Contents

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Basic Properties of Metrizable Topological Spaces

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Summary. We continue Mizar formalization of general topology according to the book [11] by Engelking. In the article, we present the final theorem of Section 4.1. Namely, the paper includes the formalization of theorems on the correspondence between the cardinalities of the basis and of some open subcover, and a discreet (closed) subspaces, and the weight of that metrizable topological space. We also define Lindelöf spaces and state the above theorem in this special case. We also introduce the concept of separation among two subsets (see [12]).

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The articles [21], [13], [20], [2], [1], [3], [10], [9], [7], [16], [4], [6], [19], [23], [22], [17], [15], [14], [8], [18], and [5] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we follow the rules: T, T_1, T_2 denote topological spaces, A, B denote subsets of T, F, G denote families of subsets of T, A_1 denotes a subset of T_1, A_2 denotes a subset of T_2, T_3, T_4, T_5 denote metrizable topological spaces, A_3, B_1 denote subsets of T_3, F_1, G_1 denote families of subsets of T_3, C denotes a cardinal number, and i_1 denotes an infinite cardinal number.

Let us consider T_1, T_2, A_1, A_2. We say that A_1 and A_2 are homeomorphic if and only if:

(Def. 1) $T_1 \restriction A_1$ and $T_2 \restriction A_2$ are homeomorphic.

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Next we state four propositions:

(1) T_1 and T_2 are homeomorphic iff $\Omega_{(T_1)}$ and $\Omega_{(T_2)}$ are homeomorphic.

(2) Let f be a function from T_1 into T_2. Suppose f is homeomorphism. Let g be a function from $T_1 \restriction A_1$ into $T_2 \restriction f^\circ A_1$. If $g = f \restriction A_1$, then g is homeomorphism.

(3) For every function f from T_1 into T_2 such that f is homeomorphism holds A_1 and $f^\circ A_1$ are homeomorphic.

(4) If T_1 and T_2 are homeomorphic, then $\operatorname{weight} T_1 = \operatorname{weight} T_2$.

Note that every topological space which is empty is also metrizable and every topological space which is metrizable is also T_4 and non empty. Let M be a metric space. Note that $M_{\rm top}$ is metrizable.

Let us consider T_3, A_3. Observe that $T_3 \restriction A_3$ is metrizable.

Let us consider T_4, T_5. Observe that $T_4 \times T_5$ is metrizable.

Next we state two propositions:

(5) $\operatorname{weight} T_1 \times T_2 \subseteq \operatorname{weight} T_1 \cdot \operatorname{weight} T_2$.

(6) If T_1 is non empty and T_2 is non empty, then $\operatorname{weight} T_1 \subseteq \operatorname{weight} T_1 \times T_2$ and $\operatorname{weight} T_2 \subseteq \operatorname{weight} T_1 \times T_2$.

Let T_1, T_2 be second-countable topological spaces. One can check that $T_1 \times T_2$ is second-countable.

One can prove the following propositions:

(7) $\operatorname{Card}(F \restriction A) \subseteq \operatorname{Card} F$.

(8) For every basis B_2 of T holds $B_2 \restriction A$ is a basis of $T \restriction A$.

Let T be a second-countable topological space and let A be a subset of T. Note that $T \restriction A$ is second-countable.

Let M be a non empty metric space and let A be a non empty subset of $M_{\rm top}$. One can check that $\operatorname{dist}_{\min}(A)$ is continuous.

We now state the proposition

(9) For every subset B of T and for every subset F of $T \restriction A$ such that $F = B$ holds $T \restriction A \restriction F = T \restriction B$.

Let us consider T_3. Observe that every subset of T_3 which is open is also F_σ and every subset of T_3 which is closed is also G_δ.

The following propositions are true:

(10) For every subset F of $T \restriction B$ such that A is F_σ and $F = A \cap B$ holds F is F_σ.

(11) For every subset F of $T \restriction B$ such that A is G_δ and $F = A \cap B$ holds F is G_δ.

(12) If T is a T_1 space and A is discrete, then A is an open subset of $T \restriction \overline{A}$.

(13) Let given T. Suppose that for every F such that F is open and a cover of T there exists G such that $G \subseteq F$ and G is a cover of T and $\operatorname{Card} G \subseteq C$.

Let given A. If A is closed and discrete, then $\operatorname{Card} A \subseteq C$.

(14) Let given T_3. Suppose that for every A_3 such that A_3 is closed and discrete holds $\operatorname{Card} A_3 \subseteq i_1$. Let given A_3. If A_3 is discrete, then $\operatorname{Card} A_3 \subseteq i_1$.

(15) Let given T. Suppose that for every A such that A is discrete holds $\operatorname{Card} A \subseteq C$. Let given F. Suppose F is open and $\emptyset \notin F$ and for all A, B such that A, $B \in F$ and $A \neq B$ holds A misses B. Then $\operatorname{Card} F \subseteq C$.

(16) For every F such that F is a cover of T there exists G such that $G \subseteq F$ and G is a cover of T and $\operatorname{Card} G \subseteq \operatorname{Card}(\Omega_T)$.

(17) If A_3 is dense, then $\operatorname{weight} T_3 \subseteq \operatorname{Card} \omega \cdot \operatorname{Card} A_3$.

2. Main Properties

Next we state several propositions:

(18) $\operatorname{weight} T_3 \subseteq i_1$ if and only if for every F_1 such that F_1 is open and a cover of T_3 there exists G_1 such that $G_1 \subseteq F_1$ and G_1 is a cover of T_3 and $\operatorname{Card} G_1 \subseteq i_1$.

