (S), F be a finite sequence of elements of the carrier of $\mathfrak{X}$, and a be an element of $\mathfrak{X}$. Then ProductS($\langle a \rangle \frown F$) = $a \,\cdot$ ProductS(F).

(33) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements a_1, a_2 of $\mathfrak{X}$ holds ProductS($\langle a_1, a_2 \rangle$) = $a_1 \cdot a_2$.

(34) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements a_1, a_2, a_3 of $\mathfrak{X}$ holds ProductS($\langle a_1, a_2, a_3 \rangle$) = $a_1 \cdot a_2 \cdot a_3$.

(35) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements x, a_1, a_2 of $\mathfrak{X}$ holds $x \setminus a_1 \setminus a_2 = x \,\setminus$ ProductS($\langle a_1, a_2 \rangle$).

(36) For every BCI-algebra $\mathfrak{X}$ with condition (S) and for all elements x, a_1, a_2, a_3 of $\mathfrak{X}$ holds $x \setminus a_1 \setminus a_2 \setminus a_3 = x \,\setminus$ ProductS($\langle a_1, a_2, a_3 \rangle$).

(37) Let $\mathfrak{X}$ be a BCI-algebra with condition (S), a, b be elements of AtomSet $\mathfrak{X}$, and m_1 be an element of $\mathfrak{X}$. Suppose that for every element x of BranchV a holds $x \leq m_1$. Then there exists an element m_2 of $\mathfrak{X}$ such that for every element y of BranchV b holds $y \leq m_2$.

Let us observe that there exists a BCI-algebra with condition (S) which is strict and BCK-5.

A BCK-algebra with condition (S) is BCK-5 BCI-algebra with condition (S).

We now state four propositions:

(38) For every BCK-algebra $\mathfrak{X}$ with condition (S) and for all elements x, y of $\mathfrak{X}$ holds $x \leq x \cdot y$ and $y \leq x \cdot y$.

(39) For every BCK-algebra $\mathfrak{X}$ with condition (S) and for all elements x, y, z of $\mathfrak{X}$ holds $x \cdot y \setminus y \cdot z \setminus z \cdot x = 0_{\mathfrak{X}}$.

(40) For every BCK-algebra $\mathfrak{X}$ with condition (S) and for all elements x, y of $\mathfrak{X}$ holds $(x \setminus y) \cdot (y \setminus x) \leq x \cdot y$.

(41) For every BCK-algebra $\mathfrak{X}$ with condition (S) and for every element x of $\mathfrak{X}$ holds $(x \setminus 0_{\mathfrak{X}}) \cdot (0_{\mathfrak{X}} \setminus x) = x$.

Let $\mathfrak{B}$ be a BCK-algebra with condition (S). We say that $\mathfrak{B}$ is commutative if and only if:

(Def. 9) For all elements x, y of $\mathfrak{B}$ holds $x \setminus (x \setminus y) = y \setminus (y \setminus x)$.

One can verify that there exists a BCK-algebra with condition (S) which is commutative.

Next we state two propositions:

(42) Let $\mathfrak{X}$ be a non empty BCI structure with complements. Then $\mathfrak{X}$ is a commutative BCK-algebra with condition (S) if and only if for all elements x, y, z of $\mathfrak{X}$ holds $x \setminus (0_{\mathfrak{X}} \setminus y) = x$ and $(x \setminus z) \setminus (x \setminus y) = y \setminus z \setminus (y \setminus x)$ and $x \setminus y \setminus z = x \setminus y \cdot z$.

(43) Let $\mathfrak{X}$ be a commutative BCK-algebra with condition (S) and a be an element of $\mathfrak{X}$. If a is greatest, then for all elements x, y of $\mathfrak{X}$ holds $x \cdot y = a \setminus (a \setminus x \setminus y)$.

Let $\mathfrak{X}$ be a BCI-algebra and let a be an element of $\mathfrak{X}$. The initial section of a yields a non empty subset of $\mathfrak{X}$ and is defined by:

(Def. 10) The initial section of $a = \{t \in \mathfrak{X}\colon t \leq a\}$.

