Contents

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Dilworth's Decomposition Theorem for Posets[1]

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Summary. The following theorem is due to Dilworth [8]: Let P be a partially ordered set. If the maximal number of elements in an independent subset (anti-chain) of P is k, then P is the union of k chains (cliques).

In this article we formalize an elegant proof of the above theorem for finite posets by Perles [13]. The result is then used in proving the case of infinite posets following the original proof of Dilworth [8].

A dual of Dilworth's theorem also holds: a poset with maximum clique m is a union of m independent sets. The proof of this dual fact is considerably easier; we follow the proof by Mirsky [11]. Mirsky states also a corollary that a poset of $r \times s + 1$ elements possesses a clique of size $r + 1$ or an independent set of size $s + 1$, or both. This corollary is then used to prove the result of Erdős and Szekeres [9].

Instead of using posets, we drop reflexivity and state the facts about anti-symmetric and transitive relations.

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The articles [1], [15], [14], [7], [2], [16], [3], [12], [17], [5], [10], [4], and [6] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *FraenkelFinCard1* deals with a finite non empty set $\mathcal{A}$, a finite set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

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$\overline{\overline{\mathcal{B}}} \leq \overline{\overline{\mathcal{A}}}$

provided the following condition is satisfied:

- $\mathcal{B} = \{\mathcal{F}(w); w$ ranges over elements of $\mathcal{A} : \mathcal{P}[w]\}$.

Next we state the proposition

(1) For all sets X, Y, x such that $x \notin X$ holds $X \setminus (Y \cup \{x\}) = X \setminus Y$.

Let us note that every set which is empty is also $\subseteq$-linear and there exists a set which is empty and $\subseteq$-linear.

Let X be a $\subseteq$-linear set. Note that every subset of X is $\subseteq$-linear.

One can prove the following four propositions:

(2) Let X, Y be sets, F be a family of subsets of X, and G be a family of subsets of Y. Then $F \cup G$ is a family of subsets of $X \cup Y$.

(3) Let X, Y be sets, F be a partition of X, and G be a partition of Y. If X misses Y, then $F \cup G$ is a partition of $X \cup Y$.

(4) For all sets X, Y and for every partition F of Y such that $Y \subset X$ holds $F \cup \{X \setminus Y\}$ is a partition of X.

(5) For every infinite set X and for every natural number n there exists a finite subset Y of X such that $\overline{\overline{Y}} > n$.

2. Cliques and Stable Sets

Let R be a relational structure and let S be a subset of R. We say that S is connected if and only if:

(Def. 1) The internal relation of R is connected in S.

Let R be a relational structure and let S be a subset of R. We introduce S is a clique as a synonym of S is connected.

Let R be a relational structure. Note that every subset of R which is trivial is also a clique.

Let R be a relational structure. One can check that there exists a subset of R which is a clique.

Let R be a relational structure. A clique of R is a clique subset of R.

We now state the proposition

(6) Let R be a relational structure and S be a subset of R. Then S is a clique of R if and only if for all elements a, b of R such that $a, b \in S$ and $a \neq b$ holds $a \leq b$ or $b \leq a$.

Let R be a relational structure. Observe that there exists a clique of R which is finite.

Let R be a reflexive relational structure. One can check that every subset of R which is connected is also strongly connected.

Let R be a non empty relational structure. Observe that there exists a clique of R which is finite and non empty.

One can prove the following propositions:

(7) Let R be a non empty relational structure and a_1, a_2 be elements of R. If $a_1 \neq a_2$ and $\{a_1, a_2\}$ is a clique of R, then $a_1 \leq a_2$ or $a_2 \leq a_1$.

(8) Let R be a non empty relational structure and a_1, a_2 be elements of R. If $a_1 \leq a_2$ or $a_2 \leq a_1$, then $\{a_1, a_2\}$ is a clique of R.

(9) For every relational structure R and for every clique C of R holds every subset of C is a clique of R.