(19) $\operatorname{weight} T_3 \subseteq i_1$ iff for every A_3 such that A_3 is closed and discrete holds $\operatorname{Card} A_3 \subseteq i_1$.

(20) $\operatorname{weight} T_3 \subseteq i_1$ iff for every A_3 such that A_3 is discrete holds $\operatorname{Card} A_3 \subseteq i_1$.

(21) $\operatorname{weight} T_3 \subseteq i_1$ if and only if for every F_1 such that F_1 is open and $\emptyset \notin F_1$ and for all A_3, B_1 such that A_3, $B_1 \in F_1$ and $A_3 \neq B_1$ holds A_3 misses B_1 holds $\operatorname{Card} F_1 \subseteq i_1$.

(22) $\operatorname{weight} T_3 \subseteq i_1$ iff $\operatorname{density} T_3 \subseteq i_1$.

(23) Let B be a basis of T_3. Suppose that for every F_1 such that F_1 is open and a cover of T_3 there exists G_1 such that $G_1 \subseteq F_1$ and G_1 is a cover of T_3 and $\operatorname{Card} G_1 \subseteq i_1$. Then there exists a basis u_1 of T_3 such that $u_1 \subseteq B$ and $\operatorname{Card} u_1 \subseteq i_1$.

3. Properties of Lindelöf spaces

Let us consider T. We say that T is Lindelöf if and only if:

(Def. 2) For every F such that F is open and a cover of T there exists G such that $G \subseteq F$ and G is a cover of T and countable.

Next we state the proposition

(24) For every basis B of T_3 such that T_3 is Lindelöf there exists a basis B' of T_3 such that $B' \subseteq B$ and B' is countable.

Let us observe that every metrizable topological space which is Lindelöf is also second-countable.

Let us note that every metrizable topological space which is Lindelöf is also separable and every metrizable topological space which is separable is also Lindelöf.

One can verify the following observations:

* there exists a non empty topological space which is Lindelöf and metrizable,
* every topological space which is second-countable is also Lindelöf,
* every topological space which is T_3 and Lindelöf is also T_4, and
* every topological space which is countable is also Lindelöf.

Let n be a natural number. Note that the topological structure of $\mathcal{E}^n_{\mathrm{T}}$ is second-countable.

Let T be a Lindelöf topological space and let A be a closed subset of T. One can verify that $T \restriction A$ is Lindelöf.

Let T_3 be a Lindelöf metrizable topological space and let A be a subset of T_3. One can verify that $T_3 \restriction A$ is Lindelöf.

Let us consider T and let A, B, L be subsets of T. We say that L separates A, B if and only if:

(Def. 3) There exist open subsets U, W of T such that $A \subseteq U$ and $B \subseteq W$ and U misses W and $L = (U \cup W)^{\mathrm{c}}$.

The following two propositions are true:

(25) If A_3 and B_1 are separated, then there exists a subset L of T_3 such that L separates A_3, B_1.

(26) Let M be a subset of T_3, A_1, A_2 be closed subsets of T_3, and V_1, V_2 be open subsets of T_3. Suppose $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$ and $\overline{V_1}$ misses $\overline{V_2}$. Let m_1, m_2, m_3 be subsets of $T_3 \restriction M$. Suppose $m_1 = M \cap \overline{V_1}$ and $m_2 = M \cap \overline{V_2}$ and m_3 separates m_1, m_2. Then there exists a subset L of T_3 such that L separates A_1, A_2 and $M \cap L \subseteq m_3$.

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Small Inductive Dimension of Topological Spaces

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Summary. We present the concept and basic properties of the Menger-Urysohn small inductive dimension of topological spaces according to the books [7]. Namely, the paper includes the formalization of main theorems from Sections 1.1 and 1.2.

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The terminology and notation used here are introduced in the following articles: [17], [8], [15], [5], [16], [6], [18], [14], [1], [2], [3], [13], [11], [9], [12], [19], [20], [10], and [4].

1. Preliminaries

For simplicity, we adopt the following rules: T, T_1, T_2 denote topological spaces, A, B denote subsets of T, F denotes a subset of $T\restriction A$, G, G_1, G_2 denote families of subsets of T, U, W denote open subsets of $T\restriction A$, p denotes a point of $T\restriction A$, n denotes a natural number, and I denotes an integer.

One can prove the following propositions:

(1) $\mathrm{Fr}(B \cap A) \subseteq \mathrm{Fr}\, B \cap A$.

(2) T is a T_4 space if and only if for all closed subsets A, B of T such that A misses B there exist open subsets U, W of T such that $A \subseteq U$ and $B \subseteq W$ and $\overline{U}$ misses $\overline{W}$.

Let us consider T. The sequence of ind of T yields a sequence of subsets of $2^{\text{the carrier of } T}$ and is defined by the conditions (Def. 1).

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(Def. 1)(i) (The sequence of ind of T)(0) $= \{\emptyset_T\}$, and

(ii) $A \in$ (the sequence of ind of T)$(n+1)$ iff $A \in$ (the sequence of ind of T)(n) or for all p, U such that $p \in U$ there exists W such that $p \in W$ and $W \subseteq U$ and $\operatorname{Fr} W \in$ (the sequence of ind of T)(n).

Let us consider T. Note that the sequence of ind of T is ascending.

We now state the proposition

(3) For every F such that $F = B$ holds $F \in$ (the sequence of ind of $T \restriction A$)(n) iff $B \in$ (the sequence of ind of T)(n).

Let us consider T, A. We say that A has finite small inductive dimension if and only if:

(Def. 2) There exists n such that $A \in$ (the sequence of ind of T)(n).

Let us consider T, A. We introduce A is finite-ind as a synonym of A has finite small inductive dimension.

