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Banach Algebra of Bounded Functionals

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Summary. In this article, we describe some basic properties of the Banach algebra which is constructed from all bounded functionals.

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The notation and terminology used here are introduced in the following papers: [7], [24], [4], [2], [5], [3], [21], [16], [23], [22], [13], [15], [6], [1], [20], [25], [8], [12], [11], [10], [9], [14], [17], [19], and [18].

1. Some Properties of Rings

Let V be a non empty additive loop structure and let V_1 be a subset of V. We say that V_1 has inverse if and only if:

(Def. 1) For every element v of V such that $v \in V_1$ holds $-v \in V_1$.

Let V be a non empty additive loop structure and let V_1 be a subset of V. We say that V_1 is additively-closed if and only if:

(Def. 2) V_1 is add closed and has inverse.

Let V be a non empty additive loop structure. One can verify that Ω_V is add closed and has inverse.

Let V be a non empty double loop structure. One can verify that every subset of V which is additively-closed is also add closed and has inverse and every subset of V which is add closed and has inverse is also additively-closed.

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Let V be a non empty additive loop structure. Observe that there exists a subset of V which is add closed and non empty and has inverse.

Let V be a ring. A ring is called a subring of V if it satisfies the conditions (Def. 3).

(Def. 3)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the addition of it $=$ (the addition of V) $\restriction$ (the carrier of it),

(iii) the multiplication of it $=$ (the multiplication of V) $\restriction$ (the carrier of it),

(iv) $1_{\rm it} = 1_V$, and

(v) $0_{\rm it} = 0_V$.

For simplicity, we follow the rules: X is a non empty set, x is an element of X, d_1, d_2 are elements of X, A is a binary operation on X, M is a function from $X \times X$ into X, V is a ring, and V_1 is a subset of V.

We now state the proposition

(1) Suppose $V_1 = X$ and $A =$ (the addition of V) $\restriction$ (V_1) and $M =$ (the multiplication of V) $\restriction$ (V_1) and $d_1 = 1_V$ and $d_2 = 0_V$ and V_1 has inverse. Then $\langle X, A, M, d_1, d_2\rangle$ is a subring of V.

Let V be a ring. One can check that there exists a subring of V which is strict.

Let V be a non empty multiplicative loop with zero structure and let V_1 be a subset of V. We say that V_1 is multiplicatively-closed if and only if:

(Def. 4) $1_V \in V_1$ and for all elements v, u of V such that $v, u \in V_1$ holds $v \cdot u \in V_1$.

Let V be a non empty additive loop structure and let V_1 be a subset of V. Let us assume that V_1 is add closed and non empty. The functor $\mathrm{Add}(V_1, V)$ yielding a binary operation on V_1 is defined as follows:

(Def. 5) $\mathrm{Add}(V_1, V) =$ (the addition of V) $\restriction$ (V_1).

Let V be a non empty multiplicative loop with zero structure and let V_1 be a subset of V. Let us assume that V_1 is multiplicatively-closed and non empty. The functor $\mathrm{mult}(V_1, V)$ yields a binary operation on V_1 and is defined as follows:

(Def. 6) $\mathrm{mult}(V_1, V) =$ (the multiplication of V) $\restriction$ (V_1).

Let V be an add-associative right zeroed right complementable non empty double loop structure and let V_1 be a subset of V. Let us assume that V_1 is add closed and non empty and has inverse. The functor $\mathrm{Zero}(V_1, V)$ yields an element of V_1 and is defined by:

(Def. 7) $\mathrm{Zero}(V_1, V) = 0_V$.

Let V be a non empty multiplicative loop with zero structure and let V_1 be a subset of V. Let us assume that V_1 is multiplicatively-closed and non empty. The functor $\mathrm{One}(V_1, V)$ yields an element of V_1 and is defined as follows:

(Def. 8) $\mathrm{One}(V_1, V) = 1_V$.

