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Heron's Formula and Ptolemy's Theorem

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Summary. The goal of this article is to formalize some theorems that are in the [17] on the web. These are elementary theorems included in every handbook of Euclidean geometry and trigonometry: the law of cosines, the Heron's formula, the isosceles triangle theorem, the intersecting chords theorem and the Ptolemy's theorem.

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The terminology and notation used here are introduced in the following articles: [5], [16], [2], [1], [13], [14], [15], [18], [12], [6], [8], [7], [11], [4], [9], [10], and [3].

1. Law of Cosines and Meister-Gauss Formula

We adopt the following rules: p_1, p_2, p_3, p_4, p_5, p_6, p, p_7 denote points of $\mathcal{E}^2_{\mathrm{T}}$ and a, b, c, r, s denote real numbers.

Next we state four propositions:

(1) If $\sin \measuredangle(p_1, p_2, p_3) = \sin \measuredangle(p_4, p_5, p_6)$ and $\cos \measuredangle(p_1, p_2, p_3) = \cos \measuredangle(p_4, p_5, p_6)$, then $\measuredangle(p_1, p_2, p_3) = \measuredangle(p_4, p_5, p_6)$.

(2) $\sin \measuredangle(p_1, p_2, p_3) = -\sin \measuredangle(p_3, p_2, p_1)$.

(3) $\cos \measuredangle(p_1, p_2, p_3) = \cos \measuredangle(p_3, p_2, p_1)$.

(4) $\measuredangle(p_1, p_4, p_2) + \measuredangle(p_2, p_4, p_3) = \measuredangle(p_1, p_4, p_3)$ or $\measuredangle(p_1, p_4, p_2) + \measuredangle(p_2, p_4, p_3) = \measuredangle(p_1, p_4, p_3) + 2 \cdot \pi$.

Let us consider p_1, p_2, p_3. The area of $\triangle(p_1, p_2, p_3)$ yields a real number and is defined by:

(Def. 1) The area of $\triangle(p_1, p_2, p_3) = \frac{1}{2} \cdot (((p_1)_{\mathbf{1}} \cdot (p_2)_{\mathbf{2}} - (p_2)_{\mathbf{1}} \cdot (p_1)_{\mathbf{2}}) + ((p_2)_{\mathbf{1}} \cdot (p_3)_{\mathbf{2}} - (p_3)_{\mathbf{1}} \cdot (p_2)_{\mathbf{2}}) + ((p_3)_{\mathbf{1}} \cdot (p_1)_{\mathbf{2}} - (p_1)_{\mathbf{1}} \cdot (p_3)_{\mathbf{2}}))$.

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Let us consider p_1, p_2, p_3. The perimeter of $\triangle(p_1, p_2, p_3)$ yields a real number and is defined by:

(Def. 2) The perimeter of $\triangle(p_1, p_2, p_3) = |p_2 - p_1| + |p_3 - p_2| + |p_1 - p_3|$.

One can prove the following three propositions:

(5) The area of $\triangle(p_1, p_2, p_3) = \frac{|p_1-p_2| \cdot |p_3-p_2| \cdot \sin \measuredangle(p_3,p_2,p_1)}{2}$.

(6) If $p_2 \neq p_1$, then $|p_3 - p_2| \cdot \sin \measuredangle(p_3, p_2, p_1) = |p_3 - p_1| \cdot \sin \measuredangle(p_2, p_1, p_3)$.

(7) $(|p_3 - p_1|)^{\mathbf{2}} = ((|p_1 - p_2|)^{\mathbf{2}} + (|p_3 - p_2|)^{\mathbf{2}}) - 2 \cdot (|p_1 - p_2|) \cdot (|p_3 - p_2|) \cdot \cos \measuredangle(p_1, p_2, p_3)$.

2. Some Elementary Facts about Euclidean geometry

Next we state a number of propositions:

(8) If $p \in \mathcal{L}(p_1, p_2)$ and $p \neq p_1$ and $p \neq p_2$, then $\measuredangle(p_1, p, p_2) = \pi$.

(9) If $p \in \mathcal{L}(p_2, p_3)$ and $p \neq p_2$, then $\measuredangle(p_3, p_2, p_1) = \measuredangle(p, p_2, p_1)$.

(10) If $p \in \mathcal{L}(p_2, p_3)$ and $p \neq p_2$, then $\measuredangle(p_1, p_2, p_3) = \measuredangle(p_1, p_2, p)$.

(11) If $\measuredangle(p_1, p, p_2) = \pi$, then $p \in \mathcal{L}(p_1, p_2)$.

(12) If $p \in \mathcal{L}(p_1, p_3)$ and $p \in \mathcal{L}(p_1, p_4)$ and $p_3 \neq p_4$ and $p \neq p_1$, then $p_3 \in \mathcal{L}(p_1, p_4)$ or $p_4 \in \mathcal{L}(p_1, p_3)$.