(10) Let R be a relational structure, C be a finite clique of R, and n be a natural number. If $n \leq \overline{\overline{C}}$, then there exists a finite clique B of R such that $\overline{\overline{B}} = n$.

(11) Let R be a transitive relational structure, C be a clique of R, and x, y be elements of R. If x is maximal in C and $x \leq y$, then $C \cup \{y\}$ is a clique of R.

(12) Let R be a transitive relational structure, C be a clique of R, and x, y be elements of R. If x is minimal in C and $y \leq x$, then $C \cup \{y\}$ is a clique of R.

Let R be a relational structure and let S be a subset of R. We say that S is stable if and only if:

(Def. 2) For all elements x, y of R such that $x, y \in S$ and $x \neq y$ holds $x \not\leq y$ and $y \not\leq x$.

Let R be a relational structure. One can check that every subset of R which is trivial is also stable. Let R be a relational structure. Note that there exists a subset of R which is stable.

Let R be a relational structure. A stable set of R is a stable subset of R.

Let R be a relational structure. Note that there exists a stable set of R which is finite.

Let R be a non empty relational structure. Observe that there exists a stable set of R which is finite and non empty.

The following propositions are true:

(13) Let R be a non empty relational structure and a_1, a_2 be elements of R. If $a_1 \neq a_2$ and $\{a_1, a_2\}$ is a stable set of R, then $a_1 \not\leq a_2$ and $a_2 \not\leq a_1$.

(14) Let R be a non empty relational structure and a_1, a_2 be elements of R. If $a_1 \not\leq a_2$ and $a_2 \not\leq a_1$, then $\{a_1, a_2\}$ is a stable set of R.

(15) Let R be a relational structure, C be a clique of R, A be a stable set of R, and a, b be sets. If $a, b \in A$ and $a, b \in C$, then $a = b$.

(16) For every relational structure R and for every stable set A of R holds every subset of A is a stable set of R.

(17) Let R be a relational structure, A be a finite stable set of R, and n be a natural number. If $n \leq \overline{\overline{A}}$, then there exists a finite stable set B of R such that $\overline{\overline{B}} = n$.

3. Clique Number and Stability Number

Let R be a relational structure. We say that R has finite clique number if and only if:

(Def. 3) There exists a finite clique C of R such that for every finite clique D of R holds $\overline{\overline{D}} \leq \overline{\overline{C}}$.

Let us observe that every relational structure which is finite has also finite clique number and there exists a relational structure which is non empty, antisymmetric, and transitive and has finite clique number.

Let R be a relational structure with finite clique number. Observe that every clique of R is finite.

Let R be a relational structure with finite clique number. The functor $\omega(R)$ yields a natural number and is defined as follows:

(Def. 4) There exists a finite clique C of R such that $\overline{\overline{C}} = \omega(R)$ and for every finite clique T of R holds $\overline{\overline{T}} \leq \omega(R)$.

Let R be an empty relational structure. Note that $\omega(R)$ is empty.

Let R be a non empty relational structure with finite clique number. Observe that $\omega(R)$ is positive.

Next we state two propositions:

(18) For every non empty relational structure R with finite clique number such that Ω_R is a stable set of R holds $\omega(R) = 1$.

(19) For every relational structure R with finite clique number such that $\omega(R) = 1$ holds Ω_R is a stable set of R.

Let R be a relational structure. We say that R has finite stability number if and only if:

(Def. 5) There exists a finite stable set A of R such that for every finite stable set B of R holds $\overline{\overline{B}} \leq \overline{\overline{A}}$.

One can verify that every relational structure which is finite has also finite stability number and there exists a relational structure which is antisymmetric, transitive, and non empty and has finite stability number.

Let R be a relational structure with finite stability number. Note that every stable set of R is finite.

