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Difference and Difference Quotient. Part II

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Summary. In this article, we give some important properties of forward difference, backward difference, central difference and difference quotient and forward difference, backward difference, central difference and difference quotient formulas of some special functions [11].

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The articles [8], [1], [4], [2], [3], [5], [7], [12], [13], [6], [9], and [10] provide the notation and terminology for this paper.

We follow the rules: h, r, r_1, r_2, x_0, x_1, x_2, x_3, x_4, x_5, x, a, b, c, k denote real numbers and f, f_1, f_2 denote functions from $\mathbb{R}$ into $\mathbb{R}$.

Next we state a number of propositions:

(1)[1] $\Delta[f](x, x+h) = \frac{(\vec{\Delta}_h[f])(1)(x)}{h}$.

(2) If $h \neq 0$, then $\Delta[f](x, x+h, x+2\cdot h) = \frac{(\vec{\Delta}_h[f])(2)(x)}{2\cdot h^{\mathbf{2}}}$.

(3) $\Delta[f](x-h, x) = \frac{(\vec{\nabla}_h[f])(1)(x)}{h}$.

(4) If $h \neq 0$, then $\Delta[f](x-2\cdot h, x-h, x) = \frac{(\vec{\nabla}_h[f])(2)(x)}{2\cdot h^{\mathbf{2}}}$.

(5) $\Delta[r\, f](x_0, x_1, x_2) = r \cdot \Delta[f](x_0, x_1, x_2)$.

(6) $\Delta[f_1 + f_2](x_0, x_1, x_2) = \Delta[f_1](x_0, x_1, x_2) + \Delta[f_2](x_0, x_1, x_2)$.

[1]The notation $\Delta(f, x, y)$ has been changed to $\Delta[f](x, y)$. More in Addenda.

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(7) $\Delta[r_1 f_1 + r_2 f_2](x_0, x_1, x_2) = r_1 \cdot \Delta[f_1](x_0, x_1, x_2) + r_2 \cdot \Delta[f_2](x_0, x_1, x_2)$.

(8) $\Delta[r f](x_0, x_1, x_2, x_3) = r \cdot \Delta[f](x_0, x_1, x_2, x_3)$.

(9) $\Delta[f_1 + f_2](x_0, x_1, x_2, x_3) = \Delta[f_1](x_0, x_1, x_2, x_3) + \Delta[f_2](x_0, x_1, x_2, x_3)$.

(10) $\Delta[r_1 f_1 + r_2 f_2](x_0, x_1, x_2, x_3) = r_1 \cdot \Delta[f_1](x_0, x_1, x_2, x_3) + r_2 \cdot \Delta[f_2](x_0, x_1, x_2, x_3)$.

Let f be a real-yielding function and let x_0, x_1, x_2, x_3, x_4 be real numbers. The functor $\Delta[f](x_0, x_1, x_2, x_3, x_4)$ yielding a real number is defined as follows:

(Def. 1) $\Delta[f](x_0, x_1, x_2, x_3, x_4) = \frac{\Delta[f](x_0,x_1,x_2,x_3) - \Delta[f](x_1,x_2,x_3,x_4)}{x_0 - x_4}$.

Next we state three propositions:

(11) $\Delta[r f](x_0, x_1, x_2, x_3, x_4) = r \cdot \Delta[f](x_0, x_1, x_2, x_3, x_4)$.

(12) $\Delta[f_1 + f_2](x_0, x_1, x_2, x_3, x_4) = \Delta[f_1](x_0, x_1, x_2, x_3, x_4) + \Delta[f_2](x_0, x_1, x_2, x_3, x_4)$.

(13) $\Delta[r_1 f_1 + r_2 f_2](x_0, x_1, x_2, x_3, x_4) = r_1 \cdot \Delta[f_1](x_0, x_1, x_2, x_3, x_4) + r_2 \cdot \Delta[f_2](x_0, x_1, x_2, x_3, x_4)$.

