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The Lebesgue Monotone Convergence Theorem

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Summary. In this article we prove the Monotone Convergence Theorem [16].

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The notation and terminology used in this paper have been introduced in the following articles: [10], [20], [2], [7], [21], [6], [8], [9], [1], [17], [18], [3], [4], [5], [13], [14], [15], [19], [11], [12], and [22].

1. Preliminaries

For simplicity, we adopt the following rules: X is a non empty set, S is a σ-field of subsets of X, M is a σ-measure on S, E is an element of S, F, G are sequences of partial functions from X into $\overline{\mathbb{R}}$, I is a sequence of extended reals, f, g are partial functions from X to $\overline{\mathbb{R}}$, s_1, s_2, s_3 are sequences of extended reals, p is an extended real number, n, m are natural numbers, x is an element of X, and z, D are sets.

Next we state a number of propositions:

(1) If f is without $+\infty$ and g is without $+\infty$, then $\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$.

(2) If f is without $+\infty$ and g is without $-\infty$, then $\operatorname{dom}(f-g) = \operatorname{dom} f \cap \operatorname{dom} g$.

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(3) If f is without $-\infty$ and g is without $-\infty$, then $f+g$ is without $-\infty$.

(4) If f is without $+\infty$ and g is without $+\infty$, then $f+g$ is without $+\infty$.

(5) If f is without $-\infty$ and g is without $+\infty$, then $f-g$ is without $-\infty$.

(6) If f is without $+\infty$ and g is without $-\infty$, then $f-g$ is without $+\infty$.

(7)(i) If s_1 is convergent to finite number, then there exists a real number g such that $\lim s_1 = g$ and for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|s_1(m) - \lim s_1| < p$,

(ii) if s_1 is convergent to $+\infty$, then $\lim s_1 = +\infty$, and

(iii) if s_1 is convergent to $-\infty$, then $\lim s_1 = -\infty$.

(8) If s_1 is non-negative, then s_1 is not convergent to $-\infty$.

(9) If s_1 is convergent and for every natural number k holds $s_1(k) \leq p$, then $\lim s_1 \leq p$.

(10) If s_1 is convergent and for every natural number k holds $p \leq s_1(k)$, then $p \leq \lim s_1$.

(11) Suppose that

(i) s_2 is convergent,

(ii) s_3 is convergent,

(iii) s_2 is non-negative,

(iv) s_3 is non-negative, and

(v) for every natural number k holds $s_1(k) = s_2(k) + s_3(k)$.

Then s_1 is non-negative and convergent and $\lim s_1 = \lim s_2 + \lim s_3$.

(12) Suppose for every natural number n holds $G(n) = F(n){\restriction}D$ and $x \in D$. Then

(i) if $F\#x$ is convergent to $+\infty$, then $G\#x$ is convergent to $+\infty$,

(ii) if $F\#x$ is convergent to $-\infty$, then $G\#x$ is convergent to $-\infty$,

(iii) if $F\#x$ is convergent to finite number, then $G\#x$ is convergent to finite number, and

(iv) if $F\#x$ is convergent, then $G\#x$ is convergent.

(13) If $E = \operatorname{dom} f$ and f is measurable on E and f is non-negative and $M(E \cap \text{EQ-dom}(f, +\infty)) \neq 0$, then $\int f \, \mathrm{d}M = +\infty$.

(14) $\int \chi_{E,X} \, \mathrm{d}M = M(E)$ and $\int \chi_{E,X}{\restriction}E \, \mathrm{d}M = M(E)$.

(15) Suppose that

(i) $E \subseteq \operatorname{dom} f$,

(ii) $E \subseteq \operatorname{dom} g$,

(iii) f is measurable on E,

(iv) g is measurable on E,

(v) f is non-negative, and

(vi) for every element x of X such that $x \in E$ holds $f(x) \leq g(x)$.

Then $\int f{\restriction}E \, \mathrm{d}M \leq \int g{\restriction}E \, \mathrm{d}M$.

2. Selected Properties of Extended Real Sequence

Let f be an extended real-valued function and let x be a set. Then $f(x)$ is an element of $\overline{\mathbb{R}}$.

Let s be an extended real-valued function. The functor $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ yields a sequence of extended reals and is defined by:

(Def. 1) $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(0) = s(0)$ and for every natural number n holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n) + s(n+1)$.

Let s be an extended real-valued function. We say that s is summable if and only if:

(Def. 2) $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is convergent.