(13) If $p \in \mathcal{L}(p_1, p_3)$ and $p \neq p_1$ and $p \neq p_3$, then $\measuredangle(p_1, p, p_2) + \measuredangle(p_2, p, p_3) = \pi$ or $\measuredangle(p_1, p, p_2) + \measuredangle(p_2, p, p_3) = 3 \cdot \pi$.

(14) If $p \in \mathcal{L}(p_1, p_2)$ and $p \neq p_1$ and $p \neq p_2$ and $\measuredangle(p_3, p, p_1) = \frac{\pi}{2}$ or $\measuredangle(p_3, p, p_1) = \frac{3}{2} \cdot \pi$, then $\measuredangle(p_1, p, p_3) = \measuredangle(p_3, p, p_2)$.

(15) If $p \in \mathcal{L}(p_1, p_3)$ and $p \in \mathcal{L}(p_2, p_4)$ and $p \neq p_1$ and $p \neq p_2$ and $p \neq p_3$ and $p \neq p_4$, then $\measuredangle(p_1, p, p_2) = \measuredangle(p_3, p, p_4)$.

(16) If $|p_3 - p_1| = |p_2 - p_3|$ and $p_1 \neq p_2$, then $\measuredangle(p_3, p_1, p_2) = \measuredangle(p_1, p_2, p_3)$.

(17) For all p_1, p_2, p_3, p such that $p \in \mathcal{L}(p_1, p_2)$ and $p \neq p_2$ holds $|(p_3 - p, p_2 - p_1)| = 0$ iff $|(p_3 - p, p_2 - p)| = 0$.

(18) If $|p_1 - p_3| = |p_2 - p_3|$ and $p \in \mathcal{L}(p_1, p_2)$ and $p \neq p_3$ and $p \neq p_1$ and $\measuredangle(p_3, p, p_1) = \frac{\pi}{2}$ or $\measuredangle(p_3, p, p_1) = \frac{3}{2} \cdot \pi$, then $\measuredangle(p_1, p_3, p) = \measuredangle(p, p_3, p_2)$.

(19) Let given p_1, p_2, p_3, p such that $|p_1 - p_3| = |p_2 - p_3|$ and $p \in \mathcal{L}(p_1, p_2)$ and $p \neq p_3$. Then

(i) if $\measuredangle(p_1, p_3, p) = \measuredangle(p, p_3, p_2)$, then $|p_1 - p| = |p - p_2|$,

(ii) if $|p_1 - p| = |p - p_2|$, then $|(p_3 - p, p_2 - p_1)| = 0$, and

(iii) if $|(p_3 - p, p_2 - p_1)| = 0$, then $\measuredangle(p_1, p_3, p) = \measuredangle(p, p_3, p_2)$.

Let us consider p_1, p_2, p_3. We say that p_1, p_2 and p_3 are collinear if and only if:

(Def. 3) $p_1 \in \mathcal{L}(p_2, p_3)$ or $p_2 \in \mathcal{L}(p_3, p_1)$ or $p_3 \in \mathcal{L}(p_1, p_2)$.

Let us consider p_1, p_2, p_3. We introduce p_1, p_2, p_3 form a triangle as an antonym of p_1, p_2 and p_3 are collinear.

The following propositions are true:

(20) p_1, p_2, p_3 form a triangle iff p_1, p_2, p_3 are mutually different and $\measuredangle(p_1,p_2,p_3) \neq \pi$ and $\measuredangle(p_2,p_3,p_1) \neq \pi$ and $\measuredangle(p_3,p_1,p_2) \neq \pi$.

(21) Suppose p_1, p_2, p_3 form a triangle and p_4, p_5, p_6 form a triangle and $\measuredangle(p_1,p_2,p_3) = \measuredangle(p_4,p_5,p_6)$ and $\measuredangle(p_3,p_1,p_2) = \measuredangle(p_6,p_4,p_5)$. Then $|p_3 - p_2| \cdot |p_4 - p_6| = |p_1 - p_3| \cdot |p_6 - p_5|$ and $|p_3 - p_2| \cdot |p_5 - p_4| = |p_2 - p_1| \cdot |p_6 - p_5|$ and $|p_1 - p_3| \cdot |p_5 - p_4| = |p_2 - p_1| \cdot |p_4 - p_6|$.

(22) Suppose p_1, p_2, p_3 form a triangle and p_4, p_5, p_6 form a triangle and $\measuredangle(p_1,p_2,p_3) = \measuredangle(p_4,p_5,p_6)$ and $\measuredangle(p_3,p_1,p_2) = \measuredangle(p_5,p_6,p_4)$. Then $|p_2 - p_3| \cdot |p_4 - p_6| = |p_3 - p_1| \cdot |p_5 - p_4|$ and $|p_2 - p_3| \cdot |p_6 - p_5| = |p_1 - p_2| \cdot |p_5 - p_4|$ and $|p_3 - p_1| \cdot |p_6 - p_5| = |p_1 - p_2| \cdot |p_4 - p_6|$.