Let f be a real-yielding function and let x_0, x_1, x_2, x_3, x_4, x_5 be real numbers. The functor $\Delta[f](x_0, x_1, x_2, x_3, x_4, x_5)$ yields a real number and is defined as follows:

(Def. 2) $\Delta[f](x_0, x_1, x_2, x_3, x_4, x_5) = \frac{\Delta[f](x_0,x_1,x_2,x_3,x_4) - \Delta[f](x_1,x_2,x_3,x_4,x_5)}{x_0 - x_5}$.

We now state a number of propositions:

(14) $\Delta[r f](x_0, x_1, x_2, x_3, x_4, x_5) = r \cdot \Delta[f](x_0, x_1, x_2, x_3, x_4, x_5)$.

(15) $\Delta[f_1 + f_2](x_0, x_1, x_2, x_3, x_4, x_5) = \Delta[f_1](x_0, x_1, x_2, x_3, x_4, x_5) + \Delta[f_2](x_0, x_1, x_2, x_3, x_4, x_5)$.

(16) $\Delta[r_1 f_1 + r_2 f_2](x_0, x_1, x_2, x_3, x_4, x_5) = r_1 \cdot \Delta[f_1](x_0, x_1, x_2, x_3, x_4, x_5) + r_2 \cdot \Delta[f_2](x_0, x_1, x_2, x_3, x_4, x_5)$.

(17) If x_0, x_1, x_2 are mutually different, then $\Delta[f](x_0, x_1, x_2) = \frac{f(x_0)}{(x_0 - x_1) \cdot (x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0) \cdot (x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0) \cdot (x_2 - x_1)}$.

(18) If x_0, x_1, x_2, x_3 are mutually different, then $\Delta[f](x_0, x_1, x_2, x_3) = \Delta[f](x_1, x_2, x_3, x_0)$ and $\Delta[f](x_0, x_1, x_2, x_3) = \Delta[f](x_3, x_2, x_1, x_0)$.

(19) If x_0, x_1, x_2, x_3 are mutually different, then $\Delta[f](x_0, x_1, x_2, x_3) = \Delta[f](x_1, x_0, x_2, x_3)$ and $\Delta[f](x_0, x_1, x_2, x_3) = \Delta[f](x_1, x_2, x_0, x_3)$.

(20) If f is constant, then $\Delta[f](x_0, x_1, x_2) = 0$.

(21) If $x_0 \neq x_1$, then $\Delta[a\Box{+}b](x_0, x_1) = a$.

(22) If x_0, x_1, x_2 are mutually different, then $\Delta[a\Box{+}b](x_0, x_1, x_2) = 0$.

(23) If x_0, x_1, x_2, x_3 are mutually different, then $\Delta[a\Box{+}b](x_0, x_1, x_2, x_3) = 0$.

(24) For every x holds $(\Delta_h[a\Box{+}b])(x) = a \cdot h$.

(25) For every x holds $(\nabla_h[a\Box{+}b])(x) = a \cdot h$.

(26) For every x holds $(\delta_h[a\Box{+}b])(x) = a \cdot h$.

(27) If for every x holds $f(x) = a\cdot x^{\mathbf{2}}+b\cdot x+c$ and $x_0 \neq x_1$, then $\Delta[f](x_0, x_1) = a \cdot (x_0 + x_1) + b$.

(28) If for every x holds $f(x) = a \cdot x^{\mathbf{2}} + b \cdot x + c$ and x_0, x_1, x_2 are mutually different, then $\Delta[f](x_0, x_1, x_2) = a$.

(29) If for every x holds $f(x) = a\cdot x^{\mathbf{2}}+b\cdot x+c$ and x_0, x_1, x_2, x_3 are mutually different, then $\Delta[f](x_0, x_1, x_2, x_3) = 0$.

(30) If for every x holds $f(x) = a \cdot x^{\mathbf{2}} + b \cdot x + c$ and x_0, x_1, x_2, x_3, x_4 are mutually different, then $\Delta[f](x_0, x_1, x_2, x_3, x_4) = 0$.