Let s be an extended real-valued function. The functor $\sum s$ yielding an extended real number is defined as follows:

(Def. 3) $\sum s = \lim((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}})$.

Next we state several propositions:

(16) If s_1 is non-negative, then $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is non-negative and $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing.

(17) If for every natural number n holds $0 < s_1(n)$, then for every natural number m holds $0 < (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m)$.

(18) If F has the same dom and for every natural number n holds $G(n) = F(n)\restriction D$, then G has the same dom.

(19) Suppose that

- (i) $D \subseteq \operatorname{dom} F(0)$,
- (ii) for every natural number n holds $G(n) = F(n)\restriction D$, and
- (iii) for every element x of X such that $x \in D$ holds $F\#x$ is convergent.

Then $\lim F\restriction D = \lim G$.

(20) Suppose F has the same dom and $E \subseteq \operatorname{dom} F(0)$ and for every natural number m holds $F(m)$ is measurable on E and $G(m) = F(m)\restriction E$. Then $G(n)$ is measurable on E.

(21) Suppose that

- (i) $E \subseteq \operatorname{dom} F(0)$,
- (ii) G has the same dom,
- (iii) for every element x of X such that $x \in E$ holds $F\#x$ is summable, and
- (iv) for every natural number n holds $G(n) = F(n)\restriction E$.

Let x be an element of X. If $x \in E$, then $G\#x$ is summable.

3. Partial Sums of Functional Sequence and their Properties

Let X be a non empty set and let F be a sequence of partial functions from X into $\overline{\mathbb{R}}$. The functor $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$ yields a sequence of partial functions from X into $\overline{\mathbb{R}}$ and is defined as follows:

(Def. 4) $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(0) = F(0)$ and for every natural number n holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n) + F(n+1)$.

Let X be a set and let F be a sequence of partial functions from X into $\overline{\mathbb{R}}$. We say that F is additive if and only if:

(Def. 5) For all natural numbers n, m such that $n \neq m$ and for every set x such that $x \in \operatorname{dom} F(n) \cap \operatorname{dom} F(m)$ holds $F(n)(x) \neq +\infty$ or $F(m)(x) \neq -\infty$.

Next we state a number of propositions:

(22) If $z \in \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)$ and $m \leq n$, then $z \in \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(m)$ and $z \in \operatorname{dom} F(m)$.

(23) If $z \in \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)$ and $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)(z) = +\infty$, then there exists a natural number m such that $m \leq n$ and $F(m)(z) = +\infty$.

(24) If F is additive and $z \in \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)$ and $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)(z) = +\infty$ and $m \leq n$, then $F(m)(z) \neq -\infty$.

(25) If $z \in \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)$ and $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)(z) = -\infty$, then there exists a natural number m such that $m \leq n$ and $F(m)(z) = -\infty$.

(26) If F is additive and $z \in \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)$ and $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)(z) = -\infty$ and $m \leq n$, then $F(m)(z) \neq +\infty$.

(27) If F is additive, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)^{-1}(\{-\infty\}) \cap F(n+1)^{-1}(\{+\infty\}) = \emptyset$ and $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n)^{-1}(\{+\infty\}) \cap F(n+1)^{-1}(\{-\infty\}) = \emptyset$.

(28) If F is additive, then $\operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n) = \bigcap\{\operatorname{dom} F(k); k$ ranges over elements of $\mathbb{N}$: $k \leq n\}$.

(29) If F is additive and has the same dom, then $\operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n) = \operatorname{dom} F(0)$.

(30) If for every natural number n holds $F(n)$ is non-negative, then F is additive.

(31) If F is additive and for every n holds $G(n) = F(n){\restriction}D$, then G is additive.

(32) If F is additive and has the same dom and $D \subseteq \operatorname{dom} F(0)$ and $x \in D$, then $(\sum_{\alpha=0}^{\kappa}(F\#x)(\alpha))_{\kappa\in\mathbb{N}}(n) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\#x)(n)$.

(33) Suppose F is additive and has the same dom and $D \subseteq \operatorname{dom} F(0)$ and $x \in D$. Then

(i) $(\sum_{\alpha=0}^{\kappa}(F\#x)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent to finite number iff $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\#x$ is convergent to finite number,

(ii) $(\sum_{\alpha=0}^{\kappa}(F\#x)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent to $+\infty$ iff $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\#x$ is convergent to $+\infty$,

(iii) $(\sum_{\alpha=0}^{\kappa}(F\#x)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent to $-\infty$ iff $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}\#x$ is convergent to $-\infty$, and

(iv) $(\sum_{\alpha=0}^{\kappa}(F\#x)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent iff $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}\#x$ is convergent.