Let us consider T, G. We say that G has finite small inductive dimension if and only if:

(Def. 3) There exists n such that $G \subseteq$ (the sequence of ind of T)(n).

Let us consider T, G. We introduce G is finite-ind as a synonym of G has finite small inductive dimension.

The following proposition is true

(4) If $A \in G$ and G is finite-ind, then A is finite-ind.

Let us consider T. One can check the following observations:

* every subset of T which is finite is also finite-ind,
* there exists a subset of T which is finite-ind,
* every family of subsets of T which is empty is also finite-ind, and
* there exists a family of subsets of T which is non empty and finite-ind.

Let T be a non empty topological space. One can check that there exists a subset of T which is non empty and finite-ind.

Let us consider T. We say that T has finite small inductive dimension if and only if:

(Def. 4) Ω_T has finite small inductive dimension.

Let us consider T. We introduce T is finite-ind as a synonym of T has finite small inductive dimension.

One can verify that every topological space which is empty is also finite-ind.

Let X be a set. Note that $\{X\}_{\rm top}$ is finite-ind.

One can check that there exists a topological space which is non empty and finite-ind.

In the sequel A_1 is a finite-ind subset of T and T_3 is a finite-ind topological space.

2. Small Inductive Dimension

Let us consider T and let us consider A. Let us assume that A is finite-ind. The functor $\operatorname{ind} A$ yields an integer and is defined as follows:

(Def. 5) $A \in$ (the sequence of ind of T)($\operatorname{ind} A + 1$) and $A \notin$ (the sequence of ind of T)($\operatorname{ind} A$).

We now state two propositions:

(5) $-1 \leq \operatorname{ind} A_1$.

(6) $\operatorname{ind} A_1 = -1$ iff A_1 is empty.

Let T be a non empty topological space and let A be a non empty finite-ind subset of T. Observe that $\operatorname{ind} A$ is natural.

The following three propositions are true:

(7) $\operatorname{ind} A_1 \leq n - 1$ iff $A_1 \in$ (the sequence of ind of T)(n).

(8) For every finite subset A of T holds $\operatorname{ind} A < \overline{\overline{A}}$.

(9) $\operatorname{ind} A_1 \leq n$ if and only if for every point p of $T \restriction A_1$ and for every open subset U of $T \restriction A_1$ such that $p \in U$ there exists an open subset W of $T \restriction A_1$ such that $p \in W$ and $W \subseteq U$ and $\operatorname{Fr} W$ is finite-ind and $\operatorname{ind} \operatorname{Fr} W \leq n - 1$.

Let us consider T and let us consider G. Let us assume that G is finite-ind. The functor $\operatorname{ind} G$ yielding an integer is defined by the conditions (Def. 6).

(Def. 6)(i) $G \subseteq$ (the sequence of ind of T)($\operatorname{ind} G + 1$),

(ii) $-1 \leq \operatorname{ind} G$, and

(iii) for every integer i such that $-1 \leq i$ and $G \subseteq$ (the sequence of ind of T)($i + 1$) holds $\operatorname{ind} G \leq i$.

The following propositions are true:

(10) $\operatorname{ind} G = -1$ and G is finite-ind iff $G \subseteq \{\emptyset_T\}$.

(11) G is finite-ind and $\operatorname{ind} G \leq I$ iff $-1 \leq I$ and for every A such that $A \in G$ holds A is finite-ind and $\operatorname{ind} A \leq I$.

(12) If G_1 is finite-ind and $G_2 \subseteq G_1$, then G_2 is finite-ind and $\operatorname{ind} G_2 \leq \operatorname{ind} G_1$.

Let us consider T and let G_1, G_2 be finite-ind families of subsets of T. Observe that $G_1 \cup G_2$ is finite-ind.

The following proposition is true

(13) If G is finite-ind and G_1 is finite-ind and $\operatorname{ind} G \leq I$ and $\operatorname{ind} G_1 \leq I$, then $\operatorname{ind}(G \cup G_1) \leq I$.

Let us consider T. The functor $\operatorname{ind} T$ yields an integer and is defined as follows:

(Def. 7) $\operatorname{ind} T = \operatorname{ind}(\Omega_T)$.

Let T be a non empty finite-ind topological space. One can verify that $\operatorname{ind} T$ is natural.

The following three propositions are true:

(14) For every non empty set X holds $\operatorname{ind}(\{X\}_{\rm top}) = 0$.

(15) Given n such that let p be a point of T and U be an open subset of T. Suppose $p \in U$. Then there exists an open subset W of T such that $p \in W$ and $W \subseteq U$ and $\operatorname{Fr} W$ is finite-ind and $\operatorname{ind} \operatorname{Fr} W \leq n-1$. Then T is finite-ind.

(16) $\operatorname{ind} T_3 \leq n$ if and only if for every point p of T_3 and for every open subset U of T_3 such that $p \in U$ there exists an open subset W of T_3 such that $p \in W$ and $W \subseteq U$ and $\operatorname{Fr} W$ is finite-ind and $\operatorname{ind} \operatorname{Fr} W \leq n-1$.

3. Monotonicity of the Small Inductive Dimension

Let us consider T_3. Observe that every subset of T_3 is finite-ind.

Let us consider T, A_1. Note that $T \restriction A_1$ is finite-ind.

One can prove the following propositions:

(17) $\operatorname{ind}(T \restriction A_1) = \operatorname{ind} A_1$.

(18) If $T \restriction A$ is finite-ind, then A is finite-ind.

(19) If $A \subseteq A_1$, then A is finite-ind and $\operatorname{ind} A \leq \operatorname{ind} A_1$.

(20) For every subset A of T_3 holds $\operatorname{ind} A \leq \operatorname{ind} T_3$.