(31) If for every x holds $f(x) = a \cdot x^{\mathbf{2}} + b \cdot x + c$, then for every x holds $(\Delta_h[f])(x) = 2 \cdot a \cdot h \cdot x + a \cdot h^{\mathbf{2}} + b \cdot h$.

(32) If for every x holds $f(x) = a \cdot x^{\mathbf{2}} + b \cdot x + c$, then for every x holds $(\nabla_h[f])(x) = (2 \cdot a \cdot h \cdot x - a \cdot h^{\mathbf{2}}) + b \cdot h$.

(33) If for every x holds $f(x) = a \cdot x^{\mathbf{2}} + b \cdot x + c$, then for every x holds $(\delta_h[f])(x) = 2 \cdot a \cdot h \cdot x + b \cdot h$.

(34) If for every x holds $f(x) = \frac{k}{x}$ and $x_0 \neq x_1$ and $x_0 \neq 0$ and $x_1 \neq 0$, then $\Delta[f](x_0, x_1) = -\frac{k}{x_0 \cdot x_1}$.

(35) If for every x holds $f(x) = \frac{k}{x}$ and $x_0 \neq 0$ and $x_1 \neq 0$ and $x_2 \neq 0$ and x_0, x_1, x_2 are mutually different, then $\Delta[f](x_0, x_1, x_2) = \frac{k}{x_0 \cdot x_1 \cdot x_2}$.

(36) Suppose for every x holds $f(x) = \frac{k}{x}$ and $x_0 \neq 0$ and $x_1 \neq 0$ and $x_2 \neq 0$ and $x_3 \neq 0$ and x_0, x_1, x_2, x_3 are mutually different. Then $\Delta[f](x_0, x_1, x_2, x_3) = -\frac{k}{x_0 \cdot x_1 \cdot x_2 \cdot x_3}$.

(37) Suppose for every x holds $f(x) = \frac{k}{x}$ and $x_0 \neq 0$ and $x_1 \neq 0$ and $x_2 \neq 0$ and $x_3 \neq 0$ and $x_4 \neq 0$ and x_0, x_1, x_2, x_3, x_4 are mutually different. Then $\Delta[f](x_0, x_1, x_2, x_3, x_4) = \frac{k}{x_0 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4}$.

(38) If for every x holds $f(x) = \frac{k}{x}$ and $x \neq 0$ and $x + h \neq 0$, then for every x holds $(\Delta_h[f])(x) = \frac{-k \cdot h}{(x+h) \cdot x}$.

(39) If for every x holds $f(x) = \frac{k}{x}$ and $x \neq 0$ and $x - h \neq 0$, then for every x holds $(\nabla_h[f])(x) = \frac{-k \cdot h}{(x-h) \cdot x}$.

(40) If for every x holds $f(x) = \frac{k}{x}$ and $x + \frac{h}{2} \neq 0$ and $x - \frac{h}{2} \neq 0$, then for every x holds $(\delta_h[f])(x) = \frac{-k \cdot h}{(x-\frac{h}{2}) \cdot (x+\frac{h}{2})}$.

(41) $\Delta[\text{the function sin}](x_0, x_1) = \frac{2 \cdot \cos(\frac{x_0+x_1}{2}) \cdot \sin(\frac{x_0-x_1}{2})}{x_0-x_1}$.

(42) For every x holds $(\Delta_h[\text{the function sin}])(x) = 2 \cdot (\cos(\frac{2 \cdot x+h}{2}) \cdot \sin(\frac{h}{2}))$.

(43) For every x holds $(\nabla_h[\text{the function sin}])(x) = 2 \cdot (\cos(\frac{2 \cdot x-h}{2}) \cdot \sin(\frac{h}{2}))$.

(44) For every x holds $(\delta_h[\text{the function sin}])(x) = 2 \cdot (\cos x \cdot \sin(\frac{h}{2}))$.

(45) $\Delta[\text{the function cos}](x_0, x_1) = -\frac{2 \cdot \sin(\frac{x_0+x_1}{2}) \cdot \sin(\frac{x_0-x_1}{2})}{x_0-x_1}$.