Let R be a relational structure with finite stability number. The functor $\alpha(R)$ yielding a natural number is defined by:

(Def. 6) There exists a finite stable set A of R such that $\overline{\overline{A}} = \alpha(R)$ and for every finite stable set T of R holds $\overline{\overline{T}} \leq \alpha(R)$.

Let R be an empty relational structure. Observe that $\alpha(R)$ is empty.

Let R be a non empty relational structure with finite stability number. One can verify that $\alpha(R)$ is positive.

We now state two propositions:

(20) For every non empty relational structure R with finite stability number such that Ω_R is a clique of R holds $\alpha(R) = 1$.

(21) For every relational structure R with finite stability number such that $\alpha(R) = 1$ holds Ω_R is a clique of R.

Let us mention that every relational structure which has finite clique number and finite stability number is also finite.

4. Lower and Upper Sets, Minimal and Maximal Elements

Let R be a relational structure and let X be a subset of R. The functor $\mathrm{Lower}\, X$ yields a subset of R and is defined by:

(Def. 7) $\mathrm{Lower}\, X = X \cup {\downarrow}X$.

The functor $\mathrm{Upper}\, X$ yielding a subset of R is defined as follows:

(Def. 8) $\mathrm{Upper}\, X = X \cup {\uparrow}X$.

One can prove the following propositions:

(22) Let R be an antisymmetric transitive relational structure, A be a stable set of R, and z be a set. If $z \in \mathrm{Upper}\, A$ and $z \in \mathrm{Lower}\, A$, then $z \in A$.

(23) Let R be a relational structure with finite stability number and A be a stable set of R. If $\overline{\overline{A}} = \alpha(R)$, then $\mathrm{Upper}\, A \cup \mathrm{Lower}\, A = \Omega_R$.

(24) Let R be a transitive relational structure, x be an element of R, and S be a subset of R. If x is minimal in $\mathrm{Lower}\, S$, then x is minimal in Ω_R.

(25) Let R be a transitive relational structure, x be an element of R, and S be a subset of R. If x is maximal in $\mathrm{Upper}\, S$, then x is maximal in Ω_R.

Let R be a relational structure. The functor $\mathrm{minimals}(R)$ yielding a subset of R is defined as follows:

(Def. 9)(i) For every element x of R holds $x \in \mathrm{minimals}(R)$ iff x is minimal in Ω_R if R is non empty,

(ii) $\mathrm{minimals}(R) = \emptyset$, otherwise.

The functor $\mathrm{maximals}(R)$ yielding a subset of R is defined as follows:

(Def. 10)(i) For every element x of R holds $x \in \mathrm{maximals}(R)$ iff x is maximal in Ω_R if R is non empty,

(ii) $\mathrm{maximals}(R) = \emptyset$, otherwise.

Let R be a non empty antisymmetric transitive relational structure with finite clique number. One can verify that $\mathrm{maximals}(R)$ is non empty and $\mathrm{minimals}(R)$ is non empty.

Let R be a relational structure. Note that $\mathrm{minimals}(R)$ is stable and $\mathrm{maximals}(R)$ is stable.

The following two propositions are true:

(26) For every relational structure R and for every stable set A of R such that $\mathrm{minimals}(R) \nsubseteq A$ holds $\mathrm{minimals}(R) \nsubseteq \mathrm{Upper}\,A$.

(27) For every relational structure R and for every stable set A of R such that $\mathrm{maximals}(R) \nsubseteq A$ holds $\mathrm{maximals}(R) \nsubseteq \mathrm{Lower}\,A$.

5. Substructures

Let R be a relational structure and let X be a finite subset of R. Observe that $\mathrm{sub}(X)$ is finite.

One can prove the following propositions:

(28) For every relational structure R and for every subset S of R holds every clique of $\mathrm{sub}(S)$ is a clique of R.

(29) Let R be a relational structure, S be a subset of R, and C be a clique of R. Then $C \cap S$ is a clique of $\mathrm{sub}(S)$.

(30) For every relational structure R and for every subset S of R holds every stable set of $\mathrm{sub}(S)$ is a stable set of R.