(21) If $F = B$ and B is finite-ind, then F is finite-ind and $\operatorname{ind} F = \operatorname{ind} B$.

(22) If $F = B$ and F is finite-ind, then B is finite-ind and $\operatorname{ind} F = \operatorname{ind} B$.

(23) Let T be a non empty topological space. Suppose T is a T_3 space. Then T is finite-ind and $\operatorname{ind} T \leq n$ if and only if for every closed subset A of T and for every point p of T such that $p \notin A$ there exists a subset L of T such that L separates $\{p\}$, A and L is finite-ind and $\operatorname{ind} L \leq n-1$.

(24) If T_1 and T_2 are homeomorphic, then T_1 is finite-ind iff T_2 is finite-ind.

(25) If T_1 and T_2 are homeomorphic and T_1 is finite-ind, then $\operatorname{ind} T_1 = \operatorname{ind} T_2$.

(26) Let A_2 be a subset of T_1 and A_3 be a subset of T_2. Suppose A_2 and A_3 are homeomorphic. Then A_2 is finite-ind if and only if A_3 is finite-ind.

(27) Let A_2 be a subset of T_1 and A_3 be a subset of T_2. If A_2 and A_3 are homeomorphic and A_2 is finite-ind, then $\operatorname{ind} A_2 = \operatorname{ind} A_3$.

(28) If $T_1 \times T_2$ is finite-ind, then $T_2 \times T_1$ is finite-ind and $\operatorname{ind}(T_1 \times T_2) = \operatorname{ind}(T_2 \times T_1)$.

(29) For every family G_3 of subsets of $T \restriction A$ such that G_3 is finite-ind and $G_3 = G$ holds G is finite-ind and $\operatorname{ind} G = \operatorname{ind} G_3$.

(30) For every family G_3 of subsets of $T \restriction A$ such that G is finite-ind and $G_3 = G$ holds G_3 is finite-ind and $\operatorname{ind} G = \operatorname{ind} G_3$.

4. Basic Properties 0-dimensional Topological Spaces

Next we state several propositions:

(31) T is finite-ind and $\operatorname{ind} T \leq n$ if and only if there exists a basis B_1 of T such that for every A such that $A \in B_1$ holds $\operatorname{Fr} A$ is finite-ind and $\operatorname{ind} \operatorname{Fr} A \leq n-1$.

(32) Let given T. Suppose that

(i) T is a T_1 space, and

(ii) for all closed subsets A, B of T such that A misses B there exist closed subsets A', B' of T such that A' misses B' and $A' \cup B' = \Omega_T$ and $A \subseteq A'$ and $B \subseteq B'$.

Then T is finite-ind and $\operatorname{ind} T \leq 0$.

(33) Let X be a set and f be a sequence of subsets of X. Then there exists a sequence g of subsets of X such that

(i) $\bigcup \operatorname{rng} f = \bigcup \operatorname{rng} g$,

(ii) for all natural numbers i, j such that $i \neq j$ holds $g(i)$ misses $g(j)$, and

(iii) for every n there exists a finite family f_1 of subsets of X such that $f_1 = \{f(i); i$ ranges over elements of $\mathbb{N}$: $i < n\}$ and $g(n) = f(n) \setminus \bigcup f_1$.

(34) Let given T. Suppose T is finite-ind and $\operatorname{ind} T \leq 0$ and T is Lindelöf. Let A, B be closed subsets of T. Suppose A misses B. Then there exist closed subsets A', B' of T such that A' misses B' and $A' \cup B' = \Omega_T$ and $A \subseteq A'$ and $B \subseteq B'$.

(35) Let given T. Suppose T is a T_1 space and Lindelöf. Then T is finite-ind and $\operatorname{ind} T \leq 0$ if and only if for all closed subsets A, B of T such that A misses B holds $\emptyset_T$ separates A, B.

(36) Let given T. Suppose that

(i) T is a T_4 space, a T_1 space, and Lindelöf, and

(ii) there exists a family F of subsets of T such that F is closed, a cover of T, countable, and finite-ind and $\operatorname{ind} F \leq 0$.

Then T is finite-ind and $\operatorname{ind} T \leq 0$.

In the sequel T_4 is a metrizable topological space.

We now state four propositions:

(37) Let A, B be closed subsets of T_4. Suppose A misses B. Let N_1 be a finite-ind subset of T_4. Suppose $\operatorname{ind} N_1 \leq 0$ and $T_4 \restriction N_1$ is second-countable. Then there exists a subset L of T_4 such that L separates A, B and L misses N_1.

(38) Let N_1 be a subset of T_4. Suppose $T_4 \restriction N_1$ is second-countable. Then N_1 is finite-ind and $\operatorname{ind} N_1 \leq 0$ if and only if for every point p of T_4 and for every open subset U of T_4 such that $p \in U$ there exists an open subset W of T_4 such that $p \in W$ and $W \subseteq U$ and N_1 misses $\operatorname{Fr} W$.

(39) Let N_1 be a subset of T_4. Suppose $T_4 \restriction N_1$ is second-countable. Then N_1 is finite-ind and $\operatorname{ind} N_1 \leq 0$ if and only if there exists a basis B of T_4 such that for every subset A of T_4 such that $A \in B$ holds N_1 misses $\operatorname{Fr} A$.

(40) Let N_1, A be subsets of T_4. Suppose $T_4 \restriction N_1$ is second-countable and N_1 is finite-ind and A is finite-ind and $\operatorname{ind} N_1 \leq 0$. Then $A \cup N_1$ is finite-ind and $\operatorname{ind}(A \cup N_1) \leq \operatorname{ind} A + 1$.

