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Formaliz. Math. 17 (**1**)

FORMALIZED MATHEMATICS
Vol. **17**, No. **1**, Pages 1–9, **2009**

The Real Vector Spaces of Finite Sequences are Finite Dimensional

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Summary. In this paper we show the finite dimensionality of real linear spaces with their carriers equal $\mathcal{R}^n$. We also give the standard basis of such spaces. For the set $\mathcal{R}^n$ we introduce the concepts of linear manifold subsets and orthogonal subsets. The cardinality of orthonormal basis of discussed spaces is proved to equal n.

MML identifier: `EUCLID_7`, version: `7.11.01 4.117.1046`

The articles [32], [7], [11], [33], [9], [2], [8], [5], [31], [4], [6], [18], [13], [22], [20], [14], [1], [21], [29], [28], [26], [3], [23], [10], [12], [30], [19], [34], [16], [17], [25], [15], [24], and [27] provide the notation and terminology for this paper.

1. Preliminaries

We use the following convention: i, j, n are elements of $\mathbb{N}$, z, B_0 are sets, and f, x_0 are real-valued finite sequences.

Next we state several propositions:

(1) For all functions f, g holds $\mathrm{dom}(f \cdot g) = \mathrm{dom}\, g \cap g^{-1}(\mathrm{dom}\, f)$.

(2) For every binary relation R and for every set Y such that $\mathrm{rng}\, R \subseteq Y$ holds $R^{-1}(Y) = \mathrm{dom}\, R$.

ISSN 1426–2630(p), 1898-9934(e)

(3) Let X be a set, Y be a non empty set, and f be a function from X into Y. If f is bijective, then $\overline{\overline{X}} = \overline{\overline{Y}}$.

(4) $\langle z \rangle \cdot \langle 1 \rangle = \langle z \rangle$.

(5) For every element x of $\mathcal{R}^0$ holds $x = \varepsilon_{\mathbb{R}}$.

(6) For all elements a, b, c of $\mathcal{R}^n$ holds $(a - b) + c + b = a + c$.

Let f_1, f_2 be finite sequences. One can verify that $\langle f_1, f_2 \rangle$ is finite sequence-like.

Let D be a set and let f_1, f_2 be finite sequences of elements of D. Then $\langle f_1, f_2 \rangle$ is a finite sequence of elements of $D \times D$.

Let h be a real-valued finite sequence. Let us observe that h is increasing if and only if:

(Def. 1) For every i such that $1 \leq i < \operatorname{len} h$ holds $h(i) < h(i+1)$.

One can prove the following four propositions:

(7) Let h be a real-valued finite sequence. Suppose h is increasing. Let given i, j. If $i < j$ and $1 \leq i$ and $j \leq \operatorname{len} h$, then $h(i) < h(j)$.

(8) Let h be a real-valued finite sequence. Suppose h is increasing. Let given i, j. If $i \leq j$ and $1 \leq i$ and $j \leq \operatorname{len} h$, then $h(i) \leq h(j)$.

(9) Let h be a natural-valued finite sequence. Suppose h is increasing. Let given i. If $1 \leq i \leq \operatorname{len} h$ and $1 \leq h(1)$, then $i \leq h(i)$.

(10) Let V be a real linear space and X be a subspace of V. Suppose V is strict and X is strict and the carrier of $X =$ the carrier of V. Then $X = V$.

Let D be a set, let F be a finite sequence of elements of D, and let h be a permutation of $\operatorname{dom} F$. The functor $F \circ h$ yields a finite sequence of elements of D and is defined as follows:

(Def. 2) $F \circ h = F \cdot h$.

One can prove the following propositions:

(11) Let D be a non empty set and f be a finite sequence of elements of D. If $1 \leq i \leq \operatorname{len} f$ and $1 \leq j \leq \operatorname{len} f$, then $(\operatorname{Swap}(f, i, j))(i) = f(j)$ and $(\operatorname{Swap}(f, i, j))(j) = f(i)$.

(12) $\emptyset$ is a permutation of $\emptyset$.

(13) $\langle 1 \rangle$ is a permutation of $\{1\}$.

(14) For every finite sequence h of elements of $\mathbb{R}$ holds h is one-to-one iff $\operatorname{sort_a} h$ is one-to-one.

(15) Let h be a finite sequence of elements of $\mathbb{N}$. Suppose h is one-to-one. Then there exists a permutation h_3 of $\operatorname{dom} h$ and there exists a finite sequence h_2 of elements of $\mathbb{N}$ such that $h_2 = h \cdot h_3$ and h_2 is increasing and $\operatorname{dom} h = \operatorname{dom} h_2$ and $\operatorname{rng} h = \operatorname{rng} h_2$.

2. Orthogonal Basis

Let B_0 be a set. We say that B_0 is $\mathbb{R}$-orthogonal if and only if:

(Def. 3) For all real-valued finite sequences x, y such that $x, y \in B_0$ and $x \neq y$ holds $|(x, y)| = 0$.

Let us observe that every set which is empty is also $\mathbb{R}$-orthogonal.

We now state the proposition

(16) B_0 is $\mathbb{R}$-orthogonal if and only if for all points x, y of $\mathcal{E}^n_{\mathrm{T}}$ such that x, $y \in B_0$ and $x \neq y$ holds x, y are orthogonal.

Let B_0 be a set. We say that B_0 is $\mathbb{R}$-normal if and only if:

(Def. 4) For every real-valued finite sequence x such that $x \in B_0$ holds $|x| = 1$.

Let us observe that every set which is empty is also $\mathbb{R}$-normal.

Let us observe that there exists a set which is $\mathbb{R}$-normal.

Let B_0, B_1 be $\mathbb{R}$-normal sets. One can verify that $B_0 \cup B_1$ is $\mathbb{R}$-normal.

One can prove the following propositions:

(17) If $|f| = 1$, then $\{f\}$ is $\mathbb{R}$-normal.

(18) If B_0 is $\mathbb{R}$-normal and $|x_0| = 1$, then $B_0 \cup \{x_0\}$ is $\mathbb{R}$-normal.

Let B_0 be a set. We say that B_0 is $\mathbb{R}$-orthonormal if and only if:

(Def. 5) B_0 is $\mathbb{R}$-orthogonal and $\mathbb{R}$-normal.

Let us note that every set which is $\mathbb{R}$-orthonormal is also $\mathbb{R}$-orthogonal and $\mathbb{R}$-normal and every set which is $\mathbb{R}$-orthogonal and $\mathbb{R}$-normal is also $\mathbb{R}$-orthonormal.

Let us observe that $\{\langle 1 \rangle\}$ is $\mathbb{R}$-orthonormal.

Let us observe that there exists a set which is $\mathbb{R}$-orthonormal and non empty.

Let us consider n. One can verify that there exists a subset of $\mathcal{R}^n$ which is $\mathbb{R}$-orthonormal.

Let us consider n and let B_0 be a subset of $\mathcal{R}^n$. We say that B_0 is complete if and only if:

(Def. 6) For every $\mathbb{R}$-orthonormal subset B of $\mathcal{R}^n$ such that $B_0 \subseteq B$ holds $B = B_0$.

Let n be an element of $\mathbb{N}$ and let B_0 be a subset of $\mathcal{R}^n$. We say that B_0 is orthogonal basis if and only if:

(Def. 7) B_0 is $\mathbb{R}$-orthonormal and complete.

Let us consider n. One can verify that every subset of $\mathcal{R}^n$ which is orthogonal basis is also $\mathbb{R}$-orthonormal and complete and every subset of $\mathcal{R}^n$ which is $\mathbb{R}$-orthonormal and complete is also orthogonal basis.

The following propositions are true:

(19) For every subset B_0 of $\mathcal{R}^0$ such that B_0 is orthogonal basis holds $B_0 = \emptyset$.

(20) Let B_0 be a subset of $\mathcal{R}^n$ and y be an element of $\mathcal{R}^n$. Suppose B_0 is orthogonal basis and for every element x of $\mathcal{R}^n$ such that $x \in B_0$ holds $|(x, y)| = 0$. Then $y = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

3. Linear Manifolds

Let us consider n and let X be a subset of $\mathcal{R}^n$. We say that X is linear manifold if and only if:

(Def. 8) For all elements x, y of $\mathcal{R}^n$ and for all elements a, b of $\mathbb{R}$ such that x, $y \in X$ holds $a \cdot x + b \cdot y \in X$.

Let us consider n. Observe that $\Omega_{\mathcal{R}^n}$ is linear manifold.

The following proposition is true

(21) $\{\langle \underbrace{0, \ldots, 0}_{n} \rangle\}$ is linear manifold.

Let us consider n. Observe that $\{\langle \underbrace{0, \ldots, 0}_{n} \rangle\}$ is linear manifold.

Let us consider n and let X be a subset of $\mathcal{R}^n$. The linear span of X yielding a subset of $\mathcal{R}^n$ is defined by:

(Def. 9) The linear span of $X = \bigcap\{Y \subseteq \mathcal{R}^n \colon Y$ is linear manifold $\wedge\ X \subseteq Y\}$.

Let us consider n and let X be a subset of $\mathcal{R}^n$. Observe that the linear span of X is linear manifold.

Let us consider n and let f be a finite sequence of elements of $\mathcal{R}^n$. The functor $\sum f$ yielding an element of $\mathcal{R}^n$ is defined as follows:

(Def. 10)(i) There exists a finite sequence g of elements of $\mathcal{R}^n$ such that $\operatorname{len} f = \operatorname{len} g$ and $f(1) = g(1)$ and for every natural number i such that $1 \leq i < \operatorname{len} f$ holds $g(i+1) = g_i + f_{i+1}$ and $\sum f = g(\operatorname{len} f)$ if $\operatorname{len} f > 0$,

(ii) $\sum f = \langle \underbrace{0, \ldots, 0}_{n} \rangle$, otherwise.

Let n be a natural number and let f be a finite sequence of elements of $\mathcal{R}^n$. The functor $\operatorname{accum} f$ yields a finite sequence of elements of $\mathcal{R}^n$ and is defined as follows:

(Def. 11) $\operatorname{len} f = \operatorname{len} \operatorname{accum} f$ and $f(1) = (\operatorname{accum} f)(1)$ and for every natural number i such that $1 \leq i < \operatorname{len} f$ holds $(\operatorname{accum} f)(i+1) = (\operatorname{accum} f)_i + f_{i+1}$.

We now state several propositions:

(22) For every finite sequence f of elements of $\mathcal{R}^n$ such that $\operatorname{len} f > 0$ holds $(\operatorname{accum} f)(\operatorname{len} f) = \sum f$.

(23) For all finite sequences F, F_2 of elements of $\mathcal{R}^n$ and for every permutation h of $\operatorname{dom} F$ such that $F_2 = F \circ h$ holds $\sum F_2 = \sum F$.

(24) For every element k of $\mathbb{N}$ holds $\sum k \mapsto \langle \underbrace{0, \ldots, 0}_{n} \rangle = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(25) Let g be a finite sequence of elements of $\mathcal{R}^n$, h be a finite sequence of elements of $\mathbb{N}$, and F be a finite sequence of elements of $\mathcal{R}^n$. Suppose h is increasing and $\operatorname{rng} h \subseteq \operatorname{dom} g$ and $F = g \cdot h$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} g$ and $i \notin \operatorname{rng} h$ holds $g(i) = \langle \underbrace{0, \ldots, 0}_{n} \rangle$. Then $\sum g = \sum F$.

(26) Let g be a finite sequence of elements of $\mathcal{R}^n$, h be a finite sequence of elements of $\mathbb{N}$, and F be a finite sequence of elements of $\mathcal{R}^n$. Suppose h is one-to-one and $\operatorname{rng} h \subseteq \operatorname{dom} g$ and $F = g \cdot h$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} g$ and $i \notin \operatorname{rng} h$ holds $g(i) = \langle \underbrace{0, \ldots, 0}_{n} \rangle$. Then $\sum g = \sum F$.

4. Standard Basis

Let us consider n, i. Then the base finite sequence of n and i is an element of $\mathcal{R}^n$.

The following propositions are true:

(27) Let i_1, i_2 be elements of $\mathbb{N}$. Suppose that

(i) $1 \leq i_1$,

(ii) $i_1 \leq n$,

(iii) $1 \leq i_2$,

(iv) $i_2 \leq n$, and

(v) the base finite sequence of n and i_1 = the base finite sequence of n and i_2.

Then $i_1 = i_2$.

(28) ^{2}(the base finite sequence of n and i) = the base finite sequence of n and i.

(29) If $1 \leq i \leq n$, then $\sum$ the base finite sequence of n and $i = 1$.

(30) If $1 \leq i \leq n$, then |the base finite sequence of n and $i| = 1$.

(31) Suppose $1 \leq i \leq n$ and $1 \leq j \leq n$ and $i \neq j$. Then |(the base finite sequence of n and i, the base finite sequence of n and j)| = 0.

(32) For every element x of $\mathcal{R}^n$ such that $1 \leq i \leq n$ holds $|(x,$ the base finite sequence of n and $i)| = x(i)$.

Let us consider n and let x_0 be an element of $\mathcal{R}^n$. The functor $\operatorname{ProjFinSeq} x_0$ yields a finite sequence of elements of $\mathcal{R}^n$ and is defined by the conditions (Def. 12).

(Def. 12)(i) $\operatorname{len} \operatorname{ProjFinSeq} x_0 = n$, and

(ii) for every i such that $1 \leq i \leq n$ holds $(\operatorname{ProjFinSeq} x_0)(i) = |(x_0,$ the base finite sequence of n and $i)| \cdot$ the base finite sequence of n and i.

The following proposition is true

(33) For every element x_0 of $\mathcal{R}^n$ holds $x_0 = \sum \mathrm{ProjFinSeq}\, x_0$.

Let us consider n. The functor $\mathbb{R}$N-Base n yields a subset of $\mathcal{R}^n$ and is defined by:

(Def. 13) $\mathbb{R}$N-Base n = {the base finite sequence of n and i; i ranges over elements of $\mathbb{N}$: $1 \leq i \ \wedge \ i \leq n$}.

Next we state the proposition

(34) For every non zero element n of $\mathbb{N}$ holds $\mathbb{R}$N-Base $n \neq \emptyset$.

Let us mention that $\mathbb{R}$N-Base 0 is empty.

Let n be a non zero element of $\mathbb{N}$. Note that $\mathbb{R}$N-Base n is non empty.

Let us consider n. Observe that $\mathbb{R}$N-Base n is orthogonal basis.

Let us consider n. Observe that there exists a subset of $\mathcal{R}^n$ which is orthogonal basis.

Let us consider n. An orthogonal basis of n is an orthogonal basis subset of $\mathcal{R}^n$.

Let n be a non zero element of $\mathbb{N}$. Observe that every orthogonal basis of n is non empty.

5. Finite Real Unitary Spaces and Finite Real Linear Spaces

Let n be an element of $\mathbb{N}$. Observe that $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is constituted finite sequences. Let n be an element of $\mathbb{N}$. One can check that every element of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is real-valued.

