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Extended Riemann Integral of Functions of Real Variable and One-sided Laplace Transform[1]

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Summary. In this article, we defined a variety of extended Riemann integrals and proved that such integration is linear. Furthermore, we defined the one-sided Laplace transform and proved the linearity of that operator.

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The papers [11], [1], [5], [12], [10], [2], [7], [6], [8], [9], [3], [4], and [13] provide the terminology and notation for this paper.

1. Preliminaries

In this paper a, b, r are elements of $\mathbb{R}$.

We now state three propositions:

(1) For all real numbers a, b, g_1, M such that $a < b$ and $0 < g_1$ and $0 < M$ there exists r such that $a < r < b$ and $(b - r) \cdot M < g_1$.

(2) For all real numbers a, b, g_1, M such that $a < b$ and $0 < g_1$ and $0 < M$ there exists r such that $a < r < b$ and $(r - a) \cdot M < g_1$.

(3) $\exp b - \exp a = \int\limits_a^b (\text{the function exp})(x)dx$.

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2. The Definition of Extended Riemann Integral

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let a, b be real numbers. We say that f is right extended Riemann integrable on a, b if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) For every real number d such that $a \leq d < b$ holds f is integrable on $[a, d]$ and $f\restriction[a, d]$ is bounded, and

(ii) there exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathrm{dom}\,\mathcal{I} = [a, b[$ and for every real number x such that $x \in \mathrm{dom}\,\mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_a^x f(x)dx$ and $\mathcal{I}$ is left convergent in b.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let a, b be real numbers. We say that f is left extended Riemann integrable on a, b if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) For every real number d such that $a < d \leq b$ holds f is integrable on $[d, b]$ and $f\restriction[d, b]$ is bounded, and

(ii) there exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathrm{dom}\,\mathcal{I} =]a, b]$ and for every real number x such that $x \in \mathrm{dom}\,\mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_x^b f(x)dx$ and $\mathcal{I}$ is right convergent in a.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let a, b be real numbers. Let us assume that f is right extended Riemann integrable on a, b. The functor $(R^{>})\int\limits_a^b f(x)dx$ yielding a real number is defined by the condition (Def. 3).

(Def. 3) There exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathrm{dom}\,\mathcal{I} = [a, b[$ and for every real number x such that $x \in \mathrm{dom}\,\mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_a^x f(x)dx$ and $\mathcal{I}$ is left convergent in b and $(R^{>})\int\limits_a^b f(x)dx = \lim\limits_{b^-}\mathcal{I}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let a, b be real numbers. Let us assume that f is left extended Riemann integrable on a, b. The functor $(R^{<})\int\limits_a^b f(x)dx$ yielding a real number is defined by the condition (Def. 4).

(Def. 4) There exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathrm{dom}\,\mathcal{I} =]a, b]$ and for every real number x such that $x \in \mathrm{dom}\,\mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_x^b f(x)dx$

and $\mathcal{I}$ is right convergent in a and $(R^{<})\int\limits_a^b f(x)dx = \lim\limits_{a^+}\mathcal{I}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let a be a real number. We say that f is extended Riemann integrable on a, $+\infty$ if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) For every real number b such that $a \leq b$ holds f is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded, and

(ii) there exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom}\mathcal{I} = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom}\mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_a^x f(x)dx$ and $\mathcal{I}$ is convergent in $+\infty$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let b be a real number. We say that f is extended Riemann integrable on $-\infty$, b if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) For every real number a such that $a \leq b$ holds f is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded, and

(ii) there exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom}\mathcal{I} =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom}\mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_x^b f(x)dx$ and $\mathcal{I}$ is convergent in $-\infty$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let a be a real number. Let us assume that f is extended Riemann integrable on a, $+\infty$. The functor $(R^{>})\int\limits_a^{+\infty} f(x)dx$ yielding a real number is defined by the condition (Def. 7).