(31) Let R be a relational structure, S be a subset of R, and A be a stable set of R. Then $A \cap S$ is a stable set of $\mathrm{sub}(S)$.

(32) Let R be a relational structure, S be a subset of R, B be a subset of $\mathrm{sub}(S)$, x be an element of $\mathrm{sub}(S)$, and y be an element of R. If $x = y$ and x is maximal in B, then y is maximal in B.

(33) Let R be a relational structure, S be a subset of R, B be a subset of $\mathrm{sub}(S)$, x be an element of $\mathrm{sub}(S)$, and y be an element of R. If $x = y$ and x is minimal in B, then y is minimal in B.

(34) Let R be a transitive relational structure, A be a stable set of R, C be a clique of $\mathrm{sub}(\mathrm{Lower}\,A)$, and a, b be elements of R. If $a \in A$ and $a, b \in C$, then $a = b$ or $b \leq a$.

(35) Let R be a transitive relational structure, A be a stable set of R, C be a clique of $\mathrm{sub}(\mathrm{Upper}\,A)$, and a, b be elements of R. If $a \in A$ and $a, b \in C$, then $a = b$ or $a \leq b$.

Let R be a relational structure with finite clique number and let S be a subset of R. One can verify that $\mathrm{sub}(S)$ has finite clique number.

Let R be a relational structure with finite stability number and let S be a subset of R. One can verify that $\mathrm{sub}(S)$ has finite stability number.

The following propositions are true:

(36) Let R be a non empty antisymmetric transitive relational structure with finite clique number and x be an element of R. Then there exists an element y of R such that y is minimal in Ω_R but $y = x$ or $y < x$.

(37) For every antisymmetric transitive relational structure R with finite clique number holds $\mathrm{Upper}\,\mathrm{minimals}(R) = \Omega_R$.

(38) Let R be a non empty antisymmetric transitive relational structure with finite clique number and x be an element of R. Then there exists an element y of R such that y is maximal in Ω_R but $y = x$ or $x < y$.

(39) For every antisymmetric transitive relational structure R with finite clique number holds Lower maximals$(R) = \Omega_R$.

(40) Let R be an antisymmetric transitive relational structure with finite clique number and A be a stable set of R. If minimals$(R) \subseteq A$, then $A =$ minimals(R).

(41) Let R be an antisymmetric transitive relational structure with finite clique number and A be a stable set of R. If maximals$(R) \subseteq A$, then $A =$ maximals(R).

(42) For every relational structure R with finite clique number and for every subset S of R holds $\omega(\mathrm{sub}(S)) \leq \omega(R)$.

(43) Let R be a relational structure with finite clique number, C be a clique of R, and S be a subset of R. If $\overline{\overline{C}} = \omega(R)$ and $C \subseteq S$, then $\omega(\mathrm{sub}(S)) = \omega(R)$.

(44) For every relational structure R with finite stability number and for every subset S of R holds $\alpha(\mathrm{sub}(S)) \leq \alpha(R)$.

(45) Let R be a relational structure with finite stability number, A be a stable set of R, and S be a subset of R. If $\overline{\overline{A}} = \alpha(R)$ and $A \subseteq S$, then $\alpha(\mathrm{sub}(S)) = \alpha(R)$.

6. Partitions into Cliques and Stable Sets

Let R be a relational structure and let P be a partition of the carrier of R. We say that P is clique-wise if and only if:

(Def. 11) For every set x such that $x \in P$ holds x is a clique of R.

Let R be a relational structure. Observe that there exists a partition of the carrier of R which is clique-wise.

Let R be a relational structure. A clique-partition of R is a clique-wise partition of the carrier of R.

Let R be an empty relational structure. One can verify that every partition of the carrier of R which is empty is also clique-wise.

Next we state four propositions:

(46) For every finite relational structure R and for every clique-partition C of R holds $\overline{\overline{C}} \geq \alpha(R)$.