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On Rough Subgroup of a Group

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Summary. This article describes a rough subgroup with respect to a normal subgroup of a group, and some properties of the lower and the upper approximations in a group.

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The articles [2], [3], [1], [4], and [5] provide the terminology and notation for this paper.

For simplicity, we adopt the following rules: G denotes a group, A, B denote non empty subsets of G, N, H, H_1, H_2 denote subgroups of G, and x, a, b denote elements of G.

Next we state a number of propositions:

(1) For every normal subgroup N of G and for all elements x_1, x_2 of G holds $x_1 \cdot N \cdot (x_2 \cdot N) = (x_1 \cdot x_2) \cdot N$.

(2) For every group G and for every subgroup N of G and for all elements x, y of G such that $y \in x \cdot N$ holds $x \cdot N = y \cdot N$.

(3) Let N be a subgroup of G, H be a subgroup of G, and x be an element of G. If $x \cdot N$ meets $\overline{H}$, then there exists an element y of G such that $y \in x \cdot N$ and $y \in H$.

(4) For all elements x, y of G and for every normal subgroup N of G such that $y \in N$ holds $x \cdot y \cdot x^{-1} \in N$.

(5) For every subgroup N of G such that for all elements x, y of G such that $y \in N$ holds $x \cdot y \cdot x^{-1} \in N$ holds N is normal.

(6) $x \in H_1 \cdot H_2$ iff there exist a, b such that $x = a \cdot b$ and $a \in H_1$ and $b \in H_2$.

(7) Let G be a group and N_1, N_2 be strict normal subgroups of G. Then there exists a strict subgroup M of G such that the carrier of $M = N_1 \cdot N_2$.

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(8) Let G be a group and N_1, N_2 be strict normal subgroups of G. Then there exists a strict normal subgroup M of G such that the carrier of $M = N_1 \cdot N_2$.

(9) Let G be a group and N, N_1, N_2 be subgroups of G. Suppose the carrier of $N = N_1 \cdot N_2$. Then N_1 is a subgroup of N and N_2 is a subgroup of N.

(10) Let N, N_1, N_2 be normal subgroups of G and a, b be elements of G. If the carrier of $N = N_1 \cdot N_2$, then $a \cdot N_1 \cdot (b \cdot N_2) = (a \cdot b) \cdot N$.

(11) For every normal subgroup N of G and for every x holds $x \cdot N \cdot x^{-1} \subseteq \overline{N}$.

Let G be a group, let A be a subset of G, and let N be a subgroup of G. The functor $N`A$ yielding a subset of G is defined by:

(Def. 1) $N`A = \{x \in G\colon x \cdot N \subseteq A\}$.

The functor $N{\sim}A$ yielding a subset of G is defined as follows:

(Def. 2) $N{\sim}A = \{x \in G\colon x \cdot N$ meets $A\}$.

Next we state a number of propositions:

(12) For every element x of G such that $x \in N`A$ holds $x \cdot N \subseteq A$.

(13) For every element x of G such that $x \cdot N \subseteq A$ holds $x \in N`A$.

(14) For every element x of G such that $x \in N{\sim}A$ holds $x \cdot N$ meets A.

(15) For every element x of G such that $x \cdot N$ meets A holds $x \in N{\sim}A$.

(16) $N`A \subseteq A$.

(17) $A \subseteq N{\sim}A$.

(18) $N`A \subseteq N{\sim}A$.

(19) $N{\sim}A \cup B = (N{\sim}A) \cup (N{\sim}B)$.

(20) $N`A \cap B = (N`A) \cap (N`B)$.

(21) If $A \subseteq B$, then $N`A \subseteq N`B$.

(22) If $A \subseteq B$, then $N{\sim}A \subseteq N{\sim}B$.

(23) $(N`A) \cup (N`B) \subseteq N`(A \cup B)$.

(24) $N{\sim}A \cup B = (N{\sim}A) \cup (N{\sim}B)$.

(25) If N is a subgroup of H, then $H`A \subseteq N`A$.

(26) If N is a subgroup of H, then $N{\sim}A \subseteq H{\sim}A$.

(27) For every group G and for all non empty subsets A, B of G and for every normal subgroup N of G holds $(N`A) \cdot (N`B) \subseteq N`A \cdot B$.

(28) For every element x of G such that $x \in N{\sim}A \cdot B$ holds $x \cdot N$ meets $A \cdot B$.

(29) For every group G and for all non empty subsets A, B of G and for every normal subgroup N of G holds $(N{\sim}A) \cdot (N{\sim}B) = N{\sim}A \cdot B$.

(30) For every element x of G such that $x \in N{\sim}N`(N{\sim}A)$ holds $x \cdot N$ meets $N`(N{\sim}A)$.

(31) For every element x of G such that $x \in N`(N{\sim}A)$ holds $x \cdot N \subseteq N{\sim}A$.

(32) For every element x of G such that $x \in N{\sim}N{\sim}A$ holds $x \cdot N$ meets $N{\sim}A$.

(33) For every element x of G such that $x \in N{\sim}N`A$ holds $x \cdot N$ meets $N`A$.

(34) $N`(N`A) = N`A$.

(35) $N{\sim}A = N{\sim}N{\sim}A$.

(36) $N`(N`A) \subseteq N{\sim}N{\sim}A$.

(37) $N{\sim}N`A \subseteq A$.

(38) $N`(N{\sim}N`A) = N`A$.

(39) If $A \subseteq N`(N{\sim}A)$, then $N{\sim}A \subseteq N{\sim}N`(N{\sim}A)$.

(40) $N{\sim}N`(N{\sim}A) = N{\sim}A$.

(41) For every element x of G such that $x \in N`(N`A)$ holds $x \cdot N \subseteq N`A$.