(Def. 7) There exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom}\mathcal{I} = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom}\mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_a^x f(x)dx$ and $\mathcal{I}$ is convergent in $+\infty$ and $(R^{>})\int\limits_a^{+\infty} f(x)dx = \lim\limits_{+\infty}\mathcal{I}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let b be a real number. Let us assume that f is extended Riemann integrable on $-\infty$, b. The functor $(R^{<})\int\limits_{-\infty}^{b} f(x)dx$ yields a real number and is defined by the condition (Def. 8).

(Def. 8) There exists a partial function $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom}\mathcal{I} =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom}\mathcal{I}$ holds $\mathcal{I}(x) = \int\limits_x^b f(x)dx$

and $\mathcal{I}$ is convergent in $-\infty$ and $(R^{<})\int\limits_{-\infty}^{b} f(x)dx = \lim\limits_{-\infty}\mathcal{I}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. We say that f is ∞-extended Riemann integrable if and only if:

(Def. 9) f is extended Riemann integrable on 0, $+\infty$ and extended Riemann integrable on $-\infty$, 0.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $(R)\int\limits_{-\infty}^{+\infty} f(x)dx$ yields a real number and is defined by:

(Def. 10) $(R)\int\limits_{-\infty}^{+\infty} f(x)dx = (R^{>})\int\limits_{0}^{+\infty} f(x)dx + (R^{<})\int\limits_{-\infty}^{0} f(x)dx$.

3. Linearity of Extended Riemann Integral

One can prove the following propositions:

(4) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. Suppose that

(i) $a < b$,

(ii) $[a, b] \subseteq \operatorname{dom} f$,

(iii) $[a, b] \subseteq \operatorname{dom} g$,

(iv) f is right extended Riemann integrable on a, b, and

(v) g is right extended Riemann integrable on a, b.

Then $f + g$ is right extended Riemann integrable on a, b and $(R^{>})\int\limits_{a}^{b}(f + g)(x)dx = (R^{>})\int\limits_{a}^{b} f(x)dx + (R^{>})\int\limits_{a}^{b} g(x)dx$.

(5) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. Suppose $a < b$ and $[a, b] \subseteq \operatorname{dom} f$ and f is right extended Riemann integrable on a, b. Let r be a real number. Then $r\,f$ is right extended Riemann integrable on a, b and $(R^{>})\int\limits_{a}^{b}(r\,f)(x)dx = r \cdot (R^{>})\int\limits_{a}^{b} f(x)dx$.

(6) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. Suppose that

(i) $a < b$,

(ii) $[a, b] \subseteq \operatorname{dom} f$,

(iii) $[a, b] \subseteq \operatorname{dom} g$,

(iv) f is left extended Riemann integrable on a, b, and

(v) g is left extended Riemann integrable on a, b.

Then $f+g$ is left extended Riemann integrable on a, b and $(R^{<})\int\limits_{a}^{b}(f+g)(x)dx = (R^{<})\int\limits_{a}^{b} f(x)dx + (R^{<})\int\limits_{a}^{b} g(x)dx$.

(7) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. Suppose $a<b$ and $[a,b]\subseteq \operatorname{dom} f$ and f is left extended Riemann integrable on a, b. Let r be a real number. Then $r\,f$ is left extended Riemann integrable on a, b and $(R^{<})\int\limits_{a}^{b}(r\,f)(x)dx = r\cdot(R^{<})\int\limits_{a}^{b} f(x)dx$.

(8) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and a be a real number. Suppose that

(i) $[a,+\infty[\ \subseteq \operatorname{dom} f$,

(ii) $[a,+\infty[\ \subseteq \operatorname{dom} g$,

(iii) f is extended Riemann integrable on a, $+\infty$, and

(iv) g is extended Riemann integrable on a, $+\infty$.

Then $f+g$ is extended Riemann integrable on a, $+\infty$ and $(R^{>})\int\limits_{a}^{+\infty}(f+g)(x)dx = (R^{>})\int\limits_{a}^{+\infty} f(x)dx + (R^{>})\int\limits_{a}^{+\infty} g(x)dx$.