(47) Let R be a relational structure with finite stability number, A be a stable set of R, and C be a clique-partition of R. Suppose Card $C =$ Card A. Then there exists a function f from A into C such that f is bijective and for every set x such that $x \in A$ holds $x \in f(x)$.

(48) Let R be a finite relational structure, A be a stable set of R, and C be a clique-partition of R. Suppose $\overline{\overline{C}} = \overline{\overline{A}}$. Let c be a set. If $c \in C$, then there exists an element a of A such that $c \cap A = \{a\}$.

(49) Let R be an antisymmetric transitive non empty relational structure with finite stability number, A be a stable set of R, U be a clique-partition of $\mathrm{sub}(\mathrm{Upper}\, A)$, and L be a clique-partition of $\mathrm{sub}(\mathrm{Lower}\, A)$. Suppose $\overline{\overline{A}} = \alpha(R)$ and $\mathrm{Card}\, U = \alpha(R)$ and $\mathrm{Card}\, L = \alpha(R)$. Then there exists a clique-partition C of R such that $\mathrm{Card}\, C = \alpha(R)$.

Let R be a relational structure and let P be a partition of the carrier of R. We say that P is stable-wise if and only if:

(Def. 12) For every set x such that $x \in P$ holds x is a stable set of R.

Let R be a relational structure. Observe that there exists a partition of the carrier of R which is stable-wise.

Let R be a relational structure. A coloring of R is a stable-wise partition of the carrier of R.

Let R be an empty relational structure. Note that every partition of the carrier of R is stable-wise.

We now state the proposition

(50) For every finite relational structure R and for every coloring C of R holds $\overline{\overline{C}} \geq \omega(R)$.

7. Dilworth's Theorem and a Dual

Next we state the proposition

(51) Let R be a finite antisymmetric transitive relational structure. Then there exists a clique-partition C of R such that $\overline{\overline{C}} = \alpha(R)$.

Let R be a non empty relational structure with finite stability number and let C be a subset of R. We say that C is strong-chain if and only if the condition (Def. 13) is satisfied.

(Def. 13) Let S be a finite non empty subset of R. Then there exists a clique-partition P of $\mathrm{sub}(S)$ such that $\overline{\overline{P}} \leq \alpha(R)$ and there exists a set c such that $c \in P$ and $S \cap C \subseteq c$ and for every set d such that $d \in P$ and $d \neq c$ holds $C \cap d = \emptyset$.

Let R be a non empty relational structure with finite stability number. Note that every subset of R which is strong-chain is also a clique.

Let R be an antisymmetric transitive non empty relational structure with finite stability number. Observe that every subset of R which is trivial and non empty is also strong-chain.

The following propositions are true:

(52) Let R be a non empty antisymmetric transitive relational structure with finite stability number. Then there exists a non empty subset S of R such that S is strong-chain and it is not true that there exists a subset D of R such that D is strong-chain and $S \subset D$.

(53) Let R be an antisymmetric transitive relational structure with finite stability number. Then there exists a clique-partition C of R such that $\operatorname{Card} C = \alpha(R)$.

(54) Let R be an antisymmetric transitive relational structure with finite clique number. Then there exists a coloring A of R such that $\operatorname{Card} A = \omega(R)$.

8. Erdős-Szekeres Theorem

One can prove the following two propositions:

(55) Let R be a finite antisymmetric transitive relational structure and r, s be natural numbers. Suppose $\operatorname{Card} R = r \cdot s + 1$. Then there exists a clique C of R such that $\overline{\overline{C}} \geq r + 1$ or there exists a stable set A of R such that $\overline{\overline{A}} \geq s + 1$.

(56) Let f be a real-valued finite sequence and n be a natural number. Suppose $\overline{\overline{f}} = n^{\mathbf{2}} + 1$ and f is one-to-one. Then there exists a real-valued finite subsequence g such that $g \subseteq f$ and $\overline{\overline{g}} \geq n + 1$ and g is increasing or decreasing.