(42) $N`(N`A) = N{\sim}N`A$.

(43) $N{\sim}N{\sim}A = N`(N{\sim}A)$.

(44) For all subgroups N, N_1, N_2 of G such that $N = N_1 \cap N_2$ holds $N{\sim}A \subseteq (N_1{\sim}A) \cap (N_2{\sim}A)$.

(45) For all subgroups N, N_1, N_2 of G such that $N = N_1 \cap N_2$ holds $(N_1`A) \cap (N_2`A) \subseteq N`A$.

(46) Let N_1, N_2 be strict normal subgroups of G. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and $N`A \subseteq (N_1`A) \cap (N_2`A)$.

(47) Let N_1, N_2 be strict normal subgroups of G. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and $(N_1{\sim}A) \cup (N_2{\sim}A) \subseteq N{\sim}A$.

(48) Let N_1, N_2 be strict normal subgroups of G. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and $N{\sim}A \subseteq ((N_1{\sim}A) \cdot N_2) \cap ((N_2{\sim}A) \cdot N_1)$.

In the sequel N_1, N_2 are subgroups of G.

Let G be a group and let H, N be subgroups of G. The functor $N`H$ yielding a subset of G is defined by:

(Def. 3) $N`H = \{x \in G: x \cdot N \subseteq \overline{H}\}$.

The functor $N{\sim}H$ yields a subset of G and is defined as follows:

(Def. 4) $N{\sim}H = \{x \in G: x \cdot N \text{ meets } \overline{H}\}$.

We now state a number of propositions:

(49) For every element x of G such that $x \in N`H$ holds $x \cdot N \subseteq \overline{H}$.

(50) For every element x of G such that $x \cdot N \subseteq \overline{H}$ holds $x \in N`H$.

(51) For every element x of G such that $x \in N{\sim}H$ holds $x \cdot N$ meets $\overline{H}$.

(52) For every element x of G such that $x \cdot N$ meets $\overline{H}$ holds $x \in N{\sim}H$.

(53) $N`H \subseteq \overline{H}$.

(54) $\overline{H} \subseteq N{\sim}H$.

(55) $N`H \subseteq N{\sim}H$.

(56) If H_1 is a subgroup of H_2, then $N{\sim}H_1 \subseteq N{\sim}H_2$.

(57) If N_1 is a subgroup of N_2, then $N_1{\sim}H \subseteq N_2{\sim}H$.

(58) If N_1 is a subgroup of N_2, then $N_1{\sim}N_1 \subseteq N_2{\sim}N_2$.

(59) If H_1 is a subgroup of H_2, then $N`H_1 \subseteq N`H_2$.

(60) If N_1 is a subgroup of N_2, then $N_2`H \subseteq N_1`H$.

(61) If N_1 is a subgroup of N_2, then $N_2`N_1 \subseteq N_1`N_2$.

(62) For every normal subgroup N of G holds $(N`H_1) \cdot (N`H_2) \subseteq N`H_1 \cdot H_2$.

(63) For every normal subgroup N of G holds $(N{\sim}H_1) \cdot (N{\sim}H_2) = N{\sim}H_1 \cdot H_2$.

(64) For all subgroups N, N_1, N_2 of G such that $N = N_1 \cap N_2$ holds $N{\sim}H \subseteq (N_1{\sim}H) \cap (N_2{\sim}H)$.

(65) For all subgroups N, N_1, N_2 of G such that $N = N_1 \cap N_2$ holds $(N_1`H) \cap (N_2`H) \subseteq N`H$.

(66) Let N_1, N_2 be strict normal subgroups of G. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and $N`H \subseteq (N_1`H) \cap (N_2`H)$.

(67) Let N_1, N_2 be strict normal subgroups of G. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and $(N_1{\sim}H) \cup (N_2{\sim}H) \subseteq N{\sim}H$.

(68) Let N_1, N_2 be strict normal subgroups of G. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and $(N_1`H) \cdot (N_2`H) \subseteq N`H$.

(69) Let N_1, N_2 be strict normal subgroups of G. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and $(N_1{\sim}H) \cdot (N_2{\sim}H) \subseteq N{\sim}H$.

(70) Let N_1, N_2 be strict normal subgroups of G. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and $N{\sim}H \subseteq ((N_1{\sim}H) \cdot N_2) \cap ((N_2{\sim}H) \cdot N_1)$.

(71) Let H be a subgroup of G and N be a normal subgroup of G. Then there exists a strict subgroup M of G such that the carrier of $M = N{\sim}H$.

(72) Let H be a subgroup of G and N be a normal subgroup of G. Suppose N is a subgroup of H. Then there exists a strict subgroup M of G such that the carrier of $M = N`H$.

(73) For all normal subgroups H, N of G there exists a strict normal subgroup M of G such that the carrier of $M = N{\sim}H$.

(74) Let H, N be normal subgroups of G. Suppose N is a subgroup of H. Then there exists a strict normal subgroup M of G such that the carrier of $M = N`H$.

(75) Let N, N_1 be normal subgroups of G. Suppose N_1 is a subgroup of N. Then there exist strict normal subgroups N_2, N_3 of G such that the carrier of $N_2 = N_1{\sim}N$ and the carrier of $N_3 = N_1$‘N and N_2‘$N \subseteq N_3$‘N.

(76) Let N, N_1 be normal subgroups of G. Suppose N_1 is a subgroup of N. Then there exist strict normal subgroups N_2, N_3 of G such that the carrier of $N_2 = N_1{\sim}N$ and the carrier of $N_3 = N_1$‘N and $N_3{\sim}N \subseteq N_2{\sim}N$.