We now state the proposition

(2) If V_1 is additively-closed, multiplicatively-closed, and non empty, then $\langle V_1, \mathrm{Add}(V_1, V), \mathrm{mult}(V_1, V), \mathrm{One}(V_1, V), \mathrm{Zero}(V_1, V)\rangle$ is a ring.

2. Some Properties of Algebras

In the sequel V is an algebra, V_1 is a subset of V, M_1 is a function from $\mathbb{R} \times X$ into X, and a is a real number.

Let V be an algebra. An algebra is called a subalgebra of V if it satisfies the conditions (Def. 9).

(Def. 9)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the addition of it = (the addition of V) $\restriction$ (the carrier of it),

(iii) the multiplication of it = (the multiplication of V) $\restriction$ (the carrier of it),

(iv) the external multiplication of it = (the external multiplication of V)$\restriction$($\mathbb{R}$ $\times$ the carrier of it),

(v) $1_{\mathrm{it}} = 1_V$, and

(vi) $0_{\mathrm{it}} = 0_V$.

The following proposition is true

(3) Suppose that $V_1 = X$ and $d_1 = 0_V$ and $d_2 = 1_V$ and A = (the addition of V) $\restriction$ (V_1) and M = (the multiplication of V) $\restriction$ (V_1) and M_1 = (the external multiplication of V)$\restriction$($\mathbb{R} \times V_1$) and V_1 has inverse. Then $\langle X, M, A, M_1, d_2, d_1\rangle$ is a subalgebra of V.

Let V be an algebra. Observe that there exists a subalgebra of V which is strict.

Let V be an algebra and let V_1 be a subset of V. We say that V_1 is additively-linearly-closed if and only if:

(Def. 10) V_1 is add closed and has inverse and for every real number a and for every element v of V such that $v \in V_1$ holds $a \cdot v \in V_1$.

Let V be an algebra. One can check that every subset of V which is additively-linearly-closed is also additively-closed.

Let V be an algebra and let V_1 be a subset of V. Let us assume that V_1 is additively-linearly-closed and non empty. The functor $\mathrm{Mult}(V_1, V)$ yielding a function from $\mathbb{R} \times V_1$ into V_1 is defined by:

(Def. 11) $\mathrm{Mult}(V_1, V)$ = (the external multiplication of V)$\restriction$($\mathbb{R} \times V_1$).

Let V be a non empty RLS structure. We say that V is scalar-multiplcation-cancelable if and only if:

(Def. 12) For every real number a and for every element v of V such that $a \cdot v = 0_V$ holds $a = 0$ or $v = 0_V$.

One can prove the following propositions:

(4) Let V be an add-associative right zeroed right complementable algebra-like non empty algebra structure and a be a real number. Then $a \cdot 0_V = 0_V$.

(5) Let V be an Abelian add-associative right zeroed right complementable algebra-like non empty algebra structure. Suppose V is scalar-multiplcation-cancelable. Then V is a real linear space.

(6) Suppose V_1 is additively-linearly-closed, multiplicatively-closed, and non empty.
Then $\langle V_1, \mathrm{mult}(V_1, V), \mathrm{Add}(V_1, V), \mathrm{Mult}(V_1, V), \mathrm{One}(V_1, V), \mathrm{Zero}(V_1, V)\rangle$ is a subalgebra of V.

Let X be a non empty set. Observe that RAlgebra X is Abelian, add-associative, right zeroed, right complementable, commutative, associative, right unital, right distributive, and algebra-like.

One can prove the following two propositions:

(7) RAlgebra X is a real linear space.

