FORMALIZED MATHEMATICS
Vol. 16, No. 4, Pages 401–403, 2008

Open Mapping Theorem

Hideki Sakurai
Shinshu University
Nagano, Japan

Hisayoshi Kunimune
Shinshu University
Nagano, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article we formalize one of the most important theorems of linear operator theory the Open Mapping Theorem commonly used in a standard book such as [8] in chapter 2.4.2. It states that a surjective continuous linear operator between Banach spaces is an open map.

MML identifier: LOPBAN_6, version: 7.10.01 4.111.1036

The notation and terminology used here are introduced in the following papers: [13], [14], [3], [9], [2], [7], [1], [4], [5], [10], [6], [12], [11], and [15].

The following proposition is true

(1) For all real numbers x, y such that $0 \leq x < y$ there exists a real number s_0 such that $0 < s_0$ and $x < \frac{y}{1+s_0} < y$.

The scheme *RecExD3* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, an element $\mathcal{C}$ of $\mathcal{A}$, and a 4-ary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and $f(1) = \mathcal{C}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), f(n+1), f(n+2)]$

provided the parameters meet the following requirement:

- For every element n of $\mathbb{N}$ and for all elements x, y of $\mathcal{A}$ there exists an element z of $\mathcal{A}$ such that $\mathcal{P}[n, x, y, z]$.

In the sequel X, Y denote real normed spaces.

The following propositions are true:

(2) For every point y_1 of X and for every real number r holds $\mathrm{Ball}(y_1, r) = y_1 + \mathrm{Ball}(0_X, r)$.

(3) For every real number r and for every real number a such that $0 < a$ holds $\mathrm{Ball}(0_X, a \cdot r) = a \cdot \mathrm{Ball}(0_X, r)$.

ISSN 1426–2630(p), 1898-9934(e)

(4) For every linear operator T from X into Y and for all subsets B_0, B_1 of X holds $T^\circ(B_0 + B_1) = T^\circ B_0 + T^\circ B_1$.

(5) Let T be a linear operator from X into Y, B_0 be a subset of X, and a be a real number. Then $T^\circ(a \cdot B_0) = a \cdot T^\circ B_0$.

(6) Let T be a linear operator from X into Y, B_0 be a subset of X, and x_1 be a point of X. Then $T^\circ(x_1 + B_0) = T(x_1) + T^\circ B_0$.

(7) For all subsets V, W of X and for all subsets V_1, W_1 of LinearTopSpaceNorm X such that $V = V_1$ and $W = W_1$ holds $V + W = V_1 + W_1$.

(8) Let V be a subset of X, x be a point of X, V_1 be a subset of LinearTopSpaceNorm X, and x_1 be a point of LinearTopSpaceNorm X. If $V = V_1$ and $x = x_1$, then $x + V = x_1 + V_1$.

(9) For every subset V of X and for every real number a and for every subset V_1 of LinearTopSpaceNorm X such that $V = V_1$ holds $a \cdot V = a \cdot V_1$.

(10) For every subset V of TopSpaceNorm X and for every subset V_1 of LinearTopSpaceNorm X such that $V = V_1$ holds $\overline{V} = \overline{V_1}$.

(11) For every point x of X and for every real number r holds $\mathrm{Ball}(0_X, r) = (-1) \cdot \mathrm{Ball}(0_X, r)$.

(12) For every point x of X and for every real number r and for every subset V of LinearTopSpaceNorm X such that $V = \mathrm{Ball}(x, r)$ holds V is convex.

(13) Let x be a point of X, r be a real number, T be a linear operator from X into Y, and V be a subset of LinearTopSpaceNorm Y. If $V = T^\circ \mathrm{Ball}(x, r)$, then V is convex.

(14) For every point x of X and for all real numbers r, s such that $r \leq s$ holds $\mathrm{Ball}(x, r) \subseteq \mathrm{Ball}(x, s)$.

(15) Let X be a real Banach space, Y be a real normed space, T be a bounded linear operator from X into Y, r be a real number, B_2 be a subset of LinearTopSpaceNorm X, and T_1, B_3 be subsets of LinearTopSpaceNorm Y. If $r > 0$ and $B_2 = \mathrm{Ball}(0_X, 1)$ and $B_3 = \mathrm{Ball}(0_Y, r)$ and $T_1 = T^\circ \mathrm{Ball}(0_X, 1)$ and $B_3 \subseteq \overline{T_1}$, then $B_3 \subseteq T_1$.

(16) Let X, Y be real Banach spaces, T be a bounded linear operator from X into Y, and T_2 be a function from LinearTopSpaceNorm X into LinearTopSpaceNorm Y. If $T_2 = T$ and T_2 is onto, then T_2 is open.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Noboru Endou, Takashi Mitsuishi, and Yasunari Shidama. Convex sets and convex combinations. *Formalized Mathematics*, 11(**1**):53–58, 2003.

[5] Noboru Endou, Takashi Mitsuishi, and Yasunari Shidama. Dimension of real unitary space. *Formalized Mathematics*, 11(**1**):23–28, 2003.
[6] Noboru Endou, Yasunari Shidama, and Katsumasa Okamura. Baire's category theorem and some spaces generated from real normed space. *Formalized Mathematics*, 14(**4**):213–219, 2006.
[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[8] Isao Miyadera. *Functional Analysis*. Riko-Gaku-Sya, 1972.
[9] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[10] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[11] Hideki Sakurai, Hisayoshi Kunimune, and Yasunari Shidama. Uniform boundedness principle. *Formalized Mathematics*, 16(**1**):19–21, 2008.
[12] Yasunari Shidama. Banach space of bounded linear operators. *Formalized Mathematics*, 12(**1**):39–48, 2004.
[13] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[14] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[15] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Formalized Mathematics*, 5(**1**):75–77, 1996.

Received September 23, 2008