(23) If p_1, p_2, p_3 are mutually different and $\measuredangle(p_1,p_2,p_3) \leq \pi$, then $\measuredangle(p_2,p_3,p_1) \leq \pi$ and $\measuredangle(p_3,p_1,p_2) \leq \pi$.

(24) If p_1, p_2, p_3 are mutually different and $\measuredangle(p_1,p_2,p_3) > \pi$, then $\measuredangle(p_2,p_3,p_1) > \pi$ and $\measuredangle(p_3,p_1,p_2) > \pi$.

(25) If $p \in \mathcal{L}(p_1,p_2)$ and p_1, p_2, p_3 form a triangle and $\measuredangle(p_1,p_3,p_2) = \measuredangle(p,p_3,p_2)$, then $p = p_1$.

(26) If $p \in \mathcal{L}(p_1,p_2)$ and $p_3 \notin \mathcal{L}(p_1,p_2)$ and $\measuredangle(p_1,p_3,p_2) \leq \pi$, then $\measuredangle(p,p_3,p_2) \leq \measuredangle(p_1,p_3,p_2)$.

(27) If $p \in \mathcal{L}(p_1,p_2)$ and $p_3 \notin \mathcal{L}(p_1,p_2)$ and $\measuredangle(p_1,p_3,p_2) > \pi$ and $p \neq p_2$, then $\measuredangle(p,p_3,p_2) \geq \measuredangle(p_1,p_3,p_2)$.

(28) If $p \in \mathcal{L}(p_1,p_2)$ and $p_3 \notin \mathcal{L}(p_1,p_2)$, then there exists p_4 such that $p_4 \in \mathcal{L}(p_1,p_2)$ and $\measuredangle(p_1,p_3,p_4) = \measuredangle(p,p_3,p_2)$.

(29) If $p_1 \in \mathrm{InsideOfCircle}(a,b,r)$ and $p_2 \in \mathrm{OutsideOfCircle}(a,b,r)$, then there exists p such that $p \in \mathcal{L}(p_1,p_2) \cap \mathrm{Circle}(a,b,r)$.

(30) If p_1, p_3, $p_4 \in \mathrm{Circle}(a,b,r)$ and $p \in \mathcal{L}(p_1,p_3)$ and $p \in \mathcal{L}(p_1,p_4)$ and $p_3 \neq p_4$, then $p = p_1$.

(31) If p_1, p_2, $p \in \mathrm{Circle}(a,b,r)$ and $p_7 = [a,b]$ and $p_7 \in \mathcal{L}(p,p_2)$ and $p_1 \neq p$, then $2 \cdot \measuredangle(p_1,p,p_2) = \measuredangle(p_1,p_7,p_2)$ or $2 \cdot (\measuredangle(p_1,p,p_2) - \pi) = \measuredangle(p_1,p_7,p_2)$.

(32) If $p_1 \in \mathrm{Circle}(a,b,r)$ and $r > 0$, then there exists p_2 such that $p_1 \neq p_2$ and $p_2 \in \mathrm{Circle}(a,b,r)$ and $[a,b] \in \mathcal{L}(p_1,p_2)$.

(33) If p_1, p_2, $p \in \mathrm{Circle}(a,b,r)$ and $p_7 = [a,b]$ and $p_1 \neq p$ and $p_2 \neq p$, then $2 \cdot \measuredangle(p_1,p,p_2) = \measuredangle(p_1,p_7,p_2)$ or $2 \cdot (\measuredangle(p_1,p,p_2) - \pi) = \measuredangle(p_1,p_7,p_2)$.

(34) Suppose p_1, p_2, p_3, $p_4 \in \mathrm{Circle}(a,b,r)$ and $p_1 \neq p_3$ and $p_1 \neq p_4$ and $p_2 \neq p_3$ and $p_2 \neq p_4$. Then $\measuredangle(p_1,p_3,p_2) = \measuredangle(p_1,p_4,p_2)$ or $\measuredangle(p_1,p_3,p_2) = \measuredangle(p_1,p_4,p_2) - \pi$ or $\measuredangle(p_1,p_3,p_2) = \measuredangle(p_1,p_4,p_2) + \pi$.

(35) If p_1, p_2, $p_3 \in \mathrm{Circle}(a,b,r)$ and $p_1 \neq p_2 \neq p_3$, then $\measuredangle(p_1,p_2,p_3) \neq \pi$.