(8) Let V be an algebra and V_1 be a subalgebra of V. Then

(i) for all elements v_1, w_1 of V_1 and for all elements v, w of V such that $v_1 = v$ and $w_1 = w$ holds $v_1 + w_1 = v + w$,

(ii) for all elements v_1, w_1 of V_1 and for all elements v, w of V such that $v_1 = v$ and $w_1 = w$ holds $v_1 \cdot w_1 = v \cdot w$,

(iii) for every element v_1 of V_1 and for every element v of V and for every real number a such that $v_1 = v$ holds $a \cdot v_1 = a \cdot v$,

(iv) $\mathbf{1}_{(V_1)} = \mathbf{1}_V$, and

(v) $0_{(V_1)} = 0_V$.

3. Banach Algebra of Bounded Functionals

Let X be a non empty set. The functor BoundedFunctions X yielding a non empty subset of RAlgebra X is defined as follows:

(Def. 13) BoundedFunctions $X = \{f : X \to \mathbb{R}$: f is bounded on $X\}$.

We now state the proposition

(9) BoundedFunctions X is additively-linearly-closed and multiplicatively-closed.

Let us consider X. Note that BoundedFunctions X is additively-linearly-closed and multiplicatively-closed.

The following proposition is true

(10) $\langle$BoundedFunctions X, mult(BoundedFunctions X, RAlgebra X), Add(BoundedFunctions X, RAlgebra X), Mult(BoundedFunctions X, RAlgebra X), One(BoundedFunctions X, RAlgebra X), Zero(BoundedFunctions X, RAlgebra X)$\rangle$ is a subalgebra of RAlgebra X.

Let X be a non empty set. The $\mathbb{R}$-algebra of bounded functions on X yields an algebra and is defined by:

(Def. 14) The $\mathbb{R}$-algebra of bounded functions on $X = \langle$BoundedFunctions X, mult(BoundedFunctions X, RAlgebra X), Add(BoundedFunctions X, RAlgebra X), Mult(BoundedFunctions X, RAlgebra X), One(Bounded Functions X, RAlgebra X), Zero(BoundedFunctions X, RAlgebra X)$\rangle$.

The following proposition is true

(11) The $\mathbb{R}$-algebra of bounded functions on X is a real linear space.

We adopt the following rules: F, G, H are vectors of the $\mathbb{R}$-algebra of bounded functions on X and f, g, h are functions from X into $\mathbb{R}$.

Next we state several propositions:

(12) If $f = F$ and $g = G$ and $h = H$, then $H = F + G$ iff for every element x of X holds $h(x) = f(x) + g(x)$.

(13) If $f = F$ and $g = G$, then $G = a \cdot F$ iff for every element x of X holds $g(x) = a \cdot f(x)$.

(14) If $f = F$ and $g = G$ and $h = H$, then $H = F \cdot G$ iff for every element x of X holds $h(x) = f(x) \cdot g(x)$.

(15) $0_{\text{the } \mathbb{R}\text{-algebra of bounded functions on } X} = X \longmapsto 0$.

(16) $\mathbf{1}_{\text{the } \mathbb{R}\text{-algebra of bounded functions on } X} = X \longmapsto 1$.

Let X be a non empty set and let F be a set. Let us assume that $F \in$ BoundedFunctions X. The functor modetrans(F, X) yielding a function from X into $\mathbb{R}$ is defined by:

(Def. 15) modetrans$(F, X) = F$ and modetrans(F, X) is bounded on X.

Let X be a non empty set and let f be a function from X into $\mathbb{R}$. The functor PreNorms(f) yielding a non empty subset of $\mathbb{R}$ is defined as follows:

(Def. 16) PreNorms$(f) = \{|f(x)| : x$ ranges over elements of $X\}$.

Next we state three propositions:

(17) If f is bounded on X, then PreNorms(f) is non empty and upper bounded.

(18) f is bounded on X iff PreNorms(f) is upper bounded.

(19) There exists a function N_1 from BoundedFunctions X into $\mathbb{R}$ such that for every set F such that $F \in$ BoundedFunctions X holds $N_1(F) =$ sup PreNorms(modetrans(F, X)).