Let n be an element of $\mathbb{N}$, let x, y be vectors of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and let a, b be real-valued functions. One can verify that $x+y$ and $a+b$ can be identified when $x = a$ and $y = b$.

Let n be an element of $\mathbb{N}$, let x be a vector of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, let y be a real-valued function, and let a, b be elements of $\mathbb{R}$. Observe that $a \cdot x$ and $b\, y$ can be identified when $a = b$ and $x = y$.

Let n be an element of $\mathbb{N}$, let x be a vector of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and let a be a real-valued function. Observe that $-x$ and $-a$ can be identified when $x = a$.

Let n be an element of $\mathbb{N}$, let x, y be vectors of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and let a, b be real-valued functions. One can check that $x - y$ and $a - b$ can be identified when $x = a$ and $y = b$. The following three propositions are true:

(35) Let n be an element of $\mathbb{N}$, x, y be elements of $\mathcal{R}^n$, and u, v be points of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $x = u$ and $y = v$, then $\otimes_{\mathcal{E}^n}(\langle u, v \rangle) = |(x, y)|$.

(36) Let n, j be elements of $\mathbb{N}$, F be a finite sequence of elements of the carrier of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, B_2 be a subset of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, v_0 be an element of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and l be a linear combination of B_2. Suppose F is one-to-one and B_2 is $\mathbb{R}$-orthogonal and $\mathrm{rng}\, F$ = the support of l and $v_0 \in B_2$ and $j \in \mathrm{dom}(l\, F)$ and $v_0 = F(j)$. Then $\otimes_{\mathcal{E}^n}(\langle v_0, \sum l\, F \rangle) = \otimes_{\mathcal{E}^n}(\langle v_0, l(F_j) \cdot v_0 \rangle)$.

(37) Let n be an element of $\mathbb{N}$, f be a finite sequence of elements of $\mathcal{R}^n$, and g be a finite sequence of elements of the carrier of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $f = g$, then $\sum f = \sum g$.

Let A be a set. Note that $\mathbb{R}^A_{\mathbb{R}}$ is constituted functions.

Let us consider n. Observe that $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$ is constituted finite sequences.

Let A be a set. One can verify that every element of $\mathbb{R}^A_{\mathbb{R}}$ is real-valued.

Let A be a set, let x, y be vectors of $\mathbb{R}^A_{\mathbb{R}}$, and let a, b be real-valued functions. Observe that $x + y$ and $a + b$ can be identified when $x = a$ and $y = b$.

Let A be a set, let x be a vector of $\mathbb{R}^A_{\mathbb{R}}$, let y be a real-valued function, and let a, b be elements of $\mathbb{R}$. Observe that $a \cdot x$ and $b\, y$ can be identified when $a = b$ and $x = y$.

Let A be a set, let x be a vector of $\mathbb{R}^A_{\mathbb{R}}$, and let a be a real-valued function. One can check that $-x$ and $-a$ can be identified when $x = a$.

Let A be a set, let x, y be vectors of $\mathbb{R}^A_{\mathbb{R}}$, and let a, b be real-valued functions. Observe that $x - y$ and $a - b$ can be identified when $x = a$ and $y = b$.

The following propositions are true:

(38) Let X be a subspace of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, x be an element of $\mathcal{R}^n$, and a be a real number. If $x \in$ the carrier of X, then $a \cdot x \in$ the carrier of X.

(39) Let X be a subspace of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$ and x, y be elements of $\mathcal{R}^n$. Suppose $x \in$ the carrier of X and $y \in$ the carrier of X. Then $x + y \in$ the carrier of X.

(40) Let X be a subspace of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, x, y be elements of $\mathcal{R}^n$, and a, b be real numbers. Suppose $x \in$ the carrier of X and $y \in$ the carrier of X. Then $a \cdot x + b \cdot y \in$ the carrier of X.

(41) For all elements x, y of $\mathcal{R}^n$ and for all points u, v of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$ such that $x = u$ and $y = v$ holds $\otimes_{\mathcal{E}^n}(\langle u, v \rangle) = |(x, y)|$.

(42) Let F be a finite sequence of elements of the carrier of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, B_2 be a subset of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, v_0 be an element of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, and l be a linear combination of B_2. Suppose F is one-to-one and B_2 is $\mathbb{R}$-orthogonal and $\mathrm{rng}\, F =$ the support of l and $v_0 \in B_2$ and $j \in \mathrm{dom}(l\, F)$ and $v_0 = F(j)$. Then $\otimes_{\mathcal{E}^n}(\langle v_0, \sum l\, F \rangle) = \otimes_{\mathcal{E}^n}(\langle v_0, l(F_j) \cdot v_0 \rangle)$.

Let us consider n. Note that every subset of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$ which is $\mathbb{R}$-orthonormal is also linearly independent.

Let n be an element of $\mathbb{N}$. Note that every subset of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ which is $\mathbb{R}$-orthonormal is also linearly independent. Next we state the proposition

(43) Let B_2 be a subset of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, x, y be elements of $\mathcal{R}^n$, and a be a real number. If B_2 is linearly independent and x, $y \in B_2$ and $y = a \cdot x$, then $x = y$.

6. Finite Dimensionality of the Spaces

Let us consider n. One can check that $\mathbb{R}$N-Base n is finite.

The following propositions are true:

(44) card $\mathbb{R}$N-Base $n = n$.

(45) Let f be a finite sequence of elements of $\mathcal{R}^n$ and g be a finite sequence of elements of the carrier of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$. If $f = g$, then $\sum f = \sum g$.

(46) Let x_0 be an element of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$ and B be a subset of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$. If $B =$ $\mathbb{R}$N-Base n, then there exists a linear combination l of B such that $x_0 = \sum l$.

(47) Let n be an element of $\mathbb{N}$, x_0 be an element of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$, and B be a subset of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$. If $B = \mathbb{R}$N-Base n, then there exists a linear combination l of B such that $x_0 = \sum l$.

(48) For every subset B of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$ such that $B = \mathbb{R}$N-Base n holds B is a basis of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$.

Let us consider n. Observe that $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$ is finite dimensional.

We now state several propositions:

(49) $\dim(\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}) = n$.

(50) For every subset B of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$ such that B is a basis of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$ holds $\overline{\overline{B}} = n$.

(51) $\emptyset$ is a basis of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,0}$.

(52) For every element n of $\mathbb{N}$ holds $\mathbb{R}$N-Base n is a basis of $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$.

(53) Every orthogonal basis of n is a basis of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$.

Let n be an element of $\mathbb{N}$. Note that $\langle \mathcal{E}^n, (\cdot|\cdot) \rangle$ is finite dimensional.

We now state two propositions:

(54) For every element n of $\mathbb{N}$ holds $\dim(\langle \mathcal{E}^n, (\cdot|\cdot) \rangle) = n$.

(55) For every orthogonal basis B of n holds $\overline{\overline{B}} = n$.

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Received September 23, 2008

FORMALIZED MATHEMATICS
Vol. **17**, No. **1**, Pages 11–21, **2009**

Several Integrability Formulas of Some Functions, Orthogonal Polynomials and Norm Functions

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Summary. In this article, we give several integrability formulas of some functions including the trigonometric function and the index function [3]. We also give the definitions of the orthogonal polynomial and norm function, and some of their important properties [19].

MML identifier: `INTEGRA9`, version: `7.11.01 4.117.1046`

The terminology and notation used here are introduced in the following articles: [10], [21], [17], [6], [20], [1], [9], [13], [2], [4], [18], [15], [5], [8], [11], [14], [12], [16], and [7].

For simplicity, we use the following convention: r, p, x denote real numbers, n denotes an element of $\mathbb{N}$, A denotes a closed-interval subset of $\mathbb{R}$, f, g denote partial functions from $\mathbb{R}$ to $\mathbb{R}$, and Z denotes an open subset of $\mathbb{R}$.

We now state a number of propositions:

(1) $-(\text{the function exp}) \cdot ((-1)\square{+}0)$ is differentiable on $\mathbb{R}$ and for every x holds $(-(\text{the function exp}) \cdot ((-1)\square{+}0))'_{\restriction \mathbb{R}}(x) = \exp(-x)$.

ISSN 1426–2630(p), 1898-9934(e)

(2) $\int\limits_A ((\text{the function exp}) \cdot ((-1)\square{+}0))(x)dx = -\exp(-\sup A)+\exp(-\inf A)$.

(3) $\frac{1}{2}$ ((the function exp) $\cdot(2\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{2}$ ((the function exp) $\cdot(2\square{+}0)))'_{\restriction\mathbb{R}}(x) = \exp(2 \cdot x)$.

(4) $\int\limits_A ((\text{the function exp}) \cdot (2\square{+}0))(x)dx = \frac{1}{2}\cdot\exp(2\cdot\sup A)-\frac{1}{2}\cdot\exp(2\cdot\inf A)$.

(5) Suppose $r \neq 0$. Then $\frac{1}{r}$ ((the function exp) $\cdot(r\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{r}$ ((the function exp) $\cdot(r\square{+}0)))'_{\restriction\mathbb{R}}(x) = \exp(r\cdot x)$.

(6) If $r \neq 0$, then $\int\limits_A ((\text{the function exp}) \cdot (r\square{+}0))(x)dx = \frac{1}{r}\cdot\exp(r\cdot\sup A) - \frac{1}{r}\cdot\exp(r\cdot\inf A)$.

(7) $\int\limits_A ((\text{the function sin}) \cdot (2\square{+}0))(x)dx = (-\frac{1}{2})\cdot\cos(2\cdot\sup A)-(-\frac{1}{2})\cdot\cos(2\cdot \inf A)$.

(8) Suppose $n \neq 0$. Then $(-\frac{1}{n})$ ((the function cos) $\cdot(n\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $((-\frac{1}{n})$ ((the function cos) $\cdot(n\square{+}0)))'_{\restriction\mathbb{R}}(x) = \sin(n \cdot x)$.

(9) If $n \neq 0$, then $\int\limits_A ((\text{the function sin}) \cdot (n\square{+}0))(x)dx = (-\frac{1}{n}) \cdot \cos(n \cdot \sup A) - (-\frac{1}{n}) \cdot \cos(n \cdot \inf A)$.

(10) $\frac{1}{2}$ ((the function sin) $\cdot(2\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{2}$ ((the function sin) $\cdot(2\square{+}0)))'_{\restriction\mathbb{R}}(x) = \cos(2 \cdot x)$.

(11) $\int\limits_A ((\text{the function cos}) \cdot (2\square{+}0))(x)dx = \frac{1}{2}\cdot\sin(2\cdot\sup A)-\frac{1}{2}\cdot\sin(2\cdot\inf A)$.

(12) Suppose $n \neq 0$. Then $\frac{1}{n}$ ((the function sin) $\cdot(n\square{+}0)$) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{n}$ ((the function sin) $\cdot(n\square{+}0)))'_{\restriction\mathbb{R}}(x) = \cos(n\cdot x)$.

(13) If $n \neq 0$, then $\int\limits_A ((\text{the function cos}) \cdot (n\square{+}0))(x)dx = \frac{1}{n}\cdot\sin(n\cdot\sup A) - \frac{1}{n}\cdot\sin(n\cdot\inf A)$.

(14) If $A \subseteq Z$, then $\int\limits_A (\mathrm{id}_Z\ (\text{the function sin}))(x)dx = ((-\sup A)\cdot\cos\sup A + \sin\sup A) - ((-\inf A)\cdot\cos\inf A + \sin\inf A)$.

(15) If $A \subseteq Z$, then $\int\limits_A (\mathrm{id}_Z\ (\text{the function cos}))(x)dx = (\sup A \cdot \sin\sup A + \cos\sup A) - (\inf A \cdot \sin\inf A + \cos\inf A)$.

(16) id_Z (the function cos) is differentiable on Z and for every x such that $x \in Z$ holds $(\mathrm{id}_Z$ (the function cos)$)'_{\restriction Z}(x) = \cos x - x \cdot \sin x$.

(17)(i) $-$the function $\sin + \mathrm{id}_Z$ (the function cos) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($-$the function $\sin + \mathrm{id}_Z$ (the function cos)$)'_{\restriction Z}(x) = -x \cdot \sin x$.

(18) If $A \subseteq Z$, then $\int\limits_A ((-\mathrm{id}_Z)$ (the function sin)$)(x)dx = (-\sin \sup A + \sup A \cdot \cos \sup A) - (-\sin \inf A + \inf A \cdot \cos \inf A)$.

(19)(i) $-$the function $\cos - \mathrm{id}_Z$ (the function sin) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($-$the function $\cos - \mathrm{id}_Z$ (the function sin)$)'_{\restriction Z}(x) = -x \cdot \cos x$.

(20) If $A \subseteq Z$, then $\int\limits_A ((-\mathrm{id}_Z)$ (the function cos)$)(x)dx = -\cos \sup A - \sup A \cdot \sin \sup A - (-\cos \inf A - \inf A \cdot \sin \inf A)$.

(21) If $A \subseteq Z$, then $\int\limits_A$((the function sin) $+ \mathrm{id}_Z$ (the function cos)$)(x)dx = \sup A \cdot \sin \sup A - \inf A \cdot \sin \inf A$.

(22) If $A \subseteq Z$, then $\int\limits_A$($-$the function $\cos + \mathrm{id}_Z$ (the function sin)$)(x)dx = (-\sup A) \cdot \cos \sup A - (-\inf A) \cdot \cos \inf A$.

(23) $\int\limits_A ((1\square{+}0)$ (the function exp)$)(x)dx = \exp(\sup A - 1) - \exp(\inf A - 1)$.

(24) $\frac{1}{n+1}\,(\square^{n+1})$ is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{n+1}\,(\square^{n+1}))'_{\restriction \mathbb{R}}(x) = x^n$.

(25) $\int\limits_A (\square^n)(x)dx = \frac{1}{n+1} \cdot (\sup A)^{n+1} - \frac{1}{n+1} \cdot (\inf A)^{n+1}$.

(26) For all partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $(f - g) \restriction C = f \restriction C - g \restriction C$.

(27) For all partial functions f_1, f_2, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $((f_1 + f_2) \restriction C)\,(g \restriction C) = (f_1\,g + f_2\,g) \restriction C$.

(28) For all partial functions f_1, f_2, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $((f_1 - f_2) \restriction C)\,(g \restriction C) = (f_1\,g - f_2\,g) \restriction C$.

(29) For all partial functions f_1, f_2, g from $\mathbb{R}$ to $\mathbb{R}$ and for every non empty subset C of $\mathbb{R}$ holds $((f_1\,f_2) \restriction C)\,(g \restriction C) = (f_1 \restriction C)\,((f_2\,g) \restriction C)$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$. The functor $\langle f, g\rangle_A$ yielding a real number is defined by:

(Def. 1) $\langle f, g\rangle_A = \int\limits_A (f\,g)(x)dx$.

The following propositions are true:

(30) For all partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$ and for every closed-interval subset A of $\mathbb{R}$ holds $\langle f, g\rangle_A = \langle g, f\rangle_A$.