(46) For every x holds $(\Delta_h[\text{the function cos}])(x) = -2 \cdot (\sin(\frac{2 \cdot x+h}{2}) \cdot \sin(\frac{h}{2}))$.

(47) For every x holds $(\nabla_h[\text{the function cos}])(x) = -2 \cdot (\sin(\frac{2 \cdot x-h}{2}) \cdot \sin(\frac{h}{2}))$.

(48) For every x holds $(\delta_h[\text{the function cos}])(x) = -2 \cdot (\sin x \cdot \sin(\frac{h}{2}))$.

(49) $\Delta[(\text{the function sin})\ (\text{the function sin})](x_0, x_1) = \frac{\frac{1}{2} \cdot (\cos(2 \cdot x_1) - \cos(2 \cdot x_0))}{x_0 - x_1}$.

(50) For every x holds $(\Delta_h[(\text{the function sin})\ (\text{the function sin})])(x) = \frac{1}{2} \cdot (\cos(2 \cdot x) - \cos(2 \cdot (x + h)))$.

(51) For every x holds $(\nabla_h[(\text{the function sin})\ (\text{the function sin})])(x) = \frac{1}{2} \cdot (\cos(2 \cdot (x - h)) - \cos(2 \cdot x))$.

(52) For every x holds $(\delta_h[(\text{the function sin})\ (\text{the function sin})])(x) = \frac{1}{2} \cdot (\cos(2 \cdot x - h) - \cos(2 \cdot x + h))$.

(53) $\Delta[(\text{the function sin})\ (\text{the function cos})](x_0, x_1) = \frac{\frac{1}{2} \cdot (\sin(2 \cdot x_0) - \sin(2 \cdot x_1))}{x_0 - x_1}$.

(54) For every x holds $(\Delta_h[(\text{the function sin})\ (\text{the function cos})])(x) = \frac{1}{2} \cdot (\sin(2 \cdot (x + h)) - \sin(2 \cdot x))$.

(55) For every x holds $(\nabla_h[(\text{the function sin})\ (\text{the function cos})])(x) = \frac{1}{2} \cdot (\sin(2 \cdot x) - \sin(2 \cdot (x - h)))$.

(56) For every x holds $(\delta_h[(\text{the function sin})\ (\text{the function cos})])(x) = \frac{1}{2} \cdot (\sin(2 \cdot x + h) - \sin(2 \cdot x - h))$.

(57) $\Delta[(\text{the function cos})\ (\text{the function cos})](x_0, x_1) = \frac{\frac{1}{2} \cdot (\cos(2 \cdot x_0) - \cos(2 \cdot x_1))}{x_0 - x_1}$.

(58) For every x holds $(\Delta_h[(\text{the function cos})\ (\text{the function cos})])(x) = \frac{1}{2} \cdot (\cos(2 \cdot (x + h)) - \cos(2 \cdot x))$.

(59) For every x holds $(\nabla_h[(\text{the function cos})\ (\text{the function cos})])(x) = \frac{1}{2} \cdot (\cos(2 \cdot x) - \cos(2 \cdot (x - h)))$.

(60) For every x holds $(\delta_h[(\text{the function cos})\ (\text{the function cos})])(x) = \frac{1}{2} \cdot (\cos(2 \cdot x + h) - \cos(2 \cdot x - h))$.

(61) $\Delta[(\text{the function sin})\ (\text{the function sin})\ (\text{the function cos})](x_0, x_1) = -\frac{\frac{1}{2} \cdot (\sin(\frac{3 \cdot (x_1 + x_0)}{2}) \cdot \sin(\frac{3 \cdot (x_1 - x_0)}{2}) + \sin(\frac{x_0 + x_1}{2}) \cdot \sin(\frac{x_0 - x_1}{2}))}{x_0 - x_1}$.

(62) Let given x. Then $(\Delta_h[(\text{the function sin})\ (\text{the function sin})\ (\text{the function cos})])(x) = \frac{1}{2} \cdot (\sin(\frac{6 \cdot x + 3 \cdot h}{2}) \cdot \sin(\frac{3 \cdot h}{2}) - \sin(\frac{2 \cdot x + h}{2}) \cdot \sin(\frac{h}{2}))$.