The following proposition is true

(44) Let $\mathfrak{X}$ be a commutative BCK-algebra with condition (S) and a, b, c be elements of $\mathfrak{X}$. Suppose ConditionS$(a, b) \subseteq$ the initial section of c. Let x be an element of ConditionS(a, b). Then $x \leq c \setminus (c \setminus a \setminus b)$.

Let $\mathfrak{B}$ be a BCK-algebra with condition (S). We say that $\mathfrak{B}$ is positive-implicative if and only if:

(Def. 11) For all elements x, y of $\mathfrak{B}$ holds $x \setminus y \setminus y = x \setminus y$.

Let us note that there exists a BCK-algebra with condition (S) which is positive-implicative.

The following propositions are true:

(45) Let $\mathfrak{X}$ be a BCK-algebra with condition (S). Then $\mathfrak{X}$ is positive-implicative if and only if for every element x of $\mathfrak{X}$ holds $x \cdot x = x$.

(46) Let $\mathfrak{X}$ be a BCK-algebra with condition (S). Then $\mathfrak{X}$ is positive-implicative if and only if for all elements x, y of $\mathfrak{X}$ such that $x \leq y$ holds $x \cdot y = y$.

(47) Let $\mathfrak{X}$ be a BCK-algebra with condition (S). Then $\mathfrak{X}$ is positive-implicative if and only if for all elements x, y, z of $\mathfrak{X}$ holds $x \cdot y \setminus z = (x \setminus z) \cdot (y \setminus z)$.

(48) Let $\mathfrak{X}$ be a BCK-algebra with condition (S). Then $\mathfrak{X}$ is positive-implicative if and only if for all elements x, y of $\mathfrak{X}$ holds $x \cdot y = x \cdot (y \setminus x)$.

(49) Let $\mathfrak{X}$ be a positive-implicative BCK-algebra with condition (S) and x, y be elements of $\mathfrak{X}$. Then $x = (x \setminus y) \cdot (x \setminus (x \setminus y))$.

Let $\mathfrak{B}$ be a non empty BCI structure with complements. We say that $\mathfrak{B}$ is SB-1 if and only if:

(Def. 12) For every element x of $\mathfrak{B}$ holds $x \cdot x = x$.

We say that $\mathfrak{B}$ is SB-2 if and only if:

(Def. 13) For all elements x, y of $\mathfrak{B}$ holds $x \cdot y = y \cdot x$.

We say that $\mathfrak{B}$ is SB-4 if and only if:

(Def. 14) For all elements x, y of $\mathfrak{B}$ holds $(x \setminus y) \cdot y = x \cdot y$.

Let us note that the BCI S-example is SB-1, SB-2, SB-4, and I and satisfies condition (S).

Let us note that there exists a non empty BCI structure with complements which is strict, SB-1, SB-2, SB-4, and I and satisfies condition (S).

A semi-Brouwerian algebra is SB-1 SB-2 SB-4 I non empty BCI structure with complements satisfying condition (S).

One can prove the following proposition

(50) Let $\mathfrak{X}$ be a non empty BCI structure with complements. Then $\mathfrak{X}$ is a positive-implicative BCK-algebra with condition (S) if and only if $\mathfrak{X}$ is a semi-Brouwerian algebra.

Let $\mathfrak{B}$ be a BCK-algebra with condition (S). We say that $\mathfrak{B}$ is implicative if and only if:

(Def. 15) For all elements x, y of $\mathfrak{B}$ holds $x \setminus (y \setminus x) = x$.

Let us observe that there exists a BCK-algebra with condition (S) which is implicative.

Next we state two propositions:

(51) Let $\mathfrak{X}$ be a BCK-algebra with condition (S). Then $\mathfrak{X}$ is implicative if and only if $\mathfrak{X}$ is commutative and positive-implicative.

(52) Let $\mathfrak{X}$ be a BCK-algebra with condition (S). Then $\mathfrak{X}$ is implicative if and only if for all elements x, y, z of $\mathfrak{X}$ holds $x \setminus (y \setminus z) = (x \setminus y \setminus z) \cdot (z \setminus (z \setminus x))$.

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