(31) Let f_1, f_2, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that

(i) $(f_1\, g) \restriction A$ is total,

(ii) $(f_2\, g) \restriction A$ is total,

(iii) $(f_1\, g) \restriction A$ is bounded,

(iv) $f_1\, g$ is integrable on A,

(v) $(f_2\, g) \restriction A$ is bounded, and

(vi) $f_2\, g$ is integrable on A.

Then $\langle f_1 + f_2, g\rangle_A = \langle (f_1), g\rangle_A + \langle (f_2), g\rangle_A$.

(32) Let f_1, f_2, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that

(i) $(f_1\, g) \restriction A$ is total,

(ii) $(f_2\, g) \restriction A$ is total,

(iii) $(f_1\, g) \restriction A$ is bounded,

(iv) $f_1\, g$ is integrable on A,

(v) $(f_2\, g) \restriction A$ is bounded, and

(vi) $f_2\, g$ is integrable on A.

Then $\langle f_1 - f_2, g\rangle_A = \langle (f_1), g\rangle_A - \langle (f_2), g\rangle_A$.

(33) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\, g)\restriction A$ is bounded and $f\, g$ is integrable on A and $A \subseteq \mathrm{dom}(f\, g)$. Then $\langle -f, g\rangle_A = -\langle f, g\rangle_A$.

(34) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\, g)\restriction A$ is bounded and $f\, g$ is integrable on A and $A \subseteq \mathrm{dom}(f\, g)$. Then $\langle r\, f, g\rangle_A = r \cdot \langle f, g\rangle_A$.

(35) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\, g)\restriction A$ is bounded and $f\, g$ is integrable on A and $A \subseteq \mathrm{dom}(f\, g)$. Then $\langle r\, f, p\, g\rangle_A = r \cdot p \cdot \langle f, g\rangle_A$.

(36) For all partial functions f, g, h from $\mathbb{R}$ to $\mathbb{R}$ and for every closed-interval subset A of $\mathbb{R}$ holds $\langle f\, g, h\rangle_A = \langle f, g\, h\rangle_A$.

(37) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that $(f\, f) \restriction A$ is total and $(f\, g) \restriction A$ is total and $(g\, g) \restriction A$ is total and $(f\, f) \restriction A$ is bounded and $(f\, g) \restriction A$ is bounded and $(g\, g) \restriction A$ is bounded and $f\, f$ is integrable on A and $f\, g$ is integrable on A and $g\, g$ is integrable on A. Then $\langle f + g, f + g\rangle_A = \langle f, f\rangle_A + 2 \cdot \langle f, g\rangle_A + \langle g, g\rangle_A$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$. We say that f is orthogonal with g in A if and only if:

(Def. 2) $\langle f, g\rangle_A = 0$.

The following propositions are true:

(38) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that $(f\,f) \upharpoonright A$ is total and $(f\,g) \upharpoonright A$ is total and $(g\,g) \upharpoonright A$ is total and $(f\,f) \upharpoonright A$ is bounded and $(f\,g) \upharpoonright A$ is bounded and $(g\,g) \upharpoonright A$ is bounded and $f\,f$ is integrable on A and $f\,g$ is integrable on A and $g\,g$ is integrable on A and f is orthogonal with g in A. Then $\langle f+g, f+g\rangle_A = \langle f, f\rangle_A + \langle g, g\rangle_A$.

(39) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\,f) \upharpoonright A$ is total and $(f\,f) \upharpoonright A$ is bounded and $f\,f$ is integrable on A and for every x such that $x \in A$ holds $((f\,f) \upharpoonright A)(x) \geq 0$. Then $\langle f, f\rangle_A \geq 0$.

(40) The function sin is orthogonal with the function cos in $[0, \pi]$.

(41) The function sin is orthogonal with the function cos in $[0, \pi \cdot 2]$.

(42) The function sin is orthogonal with the function cos in $[2 \cdot n \cdot \pi, (2 \cdot n+1) \cdot \pi]$.

(43) The function sin is orthogonal with the function cos in $[x + 2 \cdot n \cdot \pi, x + (2 \cdot n + 1) \cdot \pi]$.

(44) The function sin is orthogonal with the function cos in $[-\pi, \pi]$.

(45) The function sin is orthogonal with the function cos in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(46) The function sin is orthogonal with the function cos in $[-2 \cdot \pi, 2 \cdot \pi]$.

(47) The function sin is orthogonal with the function cos in $[-2 \cdot n \cdot \pi, 2 \cdot n \cdot \pi]$.

(48) The function sin is orthogonal with the function cos in $[x - 2 \cdot n \cdot \pi, x + 2 \cdot n \cdot \pi]$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $||f||_A$ yields a real number and is defined by:

(Def. 3) $||f||_A = \sqrt{\langle f, f\rangle_A}$.

Next we state three propositions:

(49) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose $(f\,f) \upharpoonright A$ is total and $(f\,f) \upharpoonright A$ is bounded and $f\,f$ is integrable on A and for every x such that $x \in A$ holds $((f\,f) \upharpoonright A)(x) \geq 0$. Then $0 \leq ||f||_A$.

(50) For every partial function f from $\mathbb{R}$ to $\mathbb{R}$ and for every closed-interval subset A of $\mathbb{R}$ holds $||1\,f||_A = ||f||_A$.

(51) Let f, g be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and A be a closed-interval subset of $\mathbb{R}$. Suppose that $(f\,f) \upharpoonright A$ is total and $(f\,g) \upharpoonright A$ is total and $(g\,g) \upharpoonright A$ is total and $(f\,f) \upharpoonright A$ is bounded and $(f\,g) \upharpoonright A$ is bounded and $(g\,g) \upharpoonright A$ is bounded and $f\,f$ is integrable on A and $f\,g$ is integrable on A and $g\,g$ is integrable on A and f is orthogonal with g in A and for every x such that $x \in A$ holds $((f\,f) \upharpoonright A)(x) \geq 0$ and for every x such that $x \in A$ holds $((g\,g) \upharpoonright A)(x) \geq 0$. Then $(||f+g||_A)^{\mathbf{2}} = (||f||_A)^{\mathbf{2}} + (||g||_A)^{\mathbf{2}}$.

For simplicity, we follow the rules: a, b, x are real numbers, n is an element of $\mathbb{N}$, A is a closed-interval subset of $\mathbb{R}$, f, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$, and Z is an open subset of $\mathbb{R}$.

Next we state several propositions:

(52) If $-a \notin A$, then $\frac{1}{1\Box + a} \restriction A$ is continuous.

(53) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) \neq 0$,

(iii) $Z = \operatorname{dom} f$,

(iv) $\operatorname{dom} f = \operatorname{dom} f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = -\frac{1}{(a+x)^{\mathbf{2}}}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = f(\sup A)^{-1} - f(\inf A)^{-1}$.

(54) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) \neq 0$,

(iii) $\operatorname{dom}((-1)\,\frac{1}{f}) = Z$,

(iv) $\operatorname{dom}((-1)\,\frac{1}{f}) = \operatorname{dom} f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{(a+x)^{\mathbf{2}}}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = -f(\sup A)^{-1} + f(\inf A)^{-1}$.

(55) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) \neq 0$,

(iii) $\operatorname{dom} f = Z$,

(iv) $\operatorname{dom} f = \operatorname{dom} f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{(a-x)^{\mathbf{2}}}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = f(\sup A)^{-1} - f(\inf A)^{-1}$.

(56) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$,

(iii) $\operatorname{dom}((\text{the function ln}) \cdot f) = Z$,

(iv) $\operatorname{dom}((\text{the function ln}) \cdot f) = \operatorname{dom} f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{a+x}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \ln(a + \sup A) - \ln(a + \inf A)$.

Next we state a number of propositions:

(57) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = x - a$ and $f(x) > 0$,

(iii) $\operatorname{dom}((\text{the function } \ln) \cdot f) = Z$,

(iv) $\operatorname{dom}((\text{the function } \ln) \cdot f) = \operatorname{dom} f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{x-a}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \ln f(\sup A) - \ln f(\inf A)$.

(58) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$,

(iii) $\operatorname{dom}(-(\text{the function } \ln) \cdot f) = Z$,

(iv) $\operatorname{dom}(-(\text{the function } \ln) \cdot f) = \operatorname{dom} f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{a-x}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = -\ln(a - \sup A) + \ln(a - \inf A)$.

(59) Suppose that $A \subseteq Z$ and $f = (\text{the function } \ln) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_1(x) > 0$ and $\operatorname{dom}(\mathrm{id}_Z - a\, f) = Z = \operatorname{dom} f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x}{a+x}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = \sup A - a \cdot f(\sup A) - (\inf A - a \cdot f(\inf A))$.

(60) Suppose that $A \subseteq Z$ and $f = (\text{the function } \ln) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_1(x) > 0$ and $\operatorname{dom}((2 \cdot a)\, f - \mathrm{id}_Z) = Z = \operatorname{dom} f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{a-x}{a+x}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = 2 \cdot a \cdot f(\sup A) - \sup A - (2 \cdot a \cdot f(\inf A) - \inf A)$.

(61) Suppose that $A \subseteq Z$ and $f = (\text{the function } \ln) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + a$ and $f_1(x) > 0$ and $\operatorname{dom}(\mathrm{id}_Z - (2 \cdot a)\, f) = Z = \operatorname{dom} f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x-a}{x+a}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = \sup A - 2 \cdot a \cdot f(\sup A) - (\inf A - 2 \cdot a \cdot f(\inf A))$.

(62) Suppose that $A \subseteq Z$ and $f = (\text{the function } \ln) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_1(x) > 0$ and $\operatorname{dom}(\mathrm{id}_Z + (2 \cdot a)\, f) = Z = \operatorname{dom} f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x+a}{x-a}$ and $f_2 \restriction A$

is continuous. Then $\int\limits_A f_2(x)dx = (\sup A + 2 \cdot a \cdot f(\sup A)) - (\inf A + 2 \cdot a \cdot f(\inf A))$.

(63) Suppose that $A \subseteq Z$ and $f =$ (the function ln) $\cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + b$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z + (a - b)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x+a}{x+b}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = (\sup A + (a - b) \cdot f(\sup A)) - (\inf A + (a - b) \cdot f(\inf A))$.

(64) Suppose that $A \subseteq Z$ and $f =$ (the function ln) $\cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - b$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z + (a + b)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x+a}{x-b}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = (\sup A + (a + b) \cdot f(\sup A)) - (\inf A + (a + b) \cdot f(\inf A))$.

(65) Suppose that $A \subseteq Z$ and $f =$ (the function ln) $\cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x + b$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z - (a + b)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x-a}{x+b}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = \sup A - (a + b) \cdot f(\sup A) - (\inf A - (a + b) \cdot f(\inf A))$.

(66) Suppose that $A \subseteq Z$ and $f =$ (the function ln) $\cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x - b$ and $f_1(x) > 0$ and $\mathrm{dom}(\mathrm{id}_Z + (b - a)\, f) = Z = \mathrm{dom}\, f_2$ and for every x such that $x \in Z$ holds $f_2(x) = \frac{x-a}{x-b}$ and $f_2 \restriction A$ is continuous. Then $\int\limits_A f_2(x)dx = (\sup A + (b - a) \cdot f(\sup A)) - (\inf A + (b - a) \cdot f(\inf A))$.

(67) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) > 0$,
- (iii) dom((the function ln) $\cdot f) = Z$,
- (iv) dom((the function ln) $\cdot f) = \mathrm{dom}\, f_2$,
- (v) for every x such that $x \in Z$ holds $f_2(x) = \frac{1}{x}$, and
- (vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \ln \sup A - \ln \inf A$.

(68) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $x > 0$,
- (iii) dom((the function ln) $\cdot (\square^n)) = Z$,

(iv) dom((the function ln) $\cdot(\square^n)$) = dom f_2,
(v) for every x such that $x \in Z$ holds $f_2(x) = \frac{n}{x}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \ln((\sup A)^n) - \ln((\inf A)^n)$.

(69) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = x$,
(iii) dom((the function ln) $\cdot\frac{1}{f}$) = Z,
(iv) dom((the function ln) $\cdot\frac{1}{f}$) = dom f_2,
(v) for every x such that $x \in Z$ holds $f_2(x) = -\frac{1}{x}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = -\ln \sup A + \ln \inf A$.

(70) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$,
(iii) dom($\frac{2}{3}\, f^{\frac{3}{2}}$) = Z,
(iv) dom($\frac{2}{3}\, f^{\frac{3}{2}}$) = dom f_2,
(v) for every x such that $x \in Z$ holds $f_2(x) = (a + x)^{\frac{1}{2}}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = \frac{2}{3} \cdot (a + \sup A)^{\frac{3}{2}} - \frac{2}{3} \cdot (a + \inf A)^{\frac{3}{2}}$.

(71) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$,
(iii) dom($(-\frac{2}{3})\, f^{\frac{3}{2}}$) = Z,
(iv) dom($(-\frac{2}{3})\, f^{\frac{3}{2}}$) = dom f_2,
(v) for every x such that $x \in Z$ holds $f_2(x) = (a - x)^{\frac{1}{2}}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = -\frac{2}{3} \cdot (a - \sup A)^{\frac{3}{2}} + \frac{2}{3} \cdot (a - \inf A)^{\frac{3}{2}}$.

(72) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = a + x$ and $f(x) > 0$,
(iii) dom($2\, f^{\frac{1}{2}}$) = Z,
(iv) dom($2\, f^{\frac{1}{2}}$) = dom f_2,
(v) for every x such that $x \in Z$ holds $f_2(x) = (a + x)^{-\frac{1}{2}}$, and
(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = 2\cdot(a+\sup A)^{\frac{1}{2}} - 2\cdot(a+\inf A)^{\frac{1}{2}}$.

(73) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f(x) = a - x$ and $f(x) > 0$,

(iii) $\mathrm{dom}((-2)\, f^{\frac{1}{2}}) = Z$,

(iv) $\mathrm{dom}((-2)\, f^{\frac{1}{2}}) = \mathrm{dom}\, f_2$,

(v) for every x such that $x \in Z$ holds $f_2(x) = (a-x)^{-\frac{1}{2}}$, and

(vi) $f_2 \restriction A$ is continuous.

Then $\int\limits_A f_2(x)dx = -2\cdot(a-\sup A)^{\frac{1}{2}} + 2\cdot(a-\inf A)^{\frac{1}{2}}$.

(74) Suppose that

(i) $A \subseteq Z$,

(ii) $\mathrm{dom}((-\mathrm{id}_Z)$ (the function cos)+the function sin$) = Z$,

(iii) for every x such that $x \in Z$ holds $f(x) = x \cdot \sin x$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (-\sup A \cdot \cos \sup A + \sin \sup A) - (-\inf A \cdot \cos \inf A + \sin \inf A)$.