(34) If F is additive and has the same dom and $\operatorname{dom} f \subseteq \operatorname{dom} F(0)$ and $x \in \operatorname{dom} f$ and $F\#x$ is summable and $f(x) = \sum F\#x$, then $f(x) = \lim((\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}\#x)$.

(35) Suppose that for every natural number m holds $F(m)$ is simple function in S. Then F is additive and $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}(n)$ is simple function in S.

(36) If for every natural number m holds $F(m)$ is non-negative, then $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}(n)$ is non-negative.

(37) If F has the same dom and $x \in \operatorname{dom} F(0)$ and for every natural number k holds $F(k)$ is non-negative and $n \leq m$, then $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}(n)(x) \leq (\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}(m)(x)$.

(38) Suppose F has the same dom and $x \in \operatorname{dom} F(0)$ and for every natural number m holds $F(m)$ is non-negative. Then $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}\#x$ is non-decreasing and $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}\#x$ is convergent.

(39) If for every natural number m holds $F(m)$ is without $-\infty$, then $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}(n)$ is without $-\infty$.

(40) If for every natural number m holds $F(m)$ is without $+\infty$, then $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}(n)$ is without $+\infty$.

(41) Suppose that for every natural number n holds $F(n)$ is measurable on E and $F(n)$ is without $-\infty$. Then $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}(m)$ is measurable on E.

(42) Suppose that

(i) F is additive and has the same dom,

(ii) G is additive and has the same dom,

(iii) $x \in \operatorname{dom} F(0) \cap \operatorname{dom} G(0)$, and

(iv) for every natural number k and for every element y of X such that $y \in \operatorname{dom} F(0) \cap \operatorname{dom} G(0)$ holds $F(k)(y) \leq G(k)(y)$.

Then $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}(n)(x) \leq (\sum_{\alpha=0}^{\kappa}G(\alpha))_{\kappa\in\mathbb{N}}(n)(x)$.

(43) Let X be a non empty set and F be a sequence of partial functions from X into $\overline{\mathbb{R}}$. If F is additive and has the same dom, then $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}$ has the same dom.

(44) Suppose that

(i) $\operatorname{dom} F(0) = E$,

(ii) F is additive and has the same dom,

(iii) for every natural number n holds $(\sum_{\alpha=0}^{\kappa}F(\alpha))_{\kappa\in\mathbb{N}}(n)$ is measurable on E, and

(iv) for every element x of X such that $x \in E$ holds $F\#x$ is summable.

Then $\lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})$ is measurable on E.

(45) Suppose that for every natural number n holds $F(n)$ is integrable on M. Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ is integrable on M.

(46) Suppose that

(i) $E = \operatorname{dom} F(0)$,

(ii) F is additive and has the same dom, and

(iii) for every natural number n holds $F(n)$ is measurable on E and $F(n)$ is non-negative and $I(n) = \int F(n)\, \mathrm{d}M$.

Then $\int (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)\, \mathrm{d}M = (\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(m)$.

4. Sequence of Measurable Functions

Next we state two propositions:

(47) Suppose that

(i) $E \subseteq \operatorname{dom} f$,

(ii) f is non-negative,

(iii) f is measurable on E,

(iv) F is additive,

(v) for every n holds $F(n)$ is simple function in S and $F(n)$ is non-negative and $E \subseteq \operatorname{dom} F(n)$, and

(vi) for every x such that $x \in E$ holds $F\#x$ is summable and $f(x) = \sum F\#x$.

Then there exists a sequence I of extended reals such that for every n holds $I(n) = \int F(n)\restriction E\, \mathrm{d}M$ and I is summable and $\int f\restriction E\, \mathrm{d}M = \sum I$.

(48) Suppose $E \subseteq \operatorname{dom} f$ and f is non-negative and f is measurable on E. Then there exists a sequence g of partial functions from X into $\overline{\mathbb{R}}$ such that

(i) g is additive,

(ii) for every natural number n holds $g(n)$ is simple function in S and $g(n)$ is non-negative and $g(n)$ is measurable on E,

(iii) for every element x of X such that $x \in E$ holds $g\#x$ is summable and $f(x) = \sum g\#x$, and

(iv) there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int g(n)\restriction E\, \mathrm{d}M$ and I is summable and $\int f\restriction E\, \mathrm{d}M = \sum I$.