(77) Let N, N_1 be normal subgroups of G. Suppose N_1 is a subgroup of N. Then there exist strict normal subgroups N_2, N_3 of G such that the carrier of $N_2 = N_1{\sim}N$ and the carrier of $N_3 = N_1$‘N and N_2‘$N \subseteq N_3{\sim}N$.

(78) Let N, N_1 be normal subgroups of G. Suppose N_1 is a subgroup of N. Then there exist strict normal subgroups N_2, N_3 of G such that the carrier of $N_2 = N_1{\sim}N$ and the carrier of $N_3 = N_1$‘N and N_3‘$N \subseteq N_2{\sim}N$.

(79) Let N, N_1, N_2 be normal subgroups of G. Suppose N_1 is a subgroup of N_2. Then there exist strict normal subgroups N_3, N_4 of G such that the carrier of $N_3 = N{\sim}N_1$ and the carrier of $N_4 = N{\sim}N_2$ and $N_3{\sim}N_1 \subseteq N_4{\sim}N_1$.

(80) Let N, N_1 be normal subgroups of G. Then there exists a strict normal subgroup N_2 of G such that the carrier of $N_2 = N$‘N and N‘$N_1 \subseteq N_2$‘N_1.

(81) Let N, N_1 be normal subgroups of G. Then there exists a strict normal subgroup N_2 of G such that the carrier of $N_2 = N{\sim}N$ and $N{\sim}N_1 \subseteq N_2{\sim}N_1$.

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Small Inductive Dimension of Topological Spaces. Part II

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Summary. In this paper we present basic properties of n-dimensional topological spaces according to the book [10]. In the article the formalization of Section 1.5 is completed.

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The papers [15], [1], [3], [9], [5], [8], [16], [2], [4], [6], [13], [12], [17], [14], [18], [7], and [11] provide the terminology and notation for this paper.

1. Order of a Family of Subsets of a Set

In this paper n denotes a natural number, X denotes a set, and F_1, G_1 denote families of subsets of X.

Let us consider X, F_1. We say that F_1 is finite-order if and only if:

(Def. 1) There exists n such that for every G_1 such that $G_1 \subseteq F_1$ and $n \in \operatorname{Card} G_1$ holds $\bigcap G_1$ is empty.

Let us consider X. Observe that there exists a family of subsets of X which is finite-order and every family of subsets of X which is finite is also finite-order.

Let us consider X, F_1. The functor order F_1 yielding an extended real number is defined as follows:

(Def. 2)(i) For every G_1 such that $\operatorname{order} F_1 + 1 \in \operatorname{Card} G_1$ and $G_1 \subseteq F_1$ holds $\bigcap G_1$ is empty and there exists G_1 such that $G_1 \subseteq F_1$ but $\operatorname{Card} G_1 = \operatorname{order} F_1 + 1$ but $\bigcap G_1$ is non empty or G_1 is empty if F_1 is finite-order,

(ii) $\operatorname{order} F_1 = +\infty$, otherwise.

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Let us consider X and let F be a finite-order family of subsets of X. Observe that order $F+1$ is natural and order F is integer.

Next we state three propositions:

(1) If order $F_1 \leq n$, then F_1 is finite-order.

(2) If order $F_1 \leq n$, then for every G_1 such that $G_1 \subseteq F_1$ and $n+1 \in \operatorname{Card} G_1$ holds $\bigcap G_1$ is empty.

(3) If for every finite family G of subsets of X such that $G \subseteq F_1$ and $n+1 < \overline{\overline{G}}$ holds $\bigcap G$ is empty, then order $F_1 \leq n$.

2. Basic Properties of n-dimensional Topological Spaces

One can verify that there exists a topological space which is finite-ind, second-countable, and metrizable.

For simplicity, we adopt the following convention: T_1 is a metrizable topological space, T_2, T_3 are finite-ind second-countable metrizable topological spaces, A, B, L, H are subsets of T_1, U, W are open subsets of T_1, p is a point of T_1, F, G are finite families of subsets of T_1, and I is an integer.

We now state several propositions:

(4) Let given T_1. Suppose that

(i) T_1 is second-countable, and

(ii) there exists F such that F is closed, a cover of T_1, countable, and finite-ind and $\operatorname{ind} F \leq n$.

Then T_1 is finite-ind and $\operatorname{ind} T_1 \leq n$.

(5) Let A, B be finite-ind subsets of T_1. Suppose A is closed and $T_1 \restriction (A \cup B)$ is second-countable and $\operatorname{ind} A \leq I$ and $\operatorname{ind} B \leq I$. Then $\operatorname{ind}(A \cup B) \leq I$ and $A \cup B$ is finite-ind.

(6) Let given T_1. Suppose T_1 is second-countable and finite-ind and $\operatorname{ind} T_1 \leq n$. Then there exist A, B such that $\Omega_{(T_1)} = A \cup B$ and A misses B and $\operatorname{ind} A \leq n-1$ and $\operatorname{ind} B \leq 0$.

(7) Let given T_1. Suppose T_1 is second-countable and finite-ind and $\operatorname{ind} T_1 \leq I$. Then there exists F such that

(i) F is a cover of T_1 and finite-ind,

(ii) $\operatorname{ind} F \leq 0$,

(iii) $\overline{\overline{F}} \leq I+1$, and

(iv) for all A, B such that $A, B \in F$ and A meets B holds $A = B$.

(8) Let given T_1. Suppose T_1 is second-countable and there exists F such that F is a cover of T_1 and finite-ind and $\operatorname{ind} F \leq 0$ and $\overline{\overline{F}} \leq I+1$. Then T_1 is finite-ind and $\operatorname{ind} T_1 \leq I$.