(75) Suppose $A \subseteq Z$ and dom (the function sec) $= Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{\sin x}{(\cos x)^{\mathbf{2}}}$ and $Z = \mathrm{dom}\, f$ and $f \restriction A$ is continuous. Then $\int\limits_A f(x)dx = \sec \sup A - \sec \inf A$.

(76) Suppose $Z \subseteq$ dom($-$the function cosec). Then $-$the function cosec is differentiable on Z and for every x such that $x \in Z$ holds ($-$the function cosec$)'_{\restriction Z}(x) = \frac{\cos x}{(\sin x)^{\mathbf{2}}}$.

(77) Suppose $A \subseteq Z$ and dom($-$the function cosec) $= Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{\cos x}{(\sin x)^{\mathbf{2}}}$ and $Z = \mathrm{dom}\, f$ and $f \restriction A$ is continuous. Then $\int\limits_A f(x)dx = -\mathrm{cosec} \sup A + \mathrm{cosec} \inf A$.

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Received October 14, 2008

FORMALIZED MATHEMATICS
Vol. **17**, No. **1**, Pages 23–35, **2009**

Several Integrability Formulas of Special Functions. Part II

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Summary. In this article, we give several differentiation and integrability formulas of special and composite functions including the trigonometric function, the hyperbolic function and the polynomial function [3].

MML identifier: `INTEGR11`, version: `7.11.01 4.117.1046`

The articles [10], [23], [19], [21], [22], [1], [8], [15], [9], [2], [4], [17], [5], [13], [16], [14], [18], [7], [12], [20], [6], and [11] provide the terminology and notation for this paper.

1. Differentiation Formulas

For simplicity, we adopt the following rules: r, x, a, b denote real numbers, n, m denote elements of $\mathbb{N}$, A denotes a closed-interval subset of $\mathbb{R}$, and Z denotes an open subset of $\mathbb{R}$.

One can prove the following propositions:

(1)(i) $(\frac{1}{2}\square+0) - \frac{1}{4}\,((\text{the function sin})\cdot(2\square+0))$ is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((\frac{1}{2}\square+0) - \frac{1}{4}\,((\text{the function sin})\cdot(2\square+0)))'_{\restriction\mathbb{R}}(x) = (\sin x)^{\mathbf{2}}$.

ISSN 1426–2630(p), 1898-9934(e)

(2)(i) $(\frac{1}{2}\square+0) + \frac{1}{4}$ ((the function sin) $\cdot(2\square+0)$) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((\frac{1}{2}\square+0) + \frac{1}{4}$ ((the function sin) $\cdot(2\square+0)))'_{\restriction\mathbb{R}}(x) = (\cos x)^{\mathbf{2}}$.

(3) $\frac{1}{n+1}$ $((\square^{n+1}) \cdot$ (the function sin)) is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{n+1}$ (the function sin$)^{n+1})'_{\restriction\mathbb{R}}(x) = (\sin x)^n \cdot \cos x$.

(4)(i) $(-\frac{1}{n+1})$ $((\square^{n+1}) \cdot$ (the function cos)) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((-\frac{1}{n+1})$ (the function cos$)^{n+1})'_{\restriction\mathbb{R}}(x) = (\cos x)^n \cdot \sin x$.

(5) Suppose $m + n \neq 0$ and $m - n \neq 0$. Then

(i) $\frac{1}{2\cdot(m+n)}$ ((the function sin) $\cdot((m+n)\square+0)) + \frac{1}{2\cdot(m-n)}$ ((the function sin) $\cdot((m-n)\square+0)$) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(\frac{1}{2\cdot(m+n)}$ ((the function sin) $\cdot((m+n)\square+0)) + \frac{1}{2\cdot(m-n)}$ ((the function sin) $\cdot((m-n)\square+0)))'_{\restriction\mathbb{R}}(x) = \cos(m \cdot x) \cdot \cos(n \cdot x)$.

(6) Suppose $m + n \neq 0$ and $m - n \neq 0$. Then

(i) $\frac{1}{2\cdot(m-n)}$ ((the function sin) $\cdot((m-n)\square+0)) - \frac{1}{2\cdot(m+n)}$ ((the function sin) $\cdot((m+n)\square+0)$) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(\frac{1}{2\cdot(m-n)}$ ((the function sin) $\cdot((m-n)\square+0)) - \frac{1}{2\cdot(m+n)}$ ((the function sin) $\cdot((m+n)\square+0)))'_{\restriction\mathbb{R}}(x) = \sin(m \cdot x) \cdot \sin(n \cdot x)$.

(7) Suppose $m + n \neq 0$ and $m - n \neq 0$. Then

(i) $-\frac{1}{2\cdot(m+n)}$ ((the function cos) $\cdot ((m+n)\square+0)) - \frac{1}{2\cdot(m-n)}$ ((the function cos) $\cdot((m-n)\square+0)$) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(-\frac{1}{2\cdot(m+n)}$ ((the function cos) $\cdot ((m+n)\square+0)) - \frac{1}{2\cdot(m-n)}$ ((the function cos) $\cdot((m-n)\square+0)))'_{\restriction\mathbb{R}}(x) = \sin(m \cdot x) \cdot \cos(n \cdot x)$.

(8) Suppose $n \neq 0$. Then

(i) $\frac{1}{n^{\mathbf{2}}}$ ((the function sin) $\cdot(n\square+0)) - (\frac{1}{n}\square+0)$ ((the function cos) $\cdot(n\square+0)$) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(\frac{1}{n^{\mathbf{2}}}$ ((the function sin) $\cdot(n\square+0)) - (\frac{1}{n}\square+0)$ ((the function cos) $\cdot(n\square+0)))'_{\restriction\mathbb{R}}(x) = x \cdot \sin(n \cdot x)$.

(9) Suppose $n \neq 0$. Then

(i) $\frac{1}{n^{\mathbf{2}}}$ ((the function cos) $\cdot(n\square+0)) + (\frac{1}{n}\square+0)$ ((the function sin) $\cdot(n\square+0)$) is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(\frac{1}{n^{\mathbf{2}}}$ ((the function cos) $\cdot(n\square+0)) + (\frac{1}{n}\square+0)$ ((the function sin) $\cdot(n\square+0)))'_{\restriction\mathbb{R}}(x) = x \cdot \cos(n \cdot x)$.

(10)(i) $(1\square+0)$ (the function cosh)$-$the function sinh is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((1\square+0)$ (the function cosh)$-$the function sinh$)'_{\restriction\mathbb{R}}(x) = x \cdot \sinh x$.

(11)(i) $(1\square+0)$ (the function sinh)$-$the function cosh is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((1\square+0)$ (the function sinh)$-$the function cosh$)'_{\restriction\mathbb{R}}(x) = x \cdot \cosh x$.

(12) If $a \cdot (n+1) \neq 0$, then $\frac{1}{a\cdot(n+1)}\ (a\square+b)^{n+1}$ is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{a\cdot(n+1)}\ (a\square+b)^{n+1})'_{\restriction\mathbb{R}}(x) = (a \cdot x + b)^n$.

2. Integrability Formulas

Next we state a number of propositions:

(13) $\int\limits_{A}(\text{the function sin})^{\mathbf{2}}(x)dx = \frac{1}{2} \cdot \sup A - \frac{1}{4} \cdot \sin(2 \cdot \sup A) - (\frac{1}{2} \cdot \inf A - \frac{1}{4} \cdot \sin(2 \cdot \inf A))$.

(14) $\int\limits_{[0,\pi]}(\text{the function sin})^{\mathbf{2}}(x)dx = \frac{\pi}{2}$.

(15) $\int\limits_{[0,2\cdot\pi]}(\text{the function sin})^{\mathbf{2}}(x)dx = \pi$.

(16) $\int\limits_{A}(\text{the function cos})^{\mathbf{2}}(x)dx = (\frac{1}{2} \cdot \sup A + \frac{1}{4} \cdot \sin(2 \cdot \sup A)) - (\frac{1}{2} \cdot \inf A + \frac{1}{4} \cdot \sin(2 \cdot \inf A))$.

(17) $\int\limits_{[0,\pi]}(\text{the function cos})^{\mathbf{2}}(x)dx = \frac{\pi}{2}$.

(18) $\int\limits_{[0,2\cdot\pi]}(\text{the function cos})^{\mathbf{2}}(x)dx = \pi$.

(19) $\int\limits_{A}((\text{the function sin})^{n}\ (\text{the function cos}))(x)dx = \frac{1}{n+1} \cdot (\sin \sup A)^{n+1} - \frac{1}{n+1} \cdot (\sin \inf A)^{n+1}$.

(20) $\int\limits_{[0,\pi]}((\text{the function sin})^{n}\ (\text{the function cos}))(x)dx = 0$.

(21) $\int\limits_{[0,2\cdot\pi]}((\text{the function sin})^{n}\ (\text{the function cos}))(x)dx = 0$.

(22) $\int\limits_{A}((\text{the function cos})^{n}\ (\text{the function sin}))(x)dx = (-\frac{1}{n+1}) \cdot (\cos \sup A)^{n+1} - (-\frac{1}{n+1}) \cdot (\cos \inf A)^{n+1}$.

(23) $\int_{[0,2\cdot\pi]} ((\text{the function cos})^n\,(\text{the function sin}))(x)dx = 0.$

(24) $\int_{[-\frac{\pi}{2},\frac{\pi}{2}]} ((\text{the function cos})^n\,(\text{the function sin}))(x)dx = 0.$

(25) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then $\int_A (((\text{the function cos})\cdot(m\square{+}0))\,((\text{the function cos})\cdot(n\square{+}0)))(x)dx = (\frac{1}{2\cdot(m+n)}\cdot\sin((m+n)\cdot\sup A)+\frac{1}{2\cdot(m-n)}\cdot\sin((m-n)\cdot\sup A)) - (\frac{1}{2\cdot(m+n)}\cdot\sin((m+n)\cdot\inf A)+\frac{1}{2\cdot(m-n)}\cdot\sin((m-n)\cdot\inf A)).$

(26) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then $\int_A (((\text{the function sin})\cdot(m\square{+}0))\,((\text{the function sin})\cdot(n\square{+}0)))(x)dx = \frac{1}{2\cdot(m-n)}\cdot\sin((m-n)\cdot\sup A)-\frac{1}{2\cdot(m+n)}\cdot\sin((m+n)\cdot\sup A) - (\frac{1}{2\cdot(m-n)}\cdot\sin((m-n)\cdot\inf A)-\frac{1}{2\cdot(m+n)}\cdot\sin((m+n)\cdot\inf A)).$

(27) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then $\int_A (((\text{the function sin})\cdot(m\square{+}0))\,((\text{the function cos})\cdot(n\square{+}0)))(x)dx = -\frac{1}{2\cdot(m+n)}\cdot\cos((m+n)\cdot\sup A)-\frac{1}{2\cdot(m-n)}\cdot\cos((m-n)\cdot\sup A) - (-\frac{1}{2\cdot(m+n)}\cdot\cos((m+n)\cdot\inf A)-\frac{1}{2\cdot(m-n)}\cdot\cos((m-n)\cdot\inf A)).$

(28) If $n \neq 0$, then $\int_A ((1\square{+}0)\,((\text{the function sin})\cdot(n\square{+}0)))(x)dx = \frac{1}{n^{\mathbf{2}}}\cdot\sin(n\cdot\sup A)-\frac{1}{n}\cdot\sup A\cdot\cos(n\cdot\sup A)-(\frac{1}{n^{\mathbf{2}}}\cdot\sin(n\cdot\inf A)-\frac{1}{n}\cdot\inf A\cdot\cos(n\cdot\inf A)).$

(29) If $n \neq 0$, then $\int_A ((1\square{+}0)\,((\text{the function cos})\cdot(n\square{+}0)))(x)dx = (\frac{1}{n^{\mathbf{2}}}\cdot\cos(n\cdot\sup A)+\frac{1}{n}\cdot\sup A\cdot\sin(n\cdot\sup A))-(\frac{1}{n^{\mathbf{2}}}\cdot\cos(n\cdot\inf A)+\frac{1}{n}\cdot\inf A\cdot\sin(n\cdot\inf A)).$

(30) $\int_A ((1\square{+}0)\,(\text{the function sinh}))(x)dx = \sup A\cdot\cosh\sup A-\sinh\sup A-(\inf A\cdot\cosh\inf A-\sinh\inf A).$

(31) $\int_A ((1\square{+}0)\,(\text{the function cosh}))(x)dx = \sup A\cdot\sinh\sup A-\cosh\sup A-(\inf A\cdot\sinh\inf A-\cosh\inf A).$

(32) If $a\cdot(n+1)\neq 0$, then $\int\limits_{A}(a\square+b)^n(x)dx = \frac{1}{a\cdot(n+1)}\cdot(a\cdot\sup A+b)^{n+1} - \frac{1}{a\cdot(n+1)}\cdot(a\cdot\inf A+b)^{n+1}$.

3. Addenda

In the sequel f, f_1, f_2, f_3, g are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

The following propositions are true:

(33) If $Z\subseteq \operatorname{dom}(\frac{1}{2}\,f)$ and $f=\square^2$, then $\frac{1}{2}\,f$ is differentiable on Z and for every x such that $x\in Z$ holds $(\frac{1}{2}\,f)'_{\restriction Z}(x)=x$.

(34) If $A\subseteq Z=\operatorname{dom}(\frac{1}{2}\,(\square^2))$, then $\int\limits_{A}\mathrm{id}_Z(x)dx=\frac{1}{2}\cdot(\sup A)^2-\frac{1}{2}\cdot(\inf A)^2$.

(35) Suppose $A\subseteq Z$ and for every x such that $x\in Z$ holds $g(x)=x$ and $g(x)\neq 0$ and $f(x)=-\frac{1}{x^2}$ and $Z=\operatorname{dom} g$ and $\operatorname{dom} f=Z$ and $f\restriction A$ is continuous. Then $\int\limits_{A}f(x)dx=(\sup A)^{-1}-(\inf A)^{-1}$.

(36) Suppose that

(i) $A\subseteq Z$,

(ii) $f_1=\square^2$,

(iii) for every x such that $x\in Z$ holds $f_2(x)=1$ and $x\neq 0$ and $f(x)=\frac{2\cdot x}{(1+x^2)^2}$,

(iv) $\operatorname{dom}(\frac{f_1}{f_2+f_1})=Z$,

(v) $Z=\operatorname{dom} f$, and

(vi) $f\restriction A$ is continuous.

Then $\int\limits_{A}f(x)dx=(\frac{f_1}{f_2+f_1})(\sup A)-(\frac{f_1}{f_2+f_1})(\inf A)$.

(37) Suppose $Z\subseteq$ dom((the function tan)+(the function sec)) and for every x such that $x\in Z$ holds $1+\sin x\neq 0$ and $1-\sin x\neq 0$. Then

(i) (the function tan)+(the function sec) is differentiable on Z, and

(ii) for every x such that $x\in Z$ holds ((the function tan)+(the function sec))$'_{\restriction Z}(x)=\frac{1}{1-\sin x}$.