Let X be a non empty set. Observe that there exists a sequence of partial functions from X into $\overline{\mathbb{R}}$ which is additive and has the same dom.

Let C, D, X be non empty sets, let F be a function from $C \times D$ into $X \dot{\rightarrow} \overline{\mathbb{R}}$, let c be an element of C, and let d be an element of D. Then $F(c, d)$ is a partial function from X to $\overline{\mathbb{R}}$.

Let C, D, X be non empty sets, let F be a function from $C \times D$ into X, and let c be an element of C. The functor curry(F, c) yields a function from D into X and is defined as follows:

(Def. 6) For every element d of D holds $(\mathrm{curry}(F, c))(d) = F(c, d)$.

Let C, D, X be non empty sets, let F be a function from $C \times D$ into X, and let d be an element of D. The functor curry$'(F, d)$ yields a function from C into X and is defined as follows:

(Def. 7) For every element c of C holds $(\mathrm{curry}'(F, d))(c) = F(c, d)$.

Let X, Y be sets, let F be a function from $\mathbb{N} \times \mathbb{N}$ into $X \dot{\to} Y$, and let n be a natural number. The functor curry(F, n) yielding a sequence of partial functions from X into Y is defined by:

(Def. 8) For every natural number m holds $(\mathrm{curry}(F, n))(m) = F(n, m)$.

The functor curry$'(F, n)$ yields a sequence of partial functions from X into Y and is defined by:

(Def. 9) For every natural number m holds $(\mathrm{curry}'(F, n))(m) = F(m, n)$.

Let X be a non empty set, let F be a function from $\mathbb{N}$ into $(X \dot{\to} \overline{\mathbb{R}})^{\mathbb{N}}$, and let n be a natural number. Then $F(n)$ is a sequence of partial functions from X into $\overline{\mathbb{R}}$.

The following four propositions are true:

(49) Suppose $E = \mathrm{dom}\, F(0)$ and F has the same dom and for every natural number n holds $F(n)$ is non-negative and $F(n)$ is measurable on E. Then there exists a function F_1 from $\mathbb{N}$ into $(X \dot{\to} \overline{\mathbb{R}})^{\mathbb{N}}$ such that for every natural number n holds

(i) for every natural number m holds $F_1(n)(m)$ is simple function in S and $\mathrm{dom}\, F_1(n)(m) = \mathrm{dom}\, F(n)$,

(ii) for every natural number m holds $F_1(n)(m)$ is non-negative,

(iii) for all natural numbers j, k such that $j \leq k$ and for every element x of X such that $x \in \mathrm{dom}\, F(n)$ holds $F_1(n)(j)(x) \leq F_1(n)(k)(x)$, and

(iv) for every element x of X such that $x \in \mathrm{dom}\, F(n)$ holds $F_1(n)\#x$ is convergent and $\lim(F_1(n)\#x) = F(n)(x)$.

(50) Suppose that

(i) $E = \mathrm{dom}\, F(0)$,

(ii) F is additive and has the same dom, and

(iii) for every natural number n holds $F(n)$ is measurable on E and $F(n)$ is non-negative.

Then there exists a sequence I of extended reals such that for every natural number n holds
$I(n) = \int F(n)\, \mathrm{d}M$ and $\int (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)\, \mathrm{d}M = (\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(51) Suppose that

(i) $E \subseteq \mathrm{dom}\, F(0)$,

(ii) F is additive and has the same dom,

(iii) for every natural number n holds $F(n)$ is non-negative and $F(n)$ is measurable on E, and

(iv) for every element x of X such that $x \in E$ holds $F\#x$ is summable.

Then there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n)\restriction E \, \mathrm{d}M$ and I is summable and $\int \lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})\restriction E \, \mathrm{d}M = \sum I$.

(52) Suppose that

(i) $E = \mathrm{dom}\, F(0)$,

(ii) $F(0)$ is non-negative,

(iii) F has the same dom,

(iv) for every natural number n holds $F(n)$ is measurable on E,

(v) for all natural numbers n, m such that $n \leq m$ and for every element x of X such that $x \in E$ holds $F(n)(x) \leq F(m)(x)$, and

(vi) for every element x of X such that $x \in E$ holds $F\#x$ is convergent.

Then there exists a sequence I of extended reals such that for every natural number n holds $I(n) = \int F(n) \, \mathrm{d}M$ and I is convergent and $\int \lim F \, \mathrm{d}M = \lim I$.

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