(36) Suppose p_1, p_2, p_3, $p_4 \in \mathrm{Circle}(a,b,r)$ and $p \in \mathcal{L}(p_1,p_3)$ and $p \in \mathcal{L}(p_2,p_4)$ and p_1, p_2, p_3, p_4 are mutually different. Then $\measuredangle(p_1,p_4,p_2) = \measuredangle(p_1,p_3,p_2)$.

(37) If p_1, p_2, $p_3 \in \mathrm{Circle}(a,b,r)$ and $\measuredangle(p_1,p_2,p_3) = 0$ and $p_1 \neq p_2 \neq p_3$, then $p_1 = p_3$.

(38) If p_1, p_2, p_3, $p_4 \in \mathrm{Circle}(a,b,r)$ and $p \in \mathcal{L}(p_1,p_3)$ and $p \in \mathcal{L}(p_2,p_4)$, then $|p_1 - p| \cdot |p - p_3| = |p_2 - p| \cdot |p - p_4|$.

3. Heron's Formula and Ptolemy's Theorem

One can prove the following propositions:

(39) Suppose $a = |p_2 - p_1|$ and $b = |p_3 - p_2|$ and $c = |p_1 - p_3|$ and $s = \frac{1}{2} \cdot$ the perimeter of $\triangle(p_1,p_2,p_3)$. Then |the area of $\triangle(p_1,p_2,p_3)$| $= \sqrt{s \cdot (s-a) \cdot (s-b) \cdot (s-c)}$.

(40) If p_1, p_2, p_3, $p_4 \in \mathrm{Circle}(a,b,r)$ and $p \in \mathcal{L}(p_1,p_3)$ and $p \in \mathcal{L}(p_2,p_4)$, then $|p_3 - p_1| \cdot |p_4 - p_2| = |p_2 - p_1| \cdot |p_4 - p_3| + |p_3 - p_2| \cdot |p_4 - p_1|$.

4. Appendix

In the sequel c_1, c_2, c_3 denote elements of $\mathbb{C}$.

One can prove the following propositions:

(41) $(p_1 - p_2)_{\mathbf{1}} = (p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}}$ and $(p_1 - p_2)_{\mathbf{2}} = (p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}}$.

(42) $|p_1 - p_2| = 0$ iff $p_1 = p_2$.

(43) $|p_1 - p_2| = |p_2 - p_1|$.

(44) $\measuredangle(p_1,p_2,p_3) \neq 2 \cdot \measuredangle(p_4,p_5,p_6) + 2 \cdot \pi$.

(45) $\measuredangle(p_1,p_2,p_3) \neq 2 \cdot \measuredangle(p_4,p_5,p_6) + 4 \cdot \pi$.

(46) $\measuredangle(p_1,p_2,p_3) \neq 2 \cdot \measuredangle(p_4,p_5,p_6) - 4 \cdot \pi$.

(47) $\measuredangle(p_1,p_2,p_3) \neq 2 \cdot \measuredangle(p_4,p_5,p_6) - 6 \cdot \pi$.

(48) $\measuredangle(p_1,p_2,p_3) = \measuredangle((\mathrm{euc2cpx}(p_1 - p_2)), (\mathrm{euc2cpx}(p_3 - p_2)))$.

(49) $\measuredangle(c_1,c_2) + \measuredangle(c_2,c_3) = \measuredangle(c_1,c_3)$ or $\measuredangle(c_1,c_2) + \measuredangle(c_2,c_3) = \measuredangle(c_1,c_3) + 2 \cdot \pi$.

(50) Suppose $c_1 = \mathrm{euc2cpx}(p_1 - p_2)$ and $c_2 = \mathrm{euc2cpx}(p_3 - p_2)$. Then $\Re((c_1|c_2)) = ((p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}}) \cdot ((p_3)_{\mathbf{1}} - (p_2)_{\mathbf{1}}) + ((p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}}) \cdot ((p_3)_{\mathbf{2}} - (p_2)_{\mathbf{2}})$ and $\Im((c_1|c_2)) = -((p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}}) \cdot ((p_3)_{\mathbf{2}} - (p_2)_{\mathbf{2}}) + ((p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}}) \cdot ((p_3)_{\mathbf{1}} - (p_2)_{\mathbf{1}})$ and $|c_1| = \sqrt{((p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}})^{\mathbf{2}} + ((p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}})^{\mathbf{2}}}$ and $|p_1 - p_2| = |c_1|$.

(51) Let n be an element of $\mathbb{N}$, q_1 be a point of $\mathcal{E}^n_{\mathrm{T}}$, and f be a function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$. If for every point q of $\mathcal{E}^n_{\mathrm{T}}$ holds $f(q) = |q - q_1|$, then f is continuous.

(52) Let n be an element of $\mathbb{N}$ and q_1 be a point of $\mathcal{E}^n_{\mathrm{T}}$. Then there exists a function f from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every point q of $\mathcal{E}^n_{\mathrm{T}}$ holds $f(q) = |q - q_1|$ and f is continuous.

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