(38) Suppose that

(i) $A\subseteq Z$,

(ii) for every x such that $x\in Z$ holds $1+\sin x\neq 0$ and $1-\sin x\neq 0$ and $f(x)=\frac{1}{1-\sin x}$,

(iii) dom((the function tan)+(the function sec)) $=Z$,

(iv) $Z=\operatorname{dom} f$, and

(v) $f\restriction A$ is continuous.

Then $\int\limits_{A} f(x)dx = (\tan\sup A + \sec\sup A) - (\tan\inf A + \sec\inf A)$.

(39) Suppose $Z \subseteq$ dom((the function tan)$-$(the function sec)) and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then

(i) (the function tan)$-$(the function sec) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function tan)$-$(the function sec))$'_{\restriction Z}(x) = \frac{1}{1+\sin x}$.

(40) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{1}{1+\sin x}$,

(iii) dom((the function tan)$-$(the function sec)) $= Z$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = \tan\sup A - \sec\sup A - (\tan\inf A - \sec\inf A)$.

(41) Suppose $Z \subseteq$ dom($-$the function cot $+$ the function cosec) and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then

(i) $-$the function cot $+$ the function cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($-$the function cot $+$ the function cosec)$'_{\restriction Z}(x) = \frac{1}{1+\cos x}$.

(42) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{1}{1+\cos x}$,

(iii) dom($-$the function cot $+$ the function cosec) $= Z$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-\cot\sup A + \operatorname{cosec}\sup A) - (-\cot\inf A + \operatorname{cosec}\inf A)$.

(43) Suppose $Z \subseteq$ dom($-$the function cot $-$ the function cosec) and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then

(i) $-$the function cot $-$ the function cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($-$the function cot $-$ the function cosec)$'_{\restriction Z}(x) = \frac{1}{1-\cos x}$.

(44) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{1}{1-\cos x}$,

(iii) dom($-$the function cot $-$ the function cosec) $= Z$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = -\cot \sup A - \operatorname{cosec} \sup A - (-\cot \inf A - \operatorname{cosec} \inf A)$.

(45) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{1+x^2}$,

(iv) dom (the function arctan) $= Z$,

(v) $Z = \operatorname{dom} f$, and

(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = \arctan \sup A - \arctan \inf A$.

(46) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = \frac{r}{1+x^2}$,

(iv) dom(r the function arctan) $= Z$,

(v) $Z = \operatorname{dom} f$, and

(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = r \cdot \arctan \sup A - r \cdot \arctan \inf A$.

(47) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = -\frac{1}{1+x^2}$,

(iv) dom (the function arccot) $= Z$,

(v) $Z = \operatorname{dom} f$, and

(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = \operatorname{arccot} \sup A - \operatorname{arccot} \inf A$.

(48) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = -\frac{r}{1+x^2}$,

(iv) dom(r the function arccot) $= Z$,

(v) $Z = \operatorname{dom} f$, and

(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = r \cdot \operatorname{arccot} \sup A - r \cdot \operatorname{arccot} \inf A$.

(49) Suppose $Z \subseteq \operatorname{dom}((\mathrm{id}_Z + \text{the function cot}) - \text{the function cosec})$ and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then

(i) $(\mathrm{id}_Z +$ the function cot$)-$the function cosec is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\mathrm{id}_Z+$the function cot$)-$the function cosec$)'_{\restriction Z}(x) = \frac{\cos x}{1+\cos x}$.

(50) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{\cos x}{1+\cos x}$,
(iii) $\mathrm{dom}((\mathrm{id}_Z +$ the function cot$)-$the function cosec$) = Z$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (\sup A + \cot \sup A) - \mathrm{cosec} \sup A - ((\inf A + \cot \inf A) - \mathrm{cosec} \inf A)$.

(51) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z +$ the function cot$+$the function cosec$)$ and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then
(i) $\mathrm{id}_Z +$ the function cot$+$the function cosec is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z +$the function cot$+$the function cosec$)'_{\restriction Z}(x) = \frac{\cos x}{\cos x - 1}$.

(52) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{\cos x}{\cos x - 1}$,
(iii) $\mathrm{dom}(\mathrm{id}_Z +$ the function cot$+$the function cosec$) = Z$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = (\sup A + \cot \sup A + \mathrm{cosec} \sup A) - (\inf A + \cot \inf A + \mathrm{cosec} \inf A)$.

(53) Suppose $Z \subseteq \mathrm{dom}((\mathrm{id}_Z -$ the function tan$)+$the function sec$)$ and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then
(i) $(\mathrm{id}_Z -$ the function tan$)+$the function sec is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\mathrm{id}_Z-$the function tan$)+$the function sec$)'_{\restriction Z}(x) = \frac{\sin x}{\sin x + 1}$.

(54) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{\sin x}{1+\sin x}$,
(iii) $Z \subseteq \mathrm{dom}((\mathrm{id}_Z -$ the function tan$)+$the function sec$)$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = ((\sup A - \tan\sup A) + \sec\sup A) - ((\inf A - \tan\inf A) + \sec\inf A)$.

(55) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z -$ the function tan$-$the function sec) and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then

(i) $\mathrm{id}_Z -$ the function tan$-$the function sec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z -$ the function tan$-$the function sec$)'_{\restriction Z}(x) = \frac{\sin x}{\sin x - 1}$.

(56) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{\sin x}{\sin x - 1}$,

(iii) $Z \subseteq \mathrm{dom}(\mathrm{id}_Z -$ the function tan$-$the function sec),

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = \sup A - \tan\sup A - \sec\sup A - (\inf A - \tan\inf A - \sec\inf A)$.

(57) Suppose $Z \subseteq \mathrm{dom}(($the function tan$)-\mathrm{id}_Z)$. Then (the function tan)$-\mathrm{id}_Z$ is differentiable on Z and for every x such that $x \in Z$ holds $(($the function tan$)-\mathrm{id}_Z)'_{\restriction Z}(x) = (\tan x)^{\mathbf{2}}$.

(58) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds (the function cos)$(x) > 0$ and $f(x) = (\tan x)^{\mathbf{2}}$,

(iii) $Z \subseteq \mathrm{dom}(($the function tan$)-\mathrm{id}_Z)$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = \tan\sup A - \sup A - (\tan\inf A - \inf A)$.

(59) Suppose $Z \subseteq \mathrm{dom}(-$the function cot $- \mathrm{id}_Z)$. Then $-$the function cot $- \mathrm{id}_Z$ is differentiable on Z and for every x such that $x \in Z$ holds $(-$the function cot $- \mathrm{id}_Z)'_{\restriction Z}(x) = (\cot x)^{\mathbf{2}}$.

(60) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds (the function sin)$(x) > 0$ and $f(x) = (\cot x)^{\mathbf{2}}$,

(iii) $Z \subseteq \mathrm{dom}(-$the function cot $- \mathrm{id}_Z)$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = -\cot \sup A - \sup A - (-\cot \inf A - \inf A)$.

(61) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\cos x)^{\mathbf{2}}}$ and $\cos x \neq 0$ and dom (the function tan) $= Z = \operatorname{dom} f$ and $f \restriction A$ is continuous. Then $\int\limits_A f(x)dx = \tan \sup A - \tan \inf A$.

(62) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f(x) = -\frac{1}{(\sin x)^{\mathbf{2}}}$ and $\sin x \neq 0$ and dom (the function cot) $= Z = \operatorname{dom} f$ and $f \restriction A$ is continuous. Then $\int\limits_A f(x)dx = \cot \sup A - \cot \inf A$.

(63) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{\sin x - (\cos x)^{\mathbf{2}}}{(\cos x)^{\mathbf{2}}}$ and $Z \subseteq$ dom((the function sec)$-\mathrm{id}_Z$) and $Z = \operatorname{dom} f$ and $f \restriction A$ is continuous. Then $\int\limits_A f(x)dx = \sec \sup A - \sup A - (\sec \inf A - \inf A)$.

(64) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = \frac{\cos x - (\sin x)^{\mathbf{2}}}{(\sin x)^{\mathbf{2}}}$,
- (iii) $Z \subseteq$ dom($-$the function cosec $-\ \mathrm{id}_Z$),
- (iv) $Z = \operatorname{dom} f$, and
- (v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = -\operatorname{cosec} \sup A - \sup A - (-\operatorname{cosec} \inf A - \inf A)$.

The following propositions are true:

(65) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $\sin x > 0$,
- (iii) $Z \subseteq$ dom((the function ln) $\cdot$(the function sin)),
- (iv) $Z =$ dom (the function cot), and
- (v) (the function cot)$\restriction A$ is continuous.

Then $\int\limits_A$ (the function cot)$(x)dx = \ln \sin \sup A - \ln \sin \inf A$.

(66) Suppose that
- (i) $A \subseteq Z$,
- (ii) $Z \subseteq]-1, 1[$,
- (iii) for every x such that $x \in Z$ holds $f(x) = \frac{\arcsin x}{\sqrt{1-x^{\mathbf{2}}}}$,
- (iv) $Z \subseteq$ dom($\frac{1}{2}$ (the function arcsin)^2),
- (v) $Z = \operatorname{dom} f$, and
- (vi) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = \frac{1}{2} \cdot (\arcsin \sup A)^2 - \frac{1}{2} \cdot (\arcsin \inf A)^2$.

(67) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = -\frac{\arccos x}{\sqrt{1-x^2}}$,

(iv) $Z \subseteq \mathrm{dom}(\frac{1}{2}$ (the function arccos$)^2)$,

(v) $Z = \mathrm{dom}\, f$, and

(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = \frac{1}{2} \cdot (\arccos \sup A)^2 - \frac{1}{2} \cdot (\arccos \inf A)^2$.

(68) $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f(x) > 0$ and $x \neq 0$ and dom (the function arcsin) $= Z \subseteq \mathrm{dom}(\mathrm{id}_Z$ (the function arcsin)$+f^{\frac{1}{2}})$.

(69) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = a^2$ and $f(x) > 0$ and $f_3(x) = \frac{x}{a}$ and $-1 < f_3(x) < 1$ and $x \neq 0$ and $a > 0$ and dom((the function arcsin) $\cdot f_3) = Z \subseteq \mathrm{dom}(\mathrm{id}_Z$ ((the function arcsin) $\cdot f_3) + (\square^{\frac{1}{2}}) \cdot f)$ and ((the function arcsin) $\cdot f_3){\restriction}A$ is continuous. Then $\int\limits_A$((the function arcsin) $\cdot f_3)(x)dx =$ $(\sup A \cdot \arcsin(\frac{\sup A}{a}) + f(\sup A)^{\frac{1}{2}}) - (\inf A \cdot \arcsin(\frac{\inf A}{a}) + f(\inf A)^{\frac{1}{2}})$.

(70) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f(x) > 0$ and $x \neq 0$ and dom (the function arccos) $= Z \subseteq \mathrm{dom}(\mathrm{id}_Z$ (the function arccos)$-(\square^{\frac{1}{2}}) \cdot f)$. Then $\int\limits_A$(the function arccos)$(x)dx = \sup A \cdot \arccos \sup A - f(\sup A)^{\frac{1}{2}} -$ $(\inf A \cdot \arccos \inf A - f(\inf A)^{\frac{1}{2}})$.

(71) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = a^2$ and $f(x) > 0$ and $f_3(x) = \frac{x}{a}$ and $-1 < f_3(x) < 1$ and $x \neq 0$ and $a > 0$ and dom((the function arccos) $\cdot f_3) = Z = \mathrm{dom}(\mathrm{id}_Z$ ((the function arccos) $\cdot f_3) - (\square^{\frac{1}{2}}) \cdot f)$ and ((the function arccos) $\cdot f_3){\restriction}A$ is continuous. Then $\int\limits_A$((the function arccos) $\cdot f_3)(x)dx =$ $\sup A \cdot \arccos(\frac{\sup A}{a}) - f(\sup A)^{\frac{1}{2}} - (\inf A \cdot \arccos(\frac{\inf A}{a}) - f(\inf A)^{\frac{1}{2}})$.

(72) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) $f_2 = \square^2$,

(iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,
(v) $Z = \mathrm{dom}$ (the function arctan), and
(vi) $Z = \mathrm{dom}(\mathrm{id}_Z$ the function arctan$-\frac{1}{2}$ ((the function ln) $\cdot(f_1 + f_2)))$.

Then $\int\limits_A$ (the function arctan)$(x)dx = \sup A \cdot \arctan \sup A - \frac{1}{2} \cdot \ln(1 + (\sup A)^2) - (\inf A \cdot \arctan \inf A - \frac{1}{2} \cdot \ln(1 + (\inf A)^2))$.

(73) Suppose that
(i) $A \subseteq Z$,
(ii) $Z \subseteq]-1, 1[$,
(iii) $f_2 = \square^2$,
(iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,
(v) dom (the function arccot) $= Z$, and
(vi) $Z = \mathrm{dom}(\mathrm{id}_Z$ the function arccot$+\frac{1}{2}$ ((the function ln) $\cdot(f_1 + f_2)))$.

Then $\int\limits_A$ (the function arccot)$(x)dx = (\sup A \cdot \mathrm{arccot} \sup A + \frac{1}{2} \cdot \ln(1 + (\sup A)^2)) - (\inf A \cdot \mathrm{arccot} \inf A + \frac{1}{2} \cdot \ln(1 + (\inf A)^2))$.

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Received October 14, 2008

FORMALIZED MATHEMATICS
Vol. **17**, No. **1**, Pages 37–42, **2009**

Cell Petri Net Concepts

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Summary. Based on the Petri net definitions and theorems already formalized in [8], with this article, we developed the concept of "Cell Petri Nets". It is based on [9]. In a cell Petri net we introduce the notions of colors and colored states of a Petri net, connecting mappings for linking two Petri nets, firing rules for transitions, and the synthesis of two or more Petri nets.

MML identifier: `PETRI_2`, version: `7.11.01 4.117.1046`

The papers [11], [12], [6], [13], [14], [10], [8], [2], [5], [3], [4], [7], and [1] provide the terminology and notation for this paper.

1. Preliminaries: Thin Cylinder, Locus

Let A be a non empty set, let B be a set, let B_1 be a set, and let y_1 be a function from B_1 into A. Let us assume that $B_1 \subseteq B$. The functor $\mathrm{cylinder}_0(A, B, B_1, y_1)$ yields a non empty subset of A^B and is defined by:

(Def. 1) $\mathrm{cylinder}_0(A, B, B_1, y_1) = \{y : B \to A: y \restriction B_1 = y_1\}$.

Let A be a non empty set and let B be a set. A non empty subset of A^B is said to be a thin cylinder of A and B if:

(Def. 2) There exists a subset B_1 of B and there exists a function y_1 from B_1 into A such that B_1 is finite and it $= \mathrm{cylinder}_0(A, B, B_1, y_1)$.

The following propositions are true:

ISSN 1426–2630(p), 1898-9934(e)

(1) Let A be a non empty set, B be a set, and D be a thin cylinder of A and B. Then there exists a subset B_1 of B and there exists a function y_1 from B_1 into A such that B_1 is finite and $D = \{y : B \to A\colon y{\restriction}B_1 = y_1\}$.