(63) Let given x. Then $(\nabla_h[(\text{the function sin})\ (\text{the function sin})\ (\text{the function cos})])(x) = \frac{1}{2} \cdot (\sin(\frac{6 \cdot x - 3 \cdot h}{2}) \cdot \sin(\frac{3 \cdot h}{2})) - \frac{1}{2} \cdot (\sin(\frac{2 \cdot x - h}{2}) \cdot \sin(\frac{h}{2}))$.

(64) For every x holds $(\delta_h[(\text{the function sin})\ (\text{the function sin})\ (\text{the function cos})])(x) = -\frac{1}{2} \cdot (\sin x \cdot \sin(\frac{h}{2})) + \frac{1}{2} \cdot (\sin(3 \cdot x) \cdot \sin(\frac{3 \cdot h}{2}))$.

(65) $\Delta[(\text{the function sin})\ (\text{the function cos})\ (\text{the function cos})](x_0, x_1) = \frac{\frac{1}{2} \cdot (\cos(\frac{x_0 + x_1}{2}) \cdot \sin(\frac{x_0 - x_1}{2}) + \cos(\frac{3 \cdot (x_0 + x_1)}{2}) \cdot \sin(\frac{3 \cdot (x_0 - x_1)}{2}))}{x_0 - x_1}$.

(66) Let given x. Then $(\Delta_h[(\text{the function sin})\ (\text{the function cos})\ (\text{the function cos})])(x) = \frac{1}{2} \cdot (\cos(\frac{2 \cdot x + h}{2}) \cdot \sin(\frac{h}{2}) + \cos(\frac{6 \cdot x + 3 \cdot h}{2}) \cdot \sin(\frac{3 \cdot h}{2}))$.

(67) Let given x. Then $(\nabla_h[(\text{the function sin})\ (\text{the function cos})\ (\text{the function cos})])(x) = \frac{1}{2} \cdot (\cos(\frac{2 \cdot x - h}{2}) \cdot \sin(\frac{h}{2}) + \cos(\frac{6 \cdot x - 3 \cdot h}{2}) \cdot \sin(\frac{3 \cdot h}{2}))$.

(68) For every x holds $(\delta_h[$(the function sin) (the function cos) (the function cos)$])(x) = \frac{1}{2} \cdot (\cos x \cdot \sin(\frac{h}{2}) + \cos(3 \cdot x) \cdot \sin(\frac{3 \cdot h}{2}))$.

(69) If $x_0 \in \mathrm{dom}$ (the function tan) and $x_1 \in \mathrm{dom}$ (the function tan), then Δ[the function tan]$(x_0, x_1) = \frac{\sin(x_0 - x_1)}{\cos x_0 \cdot \cos x_1 \cdot (x_0 - x_1)}$.

(70) If $x_0 \in \mathrm{dom}$ (the function cot) and $x_1 \in \mathrm{dom}$ (the function cot), then Δ[the function cot]$(x_0, x_1) = -\frac{\sin(x_0 - x_1)}{\sin x_0 \cdot \sin x_1 \cdot (x_0 - x_1)}$.

(71) Suppose $x_0 \in \mathrm{dom}$ (the function cosec) and $x_1 \in \mathrm{dom}$ (the function cosec). Then Δ[the function cosec]$(x_0, x_1) = \frac{2 \cdot \cos(\frac{x_1 + x_0}{2}) \cdot \sin(\frac{x_1 - x_0}{2})}{\sin x_1 \cdot \sin x_0 \cdot (x_0 - x_1)}$.

(72) Suppose $x_0 \in \mathrm{dom}$ (the function sec) and $x_1 \in \mathrm{dom}$ (the function sec). Then Δ[the function sec]$(x_0, x_1) = -\frac{2 \cdot \sin(\frac{x_1 + x_0}{2}) \cdot \sin(\frac{x_1 - x_0}{2})}{\cos x_1 \cdot \cos x_0 \cdot (x_0 - x_1)}$.

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