(9) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a be a real number. Suppose $[a,+\infty[\ \subseteq \operatorname{dom} f$ and f is extended Riemann integrable on a, $+\infty$. Let r be a real number. Then $r\,f$ is extended Riemann integrable on a, $+\infty$ and $(R^{>})\int\limits_{a}^{+\infty}(r\,f)(x)dx = r\cdot(R^{>})\int\limits_{a}^{+\infty} f(x)dx$.

(10) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and b be a real number. Suppose that

(i) $]-\infty,b]\subseteq \operatorname{dom} f$,

(ii) $]-\infty,b]\subseteq \operatorname{dom} g$,

(iii) f is extended Riemann integrable on $-\infty$, b, and

(iv) g is extended Riemann integrable on $-\infty$, b.

Then $f+g$ is extended Riemann integrable on $-\infty$, b and $(R^{<})\int\limits_{-\infty}^{b}(f+g)(x)dx = (R^{<})\int\limits_{-\infty}^{b} f(x)dx + (R^{<})\int\limits_{-\infty}^{b} g(x)dx$.

(11) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and b be a real number. Suppose $]-\infty,b]\subseteq \operatorname{dom} f$ and f is extended Riemann integrable on $-\infty$, b. Let r be a real number. Then $r\,f$ is extended Riemann integrable on $-\infty$, b and $(R^{<})\int\limits_{-\infty}^{b}(r\,f)(x)dx = r\cdot(R^{<})\int\limits_{-\infty}^{b} f(x)dx$.

(12) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers.

Suppose $a < b$ and $[a,b] \subseteq \operatorname{dom} f$ and f is integrable on $[a,b]$ and $f \restriction [a,b]$ is bounded. Then $(R^{>})\int\limits_a^b f(x)dx = \int\limits_a^b f(x)dx$.

(13) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a, b be real numbers. Suppose $a < b$ and $[a,b] \subseteq \operatorname{dom} f$ and f is integrable on $[a,b]$ and $f \restriction [a,b]$ is bounded. Then $(R^{<})\int\limits_a^b f(x)dx = \int\limits_a^b f(x)dx$.

4. The Definition of One-sided Laplace Transform

Let s be a real number. The functor $e^{-s\cdot\square}$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by:

(Def. 11) For every real number t holds $e^{-s\cdot\square}(t) = (\text{the function exp})(-s \cdot t)$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The one-sided Laplace transform of f yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by the conditions (Def. 12).

(Def. 12)(i) dom (the one-sided Laplace transform of f) $=]0, +\infty[$, and

(ii) for every real number s such that $s \in$ dom (the one-sided Laplace transform of f) holds (the one-sided Laplace transform of f)$(s) = (R^{>})\int\limits_0^{+\infty} (f\, e^{-s\cdot\square})(x)dx$.

5. Linearity of One-sided Laplace Transform

Next we state two propositions:

(14) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that

(i) $\operatorname{dom} f = [0, +\infty[$,

(ii) $\operatorname{dom} g = [0, +\infty[$,

(iii) for every real number s such that $s \in]0, +\infty[$ holds $f\, e^{-s\cdot\square}$ is extended Riemann integrable on 0, $+\infty$, and

(iv) for every real number s such that $s \in]0, +\infty[$ holds $g\, e^{-s\cdot\square}$ is extended Riemann integrable on 0, $+\infty$.

Then

(v) for every real number s such that $s \in]0, +\infty[$ holds $(f + g)\, e^{-s\cdot\square}$ is extended Riemann integrable on 0, $+\infty$, and

(vi) the one-sided Laplace transform of $f + g$ = (the one-sided Laplace transform of f) + (the one-sided Laplace transform of g).

(15) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and a be a real number. Suppose $\operatorname{dom} f = [0, +\infty[$ and for every real number s such that $s \in]0, +\infty[$ holds $f\, e^{-s\cdot\square}$ is extended Riemann integrable on 0, $+\infty$. Then

(i) for every real number s such that $s \in]0, +\infty[$ holds $a\, f\, e^{-s \cdot \square}$ is extended Riemann integrable on 0, $+\infty$, and

(ii) the one-sided Laplace transform of $a\, f = a$ the one-sided Laplace transform of f.

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