(2) Let A_1, A_2 be non empty sets, B be a set, and D_1 be a thin cylinder of A_1 and B. If $A_1 \subseteq A_2$, then there exists a thin cylinder D_2 of A_2 and B such that $D_1 \subseteq D_2$.

Let A be a non empty set and let B be a set. The thin cylinders of A and B constitute a non empty family of subsets of A^B defined by:

(Def. 3) The thin cylinders of A and $B = \{D \subseteq A^B\colon D$ is a thin cylinder of A and $B\}$.

We now state three propositions:

(3) Let A be a non trivial set, B be a set, B_2 be a set, y_2 be a function from B_2 into A, B_3 be a set, and y_3 be a function from B_3 into A. If $B_2 \subseteq B$ and $B_3 \subseteq B$ and $\mathrm{cylinder}_0(A, B, B_2, y_2) = \mathrm{cylinder}_0(A, B, B_3, y_3)$, then $B_2 = B_3$ and $y_2 = y_3$.

(4) Let A_1, A_2 be non empty sets and B_4, B_5 be sets. Suppose $A_1 \subseteq A_2$ and $B_4 \subseteq B_5$. Then there exists a function F from the thin cylinders of A_1 and B_4 into the thin cylinders of A_2 and B_5 such that for every set x if $x \in$ the thin cylinders of A_1 and B_4, then there exists a subset B_1 of B_4 and there exists a function y_2 from B_1 into A_1 and there exists a function y_3 from B_1 into A_2 such that B_1 is finite and $y_2 = y_3$ and $x = \mathrm{cylinder}_0(A_1, B_4, B_1, y_2)$ and $F(x) = \mathrm{cylinder}_0(A_2, B_5, B_1, y_3)$.

(5) Let A_1, A_2 be non empty sets and B_4, B_5 be sets. Then there exists a function G from the thin cylinders of A_2 and B_5 into the thin cylinders of A_1 and B_4 such that for every set x if $x \in$ the thin cylinders of A_2 and B_5, then there exists a subset B_3 of B_5 and there exists a subset B_2 of B_4 and there exists a function y_2 from B_2 into A_1 and there exists a function y_3 from B_3 into A_2 such that B_2 is finite and B_3 is finite and $B_2 = B_4 \cap B_3 \cap {y_3}^{-1}(A_1)$ and $y_2 = y_3{\restriction}B_2$ and $x = \mathrm{cylinder}_0(A_2, B_5, B_3, y_3)$ and $G(x) = \mathrm{cylinder}_0(A_1, B_4, B_2, y_2)$.

Let A_1, A_2 be non trivial sets and let B_4, B_5 be sets. Let us assume that there exist sets x, y such that $x \neq y$ and x, $y \in A_1$ and $A_1 \subseteq A_2$ and $B_4 \subseteq B_5$. The functor $\mathrm{Extcylinders}(A_1, B_4, A_2, B_5)$ yielding a function from the thin cylinders of A_1 and B_4 into the thin cylinders of A_2 and B_5 is defined by the condition (Def. 4).

(Def. 4) Let x be a set. Suppose $x \in$ the thin cylinders of A_1 and B_4. Then there exists a subset B_1 of B_4 and there exists a function y_2 from B_1 into A_1 and there exists a function y_3 from B_1 into A_2 such that B_1 is finite and $y_2 = y_3$ and $x = \mathrm{cylinder}_0(A_1, B_4, B_1, y_2)$ and $(\mathrm{Extcylinders}(A_1, B_4, A_2, B_5))(x) = \mathrm{cylinder}_0(A_2, B_5, B_1, y_3)$.

Let A_1 be a non empty set, let A_2 be a non trivial set, and let B_4, B_5 be sets. Let us assume that $A_1 \subseteq A_2$ and $B_4 \subseteq B_5$. The functor Ristcylinders(A_1, B_4, A_2, B_5) yields a function from the thin cylinders of A_2 and B_5 into the thin cylinders of A_1 and B_4 and is defined by the condition (Def. 5).

(Def. 5) Let x be a set. Suppose $x \in$ the thin cylinders of A_2 and B_5. Then there exists a subset B_3 of B_5 and there exists a subset B_2 of B_4 and there exists a function y_2 from B_2 into A_1 and there exists a function y_3 from B_3 into A_2 such that B_2 is finite and B_3 is finite and $B_2 = B_4 \cap B_3 \cap y_3{}^{-1}(A_1)$ and $y_2 = y_3 \restriction B_2$ and $x = \mathrm{cylinder}_0(A_2, B_5, B_3, y_3)$ and $(\mathrm{Ristcylinders}(A_1, B_4, A_2, B_5))(x) = \mathrm{cylinder}_0(A_1, B_4, B_2, y_2)$.

Let A be a non trivial set, let B be a set, and let D be a thin cylinder of A and B. The functor $\mathrm{loc}\, D$ yielding a finite subset of B is defined by the condition (Def. 6).

(Def. 6) There exists a subset B_1 of B and there exists a function y_1 from B_1 into A such that B_1 is finite and $D = \{y : B \to A \colon y \restriction B_1 = y_1\}$ and $\mathrm{loc}\, D = B_1$.

2. Colored Petri Nets

Let A_1, A_2 be non trivial sets, let B_4, B_5 be sets, let C_1, C_2 be non trivial sets, let D_1, D_2 be sets, and let F be a function from the thin cylinders of A_1 and B_4 into the thin cylinders of C_1 and D_1. The functor CylinderFunc$(A_1, B_4, A_2, B_5, C_1, D_1, C_2, D_2, F)$ yielding a function from the thin cylinders of A_2 and B_5 into the thin cylinders of C_2 and D_2 is defined as follows:

(Def. 7) $\mathrm{CylinderFunc}(A_1, B_4, A_2, B_5, C_1, D_1, C_2, D_2, F) = \mathrm{Extcylinders}(C_1, D_1, C_2, D_2) \cdot F \cdot \mathrm{Ristcylinders}(A_1, B_4, A_2, B_5)$.

We consider colored place/transition net structures as extensions of place/transition net structure as systems

⟨ places, transitions, S-T arcs, T-S arcs, a colored set, a firing-rule ⟩,

where the places and the transitions constitute non empty sets, the S-T arcs constitute a non empty relation between the places and the transitions, the T-S arcs constitute a non empty relation between the transitions and the places, the colored set is a non empty finite set, and the firing-rule is a function.

Let C_3 be a colored place/transition net structure and let t_0 be a transition of C_3. We say that t_0 is outbound if and only if:

(Def. 8) $\overline{\{t_0\}} = \emptyset$.

Let C_4 be a colored place/transition net structure. The functor $\mathrm{Outbds}\, C_4$ yielding a subset of the transitions of C_4 is defined by:

(Def. 9) $\mathrm{Outbds}\, C_4 = \{x; x$ ranges over transitions of C_4: x is outbound$\}$.

Let C_3 be a colored place/transition net structure. We say that C_3 is colored-PT-net-like if and only if the conditions (Def. 10) are satisfied.

(Def. 10)(i) dom (the firing-rule of C_3) $\subseteq$ (the transitions of C_3) $\setminus$ Outbds C_3, and

(ii) for every transition t of C_3 such that $t \in$ dom (the firing-rule of C_3) there exists a non empty subset C_5 of the colored set of C_3 and there exists a subset I of $^*\{t\}$ and there exists a subset O of $\overline{\{t\}}$ such that (the firing-rule of C_3)(t) is a function from the thin cylinders of C_5 and I into the thin cylinders of C_5 and O.

We now state two propositions:

(6) Let C_3 be a colored place/transition net structure and t be a transition of C_3. Suppose C_3 is colored-PT-net-like and $t \in$ dom (the firing-rule of C_3). Then there exists a non empty subset C_5 of the colored set of C_3 and there exists a subset I of $^*\{t\}$ and there exists a subset O of $\overline{\{t\}}$ such that (the firing-rule of C_3)(t) is a function from the thin cylinders of C_5 and I into the thin cylinders of C_5 and O.

(7) Let C_4, C_6 be colored place/transition net structures, t_1 be a transition of C_4, and t_2 be a transition of C_6. Suppose that

(i) the places of $C_4 \subseteq$ the places of C_6,

(ii) the transitions of $C_4 \subseteq$ the transitions of C_6,

(iii) the S-T arcs of $C_4 \subseteq$ the S-T arcs of C_6,

(iv) the T-S arcs of $C_4 \subseteq$ the T-S arcs of C_6, and

(v) $t_1 = t_2$.

Then $^*\{t_1\} \subseteq {}^*\{t_2\}$ and $\overline{\{t_1\}} \subseteq \overline{\{t_2\}}$.

One can verify that there exists a colored place/transition net structure which is strict and colored-PT-net-like.

A colored place/transition net is a colored-PT-net-like colored place/transition net structure.

3. Color Counts of CPNT

Let C_4, C_6 be colored place/transition net structures. We say that C_4 misses C_6 if and only if:

(Def. 11) (The places of C_4) $\cap$ (the places of C_6) $= \emptyset$ and (the transitions of C_4) $\cap$ (the transitions of C_6) $= \emptyset$.

Let us note that the predicate C_4 misses C_6 is symmetric.

4. Colored States of CPNT

Let C_4 be a colored place/transition net structure and let C_6 be a colored place/transition net structure. Connecting mapping of C_4 and C_6 is defined by the condition (Def. 12).

(Def. 12) There exists a function O_{12} from Outbds C_4 into the places of C_6 and there exists a function O_{21} from Outbds C_6 into the places of C_4 such that it $= \langle O_{12}, O_{21} \rangle$.

5. Outbound Transitions of CPNT

Let C_4, C_6 be colored place/transition nets and let O be a connecting mapping of C_4 and C_6. Connecting firing rule of C_4, C_6, and O is defined by the condition (Def. 13).

(Def. 13) There exist functions q_{12}, q_{21} and there exists a function O_{12} from Outbds C_4 into the places of C_6 and there exists a function O_{21} from Outbds C_6 into the places of C_4 such that

(i) $O = \langle O_{12}, O_{21} \rangle$,

(ii) $\operatorname{dom} q_{12} = \text{Outbds}\, C_4$,

(iii) $\operatorname{dom} q_{21} = \text{Outbds}\, C_6$,

(iv) for every transition t_3 of C_4 such that t_3 is outbound holds $q_{12}(t_3)$ is a function from the thin cylinders of the colored set of C_4 and ${}^*\{t_3\}$ into the thin cylinders of the colored set of C_4 and $O_{12}{}^\circ t_3$,

(v) for every transition t_4 of C_6 such that t_4 is outbound holds $q_{21}(t_4)$ is a function from the thin cylinders of the colored set of C_6 and ${}^*\{t_4\}$ into the thin cylinders of the colored set of C_6 and $O_{21}{}^\circ t_4$, and

(vi) it $= \langle q_{12}, q_{21} \rangle$.

6. Connecting Mapping for CPNT1, CPNT2

Let C_4, C_6 be colored place/transition nets, let O be a connecting mapping of C_4 and C_6, and let q be a connecting firing rule of C_4, C_6, and O. Let us assume that C_4 misses C_6. The functor synthesis(C_4, C_6, O, q) yielding a strict colored place/transition net is defined by the condition (Def. 14).

(Def. 14) There exist functions q_{12}, q_{21} and there exists a function O_{12} from Outbds C_4 into the places of C_6 and there exists a function O_{21} from Outbds C_6 into the places of C_4 such that
$O = \langle O_{12}, O_{21} \rangle$ and $\operatorname{dom} q_{12} = \text{Outbds}\, C_4$ and $\operatorname{dom} q_{21} = \text{Outbds}\, C_6$ and for every transition t_3 of C_4 such that t_3 is outbound holds $q_{12}(t_3)$ is a function from the thin cylinders of the colored set of C_4 and ${}^*\{t_3\}$ into the thin cylinders of the colored set of C_4 and $O_{12}{}^\circ t_3$ and for every transition t_4 of C_6 such that t_4 is outbound holds $q_{21}(t_4)$ is a function from the thin cylinders of the colored set of C_6 and ${}^*\{t_4\}$ into the thin cylinders of the colored set of C_6 and $O_{21}{}^\circ t_4$ and $q = \langle q_{12}, q_{21} \rangle$ and the places of synthesis$(C_4, C_6, O, q) =$ (the places of C_4) $\cup$ (the places of C_6) and the

transitions of synthesis(C_4, C_6, O, q) = (the transitions of C_4) $\cup$ (the transitions of C_6) and the S-T arcs of synthesis(C_4, C_6, O, q) = (the S-T arcs of C_4) $\cup$ (the S-T arcs of C_6) and the T-S arcs of synthesis(C_4, C_6, O, q) = (the T-S arcs of C_4) $\cup$ (the T-S arcs of C_6) $\cup O_{12} \cup O_{21}$ and the colored set of synthesis(C_4, C_6, O, q) = (the colored set of C_4) $\cup$ (the colored set of C_6) and the firing-rule of synthesis(C_4, C_6, O, q) = (the firing-rule of C_4) $+\cdot$ (the firing-rule of C_6) $+\cdot q_{12} +\cdot q_{21}$.

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Received October 14, 2008

FORMALIZED MATHEMATICS
Vol. **17**, No. **1**, Pages 43–60, **2009**

Arithmetic Operations on Functions from Sets into Functional Sets

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Summary. In this paper we introduce sets containing number-valued functions. Different arithmetic operations on maps between any set and such functional sets are later defined.

MML identifier: `VALUED_2`, version: `7.11.01 4.117.1046`

The notation and terminology used here are introduced in the following papers: [4], [9], [10], [2], [11], [6], [3], [1], [8], [5], and [7].

1. Functional sets

In this paper x, X, X_1, X_2 are sets.

Let Y be a functional set. The functor $\mathrm{DOMS}(Y)$ is defined by:

(Def. 1) $\mathrm{DOMS}(Y) = \bigcup\{\mathrm{dom}\, f : f$ ranges over elements of $Y\}$.

Let us consider X. We say that X is complex-functions-membered if and only if:

(Def. 2) If $x \in X$, then x is a complex-valued function.

Let us consider X. We say that X is extended-real-functions-membered if and only if:

(Def. 3) If $x \in X$, then x is an extended real-valued function.

Let us consider X. We say that X is real-functions-membered if and only if:

[1]The article was written while the author visited Shinshu University, Nagano, Japan.

ISSN 1426–2630(p), 1898-9934(e)

(Def. 4) If $x \in X$, then x is a real-valued function.

Let us consider X. We say that X is rational-functions-membered if and only if:

(Def. 5) If $x \in X$, then x is a rational-valued function.

Let us consider X. We say that X is integer-functions-membered if and only if:

(Def. 6) If $x \in X$, then x is an integer-valued function.

Let us consider X. We say that X is natural-functions-membered if and only if:

(Def. 7) If $x \in X$, then x is a natural-valued function.