Let X be a non empty set. The functor BoundedFunctionsNorm X yields a function from BoundedFunctions X into $\mathbb{R}$ and is defined by:

(Def. 17) For every set x such that $x \in$ BoundedFunctions X holds (BoundedFunctionsNorm X)$(x) =$ sup PreNorms(modetrans(x, X)).

We now state two propositions:

(20) If f is bounded on X, then modetrans$(f, X) = f$.

(21) If f is bounded on X, then (BoundedFunctionsNorm X)$(f) =$ sup PreNorms(f).

Let X be a non empty set. The $\mathbb{R}$-normed algebra of bounded functions on X yielding a normed algebra structure is defined as follows:

(Def. 18) The $\mathbb{R}$-normed algebra of bounded functions on $X =$ $\langle$BoundedFunctions X, mult(BoundedFunctions X, RAlgebra X), Add(BoundedFunctions X, RAlgebra X), Mult(BoundedFunctions X, RAlgebra X), One(BoundedFunctions X, RAlgebra X), Zero(BoundedFunctions X, RAlgebra X), BoundedFunctionsNorm $X\rangle$.

Let X be a non empty set. Note that the $\mathbb{R}$-normed algebra of bounded functions on X is non empty.

Let X be a non empty set. Observe that the $\mathbb{R}$-normed algebra of bounded functions on X is unital.

We now state the proposition

(22) Let W be a normed algebra structure and V be an algebra. If the algebra structure of $W = V$ and $1_V = 1_W$, then W is an algebra.

In the sequel F, G, H denote points of the $\mathbb{R}$-normed algebra of bounded functions on X.

We now state a number of propositions:

(23) The $\mathbb{R}$-normed algebra of bounded functions on X is an algebra.

(24) (Mult(BoundedFunctions X, RAlgebra X))$(1, F) = F$.

(25) The $\mathbb{R}$-normed algebra of bounded functions on X is a real linear space.

(26) $X \longmapsto 0 = 0_{\text{the } \mathbb{R}\text{-normed algebra of bounded functions on } X}$.

(27) If $f = F$ and f is bounded on X, then $|f(x)| \leq \|F\|$.

(28) $0 \leq \|F\|$.

(29) $0 = \|(0_{\text{the } \mathbb{R}\text{-normed algebra of bounded functions on } X})\|$.

(30) If $f = F$ and $g = G$ and $h = H$, then $H = F + G$ iff for every element x of X holds $h(x) = f(x) + g(x)$.

(31) If $f = F$ and $g = G$, then $G = a \cdot F$ iff for every element x of X holds $g(x) = a \cdot f(x)$.

(32) If $f = F$ and $g = G$ and $h = H$, then $H = F \cdot G$ iff for every element x of X holds $h(x) = f(x) \cdot g(x)$.

(33)(i) $\|F\| = 0$ iff $F = 0_{\text{the } \mathbb{R}\text{-normed algebra of bounded functions on } X}$,

(ii) $\|a \cdot F\| = |a| \cdot \|F\|$, and

(iii) $\|F + G\| \leq \|F\| + \|G\|$.

(34) The $\mathbb{R}$-normed algebra of bounded functions on X is real normed space-like.

Let X be a non empty set.

Note that the $\mathbb{R}$-normed algebra of bounded functions on X is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

We now state three propositions:

(35) If $f = F$ and $g = G$ and $h = H$, then $H = F - G$ iff for every element x of X holds $h(x) = f(x) - g(x)$.

(36) Let X be a non empty set and s_1 be a sequence of the $\mathbb{R}$-normed algebra of bounded functions on X. If s_1 is Cauchy sequence by norm, then s_1 is convergent.

(37) The $\mathbb{R}$-normed algebra of bounded functions on X is a real Banach space.

Let X be a non empty set.

Observe that the $\mathbb{R}$-normed algebra of bounded functions on X is complete.

The following proposition is true

(38) The $\mathbb{R}$-normed algebra of bounded functions on X is a Banach algebra.

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