Let T_1 be a second-countable metrizable topological space and let A, B be finite-ind subsets of T_1. One can check that $A \cup B$ is finite-ind.

Next we state two propositions:

(9) If A is finite-ind and B is finite-ind and $T_1 \restriction (A \cup B)$ is second-countable, then $A \cup B$ is finite-ind and $\operatorname{ind}(A \cup B) \leq \operatorname{ind} A + \operatorname{ind} B + 1$.

(10) For all topological spaces T_4, T_5 and for every subset A_1 of T_4 and for every subset A_2 of T_5 holds $\operatorname{Fr}(A_1 \times A_2) = \operatorname{Fr} A_1 \times \overline{A_2} \cup \overline{A_1} \times \operatorname{Fr} A_2$.

Let us consider T_2, T_3. Observe that $T_2 \times T_3$ is finite-ind.

We now state several propositions:

(11) Let given A, B. Suppose A is closed and B is closed and A misses B. Let given H. Suppose $\operatorname{ind} H \leq n$ and $T_1 \restriction H$ is second-countable and finite-ind. Then there exists L such that L separates A, B and $\operatorname{ind}(L \cap H) \leq n - 1$.

(12) Let given T_1. Suppose T_1 is finite-ind and second-countable and $\operatorname{ind} T_1 \leq n$. Let given A, B. Suppose A is closed and B is closed and A misses B. Then there exists L such that L separates A, B and $\operatorname{ind} L \leq n - 1$.

(13) Let given H. Suppose $T_1 \restriction H$ is second-countable. Then H is finite-ind and $\operatorname{ind} H \leq n$ if and only if for all p, U such that $p \in U$ there exists W such that $p \in W$ and $W \subseteq U$ and $H \cap \operatorname{Fr} W$ is finite-ind and $\operatorname{ind}(H \cap \operatorname{Fr} W) \leq n - 1$.

(14) Let given H. Suppose $T_1 \restriction H$ is second-countable. Then H is finite-ind and $\operatorname{ind} H \leq n$ if and only if there exists a basis B_1 of T_1 such that for every A such that $A \in B_1$ holds $H \cap \operatorname{Fr} A$ is finite-ind and $\operatorname{ind}(H \cap \operatorname{Fr} A) \leq n - 1$.

(15) If T_2 is non empty or T_3 is non empty, then $\operatorname{ind}(T_2 \times T_3) \leq \operatorname{ind} T_2 + \operatorname{ind} T_3$.

(16) If $\operatorname{ind} T_3 = 0$, then $\operatorname{ind}(T_2 \times T_3) = \operatorname{ind} T_2$.

3. Small Inductive Dimension of Euclidean Spaces

For simplicity, we follow the rules: u denotes a point of $\mathcal{E}^1$, U denotes a point of $\mathcal{E}^1_{\mathrm{T}}$, r, u_1 denote real numbers, and s denotes a real number.

Next we state three propositions:

(17) If $\langle u_1 \rangle = u$ and $r > 0$, then $\overline{\operatorname{Ball}}(u, r) = \{\langle s \rangle : u_1 - r \leq s \ \wedge \ s \leq u_1 + r\}$.

(18) If $\langle u_1 \rangle = U$ and $r > 0$, then $\operatorname{Fr} \operatorname{Ball}(U, r) = \{\langle u_1 - r \rangle, \langle u_1 + r \rangle\}$.

(19) Let T be a topological space and A be a countable subset of T. If $T \restriction A$ is a T_4 space, then A is finite-ind and $\operatorname{ind} A \leq 0$.

Let T_1 be a metrizable topological space. Observe that every subset of T_1 which is countable is also finite-ind.

Let n be a natural number. Observe that $\mathcal{E}^n_{\mathrm{T}}$ is finite-ind.

One can prove the following propositions:

(20) If $n \leq 1$, then $\operatorname{ind}(\mathcal{E}^n_{\mathrm{T}}) = n$.

(21) $\operatorname{ind}(\mathcal{E}^n_{\mathrm{T}}) \leq n$.

(22) Let given A. Suppose $T_1 \restriction A$ is second-countable and finite-ind and $\operatorname{ind} A \leq 0$. Let given F. Suppose F is open and a cover of A. Then there exists a function g from F into $2^{\text{the carrier of } T_1}$ such that
  (i) $\operatorname{rng} g$ is open,
  (ii) $\operatorname{rng} g$ is a cover of A,
  (iii) for every set a such that $a \in F$ holds $g(a) \subseteq a$, and
  (iv) for all sets a, b such that $a, b \in F$ and $a \neq b$ holds $g(a)$ misses $g(b)$.

(23) Let given T_1. Suppose T_1 is second-countable and finite-ind and $\operatorname{ind} T_1 \leq n$. Let given F. Suppose F is open and a cover of T_1. Then there exists G such that G is open, a cover of T_1, and finer than F and $\overline{\overline{G}} \leq \overline{\overline{F}} \cdot (n+1)$ and $\operatorname{order} G \leq n$.

(24) Let given T_1. Suppose T_1 is finite-ind. Let given A. Suppose $\operatorname{ind}(A^{\mathrm{c}}) \leq n$ and $T_1 \restriction A^{\mathrm{c}}$ is second-countable. Let A_1, A_2 be closed subsets of T_1. Suppose $A = A_1 \cup A_2$. Then there exist closed subsets X_1, X_2 of T_1 such that $\Omega_{(T_1)} = X_1 \cup X_2$ and $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$ and $A_1 \cap X_2 = A_1 \cap A_2 = X_1 \cap A_2$ and $\operatorname{ind}(X_1 \cap X_2 \setminus A_1 \cap A_2) \leq n - 1$.

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