One can check the following observations:

* every set which is natural-functions-membered is also integer-functions-membered,
* every set which is integer-functions-membered is also rational-functions-membered,
* every set which is rational-functions-membered is also real-functions-membered,
* every set which is real-functions-membered is also complex-functions-membered, and
* every set which is real-functions-membered is also extended-real-functions-membered.

Let us mention that every set which is empty is also natural-functions-membered.

Let f be a complex-valued function. Observe that $\{f\}$ is complex-functions-membered.

One can verify that every set which is complex-functions-membered is also functional and every set which is extended-real-functions-membered is also functional.

One can verify that there exists a set which is natural-functions-membered and non empty.

Let X be a complex-functions-membered set. One can verify that every subset of X is complex-functions-membered.

Let X be an extended-real-functions-membered set. Note that every subset of X is extended-real-functions-membered.

Let X be a real-functions-membered set. Note that every subset of X is real-functions-membered.

Let X be a rational-functions-membered set. Observe that every subset of X is rational-functions-membered.

Let X be an integer-functions-membered set. Note that every subset of X is integer-functions-membered.

Let X be a natural-functions-membered set. Observe that every subset of X is natural-functions-membered.

Let D be a set. The functor $\mathbb{C}$-PFuncs D yields a set and is defined by:

(Def. 8) For every set f holds $f \in \mathbb{C}$-PFuncs D iff f is a partial function from D to $\mathbb{C}$.

Let D be a set. The functor $\mathbb{C}$-Funcs D yielding a set is defined by:

(Def. 9) For every set f holds $f \in \mathbb{C}$-Funcs D iff f is a function from D into $\mathbb{C}$.

Let D be a set. The functor $\overline{\mathbb{R}}$-PFuncs D yields a set and is defined by:

(Def. 10) For every set f holds $f \in \overline{\mathbb{R}}$-PFuncs D iff f is a partial function from D to $\overline{\mathbb{R}}$.

Let D be a set. The functor $\overline{\mathbb{R}}$-Funcs D yields a set and is defined as follows:

(Def. 11) For every set f holds $f \in \overline{\mathbb{R}}$-Funcs D iff f is a function from D into $\overline{\mathbb{R}}$.

Let D be a set. The functor $\mathbb{R}$-PFuncs D yielding a set is defined by:

(Def. 12) For every set f holds $f \in \mathbb{R}$-PFuncs D iff f is a partial function from D to $\mathbb{R}$.

Let D be a set. The functor $\mathbb{R}$-Funcs D yielding a set is defined by:

(Def. 13) For every set f holds $f \in \mathbb{R}$-Funcs D iff f is a function from D into $\mathbb{R}$.

Let D be a set. The functor $\mathbb{Q}$-PFuncs D yields a set and is defined as follows:

(Def. 14) For every set f holds $f \in \mathbb{Q}$-PFuncs D iff f is a partial function from D to $\mathbb{Q}$.

Let D be a set. The functor $\mathbb{Q}$-Funcs D yields a set and is defined by:

(Def. 15) For every set f holds $f \in \mathbb{Q}$-Funcs D iff f is a function from D into $\mathbb{Q}$.

Let D be a set. The functor $\mathbb{Z}$-PFuncs D yielding a set is defined by:

(Def. 16) For every set f holds $f \in \mathbb{Z}$-PFuncs D iff f is a partial function from D to $\mathbb{Z}$.

Let D be a set. The functor $\mathbb{Z}$-Funcs D yields a set and is defined as follows:

(Def. 17) For every set f holds $f \in \mathbb{Z}$-Funcs D iff f is a function from D into $\mathbb{Z}$.

Let D be a set. The functor $\mathbb{N}$-PFuncs D yields a set and is defined by:

(Def. 18) For every set f holds $f \in \mathbb{N}$-PFuncs D iff f is a partial function from D to $\mathbb{N}$.

Let D be a set. The functor $\mathbb{N}$-Funcs D yielding a set is defined by:

(Def. 19) For every set f holds $f \in \mathbb{N}$-Funcs D iff f is a function from D into $\mathbb{N}$.

The following propositions are true:

(1) $\mathbb{C}$-Funcs X is a subset of $\mathbb{C}$-PFuncs X.

(2) $\overline{\mathbb{R}}$-Funcs X is a subset of $\overline{\mathbb{R}}$-PFuncs X.

(3) $\mathbb{R}$-Funcs X is a subset of $\mathbb{R}$-PFuncs X.

(4) $\mathbb{Q}$-Funcs X is a subset of $\mathbb{Q}$-PFuncs X.

(5) $\mathbb{Z}$-Funcs X is a subset of $\mathbb{Z}$-PFuncs X.

(6) $\mathbb{N}$-Funcs X is a subset of $\mathbb{N}$-PFuncs X.

Let us consider X. One can verify the following observations:

* $\mathbb{C}$-PFuncs X is complex-functions-membered,
* $\mathbb{C}$-Funcs X is complex-functions-membered,
* $\overline{\mathbb{R}}$-PFuncs X is extended-real-functions-membered,
* $\overline{\mathbb{R}}$-Funcs X is extended-real-functions-membered,
* $\mathbb{R}$-PFuncs X is real-functions-membered,
* $\mathbb{R}$-Funcs X is real-functions-membered,
* $\mathbb{Q}$-PFuncs X is rational-functions-membered,
* $\mathbb{Q}$-Funcs X is rational-functions-membered,
* $\mathbb{Z}$-PFuncs X is integer-functions-membered,
* $\mathbb{Z}$-Funcs X is integer-functions-membered,
* $\mathbb{N}$-PFuncs X is natural-functions-membered, and
* $\mathbb{N}$-Funcs X is natural-functions-membered.

Let X be a complex-functions-membered set. Observe that every element of X is complex-valued.

Let X be an extended-real-functions-membered set. One can check that every element of X is extended real-valued.

Let X be a real-functions-membered set. One can check that every element of X is real-valued.

Let X be a rational-functions-membered set. One can check that every element of X is rational-valued.

Let X be an integer-functions-membered set. Observe that every element of X is integer-valued.

Let X be a natural-functions-membered set. Observe that every element of X is natural-valued.

Let X, x be sets, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. Observe that $f(x)$ is function-like and relation-like.

Let X, x be sets, let Y be an extended-real-functions-membered set, and let f be a partial function from X to Y. Observe that $f(x)$ is function-like and relation-like.

Let us consider X, x, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. One can check that $f(x)$ is complex-valued.

Let us consider X, x, let Y be an extended-real-functions-membered set, and let f be a partial function from X to Y. One can verify that $f(x)$ is extended real-valued.

Let us consider X, x, let Y be a real-functions-membered set, and let f be a partial function from X to Y. Note that $f(x)$ is real-valued.

Let us consider X, x, let Y be a rational-functions-membered set, and let f be a partial function from X to Y. Note that $f(x)$ is rational-valued.

Let us consider X, x, let Y be an integer-functions-membered set, and let f be a partial function from X to Y. Note that $f(x)$ is integer-valued.

Let us consider X, x, let Y be a natural-functions-membered set, and let f be a partial function from X to Y. One can check that $f(x)$ is natural-valued.

Let us consider X and let Y be a complex-membered set. One can check that $X \dot{\to} Y$ is complex-functions-membered.

Let us consider X and let Y be an extended real-membered set. Observe that $X \dot{\to} Y$ is extended-real-functions-membered.

Let us consider X and let Y be a real-membered set. Observe that $X \dot{\to} Y$ is real-functions-membered.

Let us consider X and let Y be a rational-membered set. Observe that $X \dot{\to} Y$ is rational-functions-membered.

Let us consider X and let Y be an integer-membered set. Observe that $X \dot{\to} Y$ is integer-functions-membered.

Let us consider X and let Y be a natural-membered set. One can verify that $X \dot{\to} Y$ is natural-functions-membered.

Let us consider X and let Y be a complex-membered set. Note that Y^X is complex-functions-membered.

Let us consider X and let Y be an extended real-membered set. Note that Y^X is extended-real-functions-membered.

Let us consider X and let Y be a real-membered set. Note that Y^X is real-functions-membered.

Let us consider X and let Y be a rational-membered set. Note that Y^X is rational-functions-membered.

Let us consider X and let Y be an integer-membered set. Note that Y^X is integer-functions-membered.

Let us consider X and let Y be a natural-membered set. One can check that Y^X is natural-functions-membered.

Let R be a binary relation. We say that R is complex-functions-valued if and only if:

(Def. 20) $\operatorname{rng} R$ is complex-functions-membered.

We say that R is extended-real-functions-valued if and only if:

(Def. 21) $\operatorname{rng} R$ is extended-real-functions-membered.

We say that R is real-functions-valued if and only if:

(Def. 22) $\operatorname{rng} R$ is real-functions-membered.

We say that R is rational-functions-valued if and only if:

(Def. 23) $\operatorname{rng} R$ is rational-functions-membered.

We say that R is integer-functions-valued if and only if:

(Def. 24) rng R is integer-functions-membered.

We say that R is natural-functions-valued if and only if:

(Def. 25) rng R is natural-functions-membered.

Let f be a function. Let us observe that f is complex-functions-valued if and only if:

(Def. 26) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is a complex-valued function.

Let us observe that f is extended-real-functions-valued if and only if:

(Def. 27) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is an extended real-valued function.

Let us observe that f is real-functions-valued if and only if:

(Def. 28) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is a real-valued function.

Let us observe that f is rational-functions-valued if and only if:

(Def. 29) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is a rational-valued function.

Let us observe that f is integer-functions-valued if and only if:

(Def. 30) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is an integer-valued function.

Let us observe that f is natural-functions-valued if and only if:

(Def. 31) For every set x such that $x \in \operatorname{dom} f$ holds $f(x)$ is a natural-valued function.

One can verify the following observations:

* every binary relation which is natural-functions-valued is also integer-functions-valued,
* every binary relation which is integer-functions-valued is also rational-functions-valued,
* every binary relation which is rational-functions-valued is also real-functions-valued,
* every binary relation which is real-functions-valued is also extended-real-functions-valued, and
* every binary relation which is real-functions-valued is also complex-functions-valued.

Let us note that every binary relation which is empty is also natural-functions-valued.

Let us mention that there exists a function which is natural-functions-valued.

Let R be a complex-functions-valued binary relation. Note that rng R is complex-functions-membered.

Let R be an extended-real-functions-valued binary relation. Observe that rng R is extended-real-functions-membered.

Let R be a real-functions-valued binary relation. Note that $\operatorname{rng} R$ is real-functions-membered.

Let R be a rational-functions-valued binary relation. Observe that $\operatorname{rng} R$ is rational-functions-membered.

Let R be an integer-functions-valued binary relation. One can verify that $\operatorname{rng} R$ is integer-functions-membered.

Let R be a natural-functions-valued binary relation. One can check that $\operatorname{rng} R$ is natural-functions-membered.

Let us consider X and let Y be a complex-functions-membered set. Observe that every partial function from X to Y is complex-functions-valued.

Let us consider X and let Y be an extended-real-functions-membered set. One can check that every partial function from X to Y is extended-real-functions-valued.

Let us consider X and let Y be a real-functions-membered set. One can check that every partial function from X to Y is real-functions-valued.

Let us consider X and let Y be a rational-functions-membered set. Observe that every partial function from X to Y is rational-functions-valued.

Let us consider X and let Y be an integer-functions-membered set. Observe that every partial function from X to Y is integer-functions-valued.

Let us consider X and let Y be a natural-functions-membered set. Note that every partial function from X to Y is natural-functions-valued.

Let f be a complex-functions-valued function and let us consider x. Note that $f(x)$ is function-like and relation-like.

Let f be an extended-real-functions-valued function and let us consider x. Observe that $f(x)$ is function-like and relation-like.

Let f be a complex-functions-valued function and let us consider x. One can verify that $f(x)$ is complex-valued.

Let f be an extended-real-functions-valued function and let us consider x. Note that $f(x)$ is extended real-valued.

Let f be a real-functions-valued function and let us consider x. One can verify that $f(x)$ is real-valued.

Let f be a rational-functions-valued function and let us consider x. Observe that $f(x)$ is rational-valued.

Let f be an integer-functions-valued function and let us consider x. Note that $f(x)$ is integer-valued.

Let f be a natural-functions-valued function and let us consider x. One can check that $f(x)$ is natural-valued.

2. Operations

For simplicity, we adopt the following rules: Y, Y_1, Y_2 are complex-functions-membered sets, c, c_1, c_2 are complex numbers, f is a partial function from X

to Y, f_1 is a partial function from X_1 to Y_1, f_2 is a partial function from X_2 to Y_2, and g, h, k are complex-valued functions.

We now state a number of propositions:

(7) If $g \neq \emptyset$ and $g + c_1 = g + c_2$, then $c_1 = c_2$.

(8) If $g \neq \emptyset$ and $g - c_1 = g - c_2$, then $c_1 = c_2$.

(9) If $g \neq \emptyset$ and g is non-empty and $g\,c_1 = g\,c_2$, then $c_1 = c_2$.

(10) $-(g + c) = -g - c$.

(11) $-(g - c) = -g + c$.

(12) $(g + c_1) + c_2 = g + (c_1 + c_2)$.

(13) $(g + c_1) - c_2 = g + (c_1 - c_2)$.

(14) $(g - c_1) + c_2 = g - (c_1 - c_2)$.

(15) $g - c_1 - c_2 = g - (c_1 + c_2)$.

(16) $g\,c_1\,c_2 = g\,(c_1 \cdot c_2)$.

(17) $-(g + h) = -g - h$.

(18) $g - h = -(h - g)$.

(19) $(g\,h)/k = g\,(h/k)$.

(20) $(g/h)\,k = (g\,k)/h$.

(21) $g/h/k = g/(h\,k)$.

(22) $c - g = (-c)\,g$.

(23) $c - g = -c\,g$.

(24) $(-c)\,g = -c\,g$.

(25) $-g\,h = (-g)\,h$.

(26) $-g/h = (-g)/h$.

(27) $-g/h = g/-h$.

Let f be a complex-valued function and let c be a complex number. The functor f/c yields a function and is defined as follows:

(Def. 32) $f/c = \frac{1}{c}\,f$.

Let f be a complex-valued function and let c be a complex number. Note that f/c is complex-valued.

Let f be a real-valued function and let r be a real number. Note that f/r is real-valued.

Let f be a rational-valued function and let r be a rational number. One can check that f/r is rational-valued.

Let f be a complex-valued finite sequence and let c be a complex number. One can check that f/c is finite sequence-like.

The following propositions are true:

(28) $\mathrm{dom}(g/c) = \mathrm{dom}\,g$.

(29) $(g/c)(x) = \frac{g(x)}{c}$.

(30) $(-g)/c = -g/c$.

(31) $g/-c = -g/c$.

(32) $g/-c = (-g)/c$.

(33) If $g \neq \emptyset$ and g is non-empty and $g/c_1 = g/c_2$, then $c_1 = c_2$.

(34) $(g\,c_1)/c_2 = g\,\frac{c_1}{c_2}$.

(35) $(g/c_1)\,c_2 = (g\,c_2)/c_1$.

(36) $g/c_1/c_2 = g/(c_1 \cdot c_2)$.

(37) $(g + h)/c = g/c + h/c$.

(38) $(g - h)/c = g/c - h/c$.

(39) $(g\,h)/c = g\,(h/c)$.

(40) $(g/h)/c = g/(h\,c)$.

Let us consider X, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. The functor $-f$ yields a function and is defined by:

(Def. 33) $\operatorname{dom}(-f) = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom}(-f)$ holds $(-f)(x) = -f(x)$.

Let us consider X, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. Then $-f$ is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, and let f be a partial function from X to Y. Then $-f$ is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, and let f be a partial function from X to Y. Then $-f$ is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, and let f be a partial function from X to Y. Then $-f$ is a partial function from X to $\mathbb{Z}$-PFuncs DOMS(Y).

Let Y be a complex-functions-membered set and let f be a finite sequence of elements of Y. One can check that $-f$ is finite sequence-like.

We now state two propositions:

(41) $- - f = f$.

(42) If $-f_1 = -f_2$, then $f_1 = f_2$.

Let X be a complex-functions-membered set, let Y be a set, and let f be a partial function from X to Y. The functor $f \circ -$ yielding a function is defined as follows:

(Def. 34) $\operatorname{dom}(f \circ -) = \operatorname{dom} f$ and for every complex-valued function x such that $x \in \operatorname{dom}(f \circ -)$ holds $(f \circ -)(x) = f(-x)$.

Let us consider X, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. The functor ${}^{1}/f$ yields a function and is defined as follows:

(Def. 35) $\operatorname{dom}{}^{1}/f = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom}{}^{1}/f$ holds $({}^{1}/f)(x) = f(x)^{-1}$.

Let us consider X, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. Then ${}^{1}/f$ is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, and let f be a partial function from X to Y. Then ${}^{1}/f$ is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, and let f be a partial function from X to Y. Then ${}^{1}/f$ is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

Let Y be a complex-functions-membered set and let f be a finite sequence of elements of Y. Note that ${}^{1}/f$ is finite sequence-like.

The following proposition is true

(43) ${}^{1}/{}^{1}/f = f$.

Let us consider X, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. The functor $|f|$ yields a function and is defined by:

(Def. 36) $\operatorname{dom}|f| = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom}|f|$ holds $|f|(x) = |f(x)|$.

Let us consider X, let Y be a complex-functions-membered set, and let f be a partial function from X to Y. Then $|f|$ is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, and let f be a partial function from X to Y. Then $|f|$ is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, and let f be a partial function from X to Y. Then $|f|$ is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, and let f be a partial function from X to Y. Then $|f|$ is a partial function from X to $\mathbb{N}$-PFuncs DOMS(Y).

Let Y be a complex-functions-membered set and let f be a finite sequence of elements of Y. Note that $|f|$ is finite sequence-like.

We now state the proposition

(44) $||f|| = |f|$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. The functor $f+c$

yields a function and is defined by:

(Def. 37) $\operatorname{dom}(f + c) = \operatorname{dom} f$ and for every set x such that $x \in \operatorname{dom}(f + c)$ holds $(f + c)(x) = c + f(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. Then $f + c$ is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let c be a real number. Then $f+c$ is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let c be a rational number. Then $f + c$ is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, let f be a partial function from X to Y, and let c be an integer number. Then $f + c$ is a partial function from X to $\mathbb{Z}$-PFuncs DOMS(Y).

Let us consider X, let Y be a natural-functions-membered set, let f be a partial function from X to Y, and let c be a natural number. Then $f + c$ is a partial function from X to $\mathbb{N}$-PFuncs DOMS(Y).

One can prove the following propositions:

(45) $f + c_1 + c_2 = f + (c_1 + c_2)$.

(46) If $f \neq \emptyset$ and f is non-empty and $f + c_1 = f + c_2$, then $c_1 = c_2$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. The functor $f - c$ yields a function and is defined as follows:

(Def. 38) $f - c = f + -c$.

We now state two propositions:

(47) $\operatorname{dom}(f - c) = \operatorname{dom} f$.

(48) If $x \in \operatorname{dom}(f - c)$, then $(f - c)(x) = f(x) - c$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. Then $f - c$ is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let c be a real number. Then $f-c$ is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let c be a rational number. Then $f - c$ is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, let f be a partial function from X to Y, and let c be an integer number. Then $f - c$ is a partial function from X to $\mathbb{Z}$-PFuncs DOMS(Y).

We now state four propositions:

(49) If $f \neq \emptyset$ and f is non-empty and $f - c_1 = f - c_2$, then $c_1 = c_2$.

(50) $(f + c_1) - c_2 = f + (c_1 - c_2)$.

(51) $(f - c_1) + c_2 = f - (c_1 - c_2)$.

(52) $f - c_1 - c_2 = f - (c_1 + c_2)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. The functor $f \cdot c$ yielding a function is defined as follows:

(Def. 39) $\mathrm{dom}(f \cdot c) = \mathrm{dom}\, f$ and for every set x such that $x \in \mathrm{dom}(f \cdot c)$ holds $(f \cdot c)(x) = c\, f(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. Then $f \cdot c$ is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let c be a real number. Then $f \cdot c$ is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let c be a rational number. Then $f \cdot c$ is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, let f be a partial function from X to Y, and let c be an integer number. Then $f \cdot c$ is a partial function from X to $\mathbb{Z}$-PFuncs DOMS(Y).

Let us consider X, let Y be a natural-functions-membered set, let f be a partial function from X to Y, and let c be a natural number. Then $f \cdot c$ is a partial function from X to $\mathbb{N}$-PFuncs DOMS(Y).

The following two propositions are true:

(53) $f \cdot c_1 \cdot c_2 = f \cdot (c_1 \cdot c_2)$.

(54) If $f \neq \emptyset$ and f is non-empty and for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is non-empty and $f \cdot c_1 = f \cdot c_2$, then $c_1 = c_2$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. The functor f/c yielding a function is defined as follows:

(Def. 40) $f/c = f \cdot c^{-1}$.

One can prove the following propositions:

(55) $\mathrm{dom}(f/c) = \mathrm{dom}\, f$.

(56) If $x \in \mathrm{dom}(f/c)$, then $(f/c)(x) = c^{-1}\, f(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let c be a complex number. Then f/c is a partial function from X to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let c be a real number. Then f/c is a partial function from X to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let c be a rational number. Then f/c is a partial function from X to $\mathbb{Q}$-PFuncs DOMS(Y).

The following propositions are true:

(57) $f/c_1/c_2 = f/(c_1 \cdot c_2)$.

(58) If $f \neq \emptyset$ and f is non-empty and for every x such that $x \in \operatorname{dom} f$ holds $f(x)$ is non-empty and $f/c_1 = f/c_2$, then $c_1 = c_2$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. The functor $f + g$ yielding a function is defined as follows:

(Def. 41) $\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$ and for every set x such that $x \in \operatorname{dom}(f+g)$ holds $(f+g)(x) = f(x) + g(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. Then $f + g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let g be a real-valued function. Then $f+g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let g be a rational-valued function. Then $f + g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, let f be a partial function from X to Y, and let g be an integer-valued function. Then $f + g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{Z}$-PFuncs DOMS(Y).

Let us consider X, let Y be a natural-functions-membered set, let f be a partial function from X to Y, and let g be a natural-valued function. Then $f+g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{N}$-PFuncs DOMS(Y).

Next we state two propositions:

(59) $f + g + h = f + (g + h)$.

(60) $-(f + g) = (-f) + -g$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. The functor $f - g$ yields a function and is defined by:

(Def. 42) $f - g = f + -g$.

We now state two propositions:

(61) $\operatorname{dom}(f - g) = \operatorname{dom} f \cap \operatorname{dom} g$.

(62) If $x \in \operatorname{dom}(f - g)$, then $(f - g)(x) = f(x) - g(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. Then $f - g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let g be a real-valued function. Then $f - g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let g be a rational-valued function. Then $f - g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, let f be a partial function from X to Y, and let g be an integer-valued function. Then $f - g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{Z}$-PFuncs DOMS(Y).

The following propositions are true:

(63) $f - -g = f + g$.

(64) $-(f - g) = (-f) + g$.

(65) $(f + g) - h = f + (g - h)$.

(66) $(f - g) + h = f - (g - h)$.

(67) $f - g - h = f - (g + h)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. The functor $f \cdot g$ yielding a function is defined by:

(Def. 43) $\operatorname{dom}(f \cdot g) = \operatorname{dom} f \cap \operatorname{dom} g$ and for every set x such that $x \in \operatorname{dom}(f \cdot g)$ holds $(f \cdot g)(x) = f(x)\, g(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. Then $f \cdot g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let g be a real-valued function. Then $f \cdot g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let g be a rational-valued function. Then $f \cdot g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{Q}$-PFuncs DOMS(Y).

Let us consider X, let Y be an integer-functions-membered set, let f be a partial function from X to Y, and let g be an integer-valued function. Then $f \cdot g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{Z}$-PFuncs DOMS(Y).

Let us consider X, let Y be a natural-functions-membered set, let f be a partial function from X to Y, and let g be a natural-valued function. Then $f \cdot g$ is a partial function from $X \cap \operatorname{dom} g$ to $\mathbb{N}$-PFuncs DOMS(Y).

Next we state three propositions:

(68) $f \cdot -g = (-f) \cdot g$.

(69) $f \cdot -g = -f \cdot g$.

(70) $f \cdot g \cdot h = f \cdot (g\,h)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. The functor f/g yields a function and is defined by:

(Def. 44) $f/g = f \cdot g^{-1}$.

Next we state two propositions:

(71) $\mathrm{dom}(f/g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$.

(72) If $x \in \mathrm{dom}(f/g)$, then $(f/g)(x) = f(x)/g(x)$.

Let us consider X, let Y be a complex-functions-membered set, let f be a partial function from X to Y, and let g be a complex-valued function. Then f/g is a partial function from $X \cap \mathrm{dom}\, g$ to $\mathbb{C}$-PFuncs DOMS(Y).

Let us consider X, let Y be a real-functions-membered set, let f be a partial function from X to Y, and let g be a real-valued function. Then f/g is a partial function from $X \cap \mathrm{dom}\, g$ to $\mathbb{R}$-PFuncs DOMS(Y).

Let us consider X, let Y be a rational-functions-membered set, let f be a partial function from X to Y, and let g be a rational-valued function. Then f/g is a partial function from $X \cap \mathrm{dom}\, g$ to $\mathbb{Q}$-PFuncs DOMS(Y).

Next we state the proposition

(73) $(f \cdot g)/h = f \cdot (g/h)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. The functor $f + g$ yielding a function is defined as follows:

(Def. 45) $\mathrm{dom}(f+g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and for every set x such that $x \in \mathrm{dom}(f+g)$ holds $(f+g)(x) = f(x) + g(x)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{C}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be real-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{R}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be rational-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Q}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be integer-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to

Y_2. Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Z}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be natural-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f + g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{N}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

We now state three propositions:

(74) $f_1 + f_2 = f_2 + f_1$.

(75) $(f + f_1) + f_2 = f + (f_1 + f_2)$.

(76) $-(f_1 + f_2) = (-f_1) + -f_2$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. The functor $f - g$ yields a function and is defined by:

(Def. 46) $\mathrm{dom}(f - g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and for every set x such that $x \in \mathrm{dom}(f - g)$ holds $(f - g)(x) = f(x) - g(x)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f - g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{C}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be real-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f - g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{R}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be rational-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f - g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Q}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be integer-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f - g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Z}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

One can prove the following propositions:

(77) $f_1 - f_2 = -(f_2 - f_1)$.

(78) $-(f_1 - f_2) = (-f_1) + f_2$.

(79) $(f + f_1) - f_2 = f + (f_1 - f_2)$.

(80) $(f - f_1) + f_2 = f - (f_1 - f_2)$.

(81) $f - f_1 - f_2 = f - (f_1 + f_2)$.

(82) $f - f_1 - f_2 = f - f_2 - f_1$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2.

The functor $f \cdot g$ yields a function and is defined by:

(Def. 47) $\mathrm{dom}(f \cdot g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and for every set x such that $x \in \mathrm{dom}(f \cdot g)$ holds $(f \cdot g)(x) = f(x)\, g(x)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{C}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be real-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{R}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be rational-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Q}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be integer-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{Z}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be natural-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then $f \cdot g$ is a partial function from $X_1 \cap X_2$ to $\mathbb{N}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

We now state several propositions:

(83) $f_1 \cdot f_2 = f_2 \cdot f_1$.

(84) $(f \cdot f_1) \cdot f_2 = f \cdot (f_1 \cdot f_2)$.

(85) $(-f_1) \cdot f_2 = -f_1 \cdot f_2$.

(86) $f_1 \cdot -f_2 = -f_1 \cdot f_2$.

(87) $f \cdot (f_1 + f_2) = f \cdot f_1 + f \cdot f_2$.

(88) $(f_1 + f_2) \cdot f = f_1 \cdot f + f_2 \cdot f$.

(89) $f \cdot (f_1 - f_2) = f \cdot f_1 - f \cdot f_2$.

(90) $(f_1 - f_2) \cdot f = f_1 \cdot f - f_2 \cdot f$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. The functor f/g yields a function and is defined by:

(Def. 48) $\mathrm{dom}(f/g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and for every set x such that $x \in \mathrm{dom}(f/g)$ holds $(f/g)(x) = f(x)/g(x)$.

Let X_1, X_2 be sets, let Y_1, Y_2 be complex-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2

to Y_2. Then f/g is a partial function from $X_1 \cap X_2$ to $\mathbb{C}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be real-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then f/g is a partial function from $X_1 \cap X_2$ to $\mathbb{R}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

Let X_1, X_2 be sets, let Y_1, Y_2 be rational-functions-membered sets, let f be a partial function from X_1 to Y_1, and let g be a partial function from X_2 to Y_2. Then f/g is a partial function from $X_1 \cap X_2$ to $\mathbb{Q}$-PFuncs(DOMS(Y_1) $\cap$ DOMS(Y_2)).

One can prove the following propositions:

(91) $(-f_1)/f_2 = -f_1/f_2$.

(92) $f_1/-f_2 = -f_1/f_2$.

(93) $(f \cdot f_1)/f_2 = f \cdot (f_1/f_2)$.

(94) $(f/f_1) \cdot f_2 = (f \cdot f_2)/f_1$.

(95) $f/f_1/f_2 = f/(f_1 \cdot f_2)$.

(96) $(f_1 + f_2)/f = f_1/f + f_2/f$.

(97) $(f_1 - f_2)/f = f_1/f - f_2/f$.

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Received October 15, 2008

FORMALIZED MATHEMATICS
Volume 17, Number 1, 2009

Addenda

Authors

MML Identifiers

ISSN 1426–2630(p), 1898-9934(e)