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Complex Integral[1]

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Summary. In this article, we defined complex curve and complex integral. Then we have proved the linearity for the complex integral. Furthermore, we have proved complex integral of complex curve's connection is the sum of each complex integral of individual complex curve.

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The terminology and notation used here are introduced in the following articles: [10], [2], [14], [11], [12], [3], [4], [1], [7], [15], [5], [13], [8], [17], [9], [16], and [6].

1. The Definition of Complex Curve and Complex Integral

In this paper t is an element of $\mathbb{R}$.

The function $\mathbb{R}^2 \to \mathbb{C}$ from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{C}$ is defined as follows:

(Def. 1) For every element p of $\mathbb{R} \times \mathbb{R}$ and for all elements a, b of $\mathbb{R}$ such that $a = p_{\mathbf{1}}$ and $b = p_{\mathbf{2}}$ holds $(\mathbb{R}^2 \to \mathbb{C})(\langle a, b \rangle) = a + b \cdot i$.

Let a, b be real numbers, let x, y be partial functions from $\mathbb{R}$ to $\mathbb{R}$, let Z be a subset of $\mathbb{R}$, and let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$. The functor $\int(f, x, y, a, b, Z)$ yielding a complex number is defined by the condition (Def. 2).

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(Def. 2) There exist partial functions u_0, v_0 from $\mathbb{R}$ to $\mathbb{R}$ such that $u_0 = \Re(f) \cdot (\mathbb{R}^2 \to \mathbb{C}) \cdot \langle x, y \rangle$ and $v_0 = \Im(f) \cdot (\mathbb{R}^2 \to \mathbb{C}) \cdot \langle x, y \rangle$ and $\int(f, x, y, a, b, Z) = \int\limits_a^b (u_0\, x'_{\restriction Z} - v_0\, y'_{\restriction Z})(x)dx + \int\limits_a^b (v_0\, x'_{\restriction Z} + u_0\, y'_{\restriction Z})(x)dx \cdot i$.

Let C be a partial function from $\mathbb{R}$ to $\mathbb{C}$. We say that C is C_1-curve-like if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exist real numbers a, b and there exist partial functions x, y from $\mathbb{R}$ to $\mathbb{R}$ and there exists a subset Z of $\mathbb{R}$ such that $a \leq b$ and $[a, b] = \operatorname{dom} C$ and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and Z is open and $[a, b] \subseteq Z$ and x is differentiable on Z and y is differentiable on Z and x is continuous on Z and y is continuous on Z and $C = (x + i\, y) \restriction [a, b]$.

Let us observe that there exists a partial function from $\mathbb{R}$ to $\mathbb{C}$ which is C_1-curve-like.

A C_1-curve is a C_1-curve-like partial function from $\mathbb{R}$ to $\mathbb{C}$.

Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$ and let C be a C_1-curve. Let us assume that $\operatorname{rng} C \subseteq \operatorname{dom} f$. The functor $\int\limits_C f(x)dx$ yields a complex number and is defined by the condition (Def. 4).

(Def. 4) There exist real numbers a, b and there exist partial functions x, y from $\mathbb{R}$ to $\mathbb{R}$ and there exists a subset Z of $\mathbb{R}$ such that $a \leq b$ and $[a, b] = \operatorname{dom} C$ and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and Z is open and $[a, b] \subseteq Z$ and x is differentiable on Z and y is differentiable on Z and x is continuous on Z and y is continuous on Z and $C = (x + i\, y) \restriction [a, b]$ and $\int\limits_C f(x)dx = \int(f, x, y, a, b, Z)$.

Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$ and let C be a C_1-curve. We say that f is integrable on C if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let a, b be real numbers, x, y be partial functions from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and u_0, v_0 be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $a \leq b$ and $[a, b] = \operatorname{dom} C$ and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and Z is open and $[a, b] \subseteq Z$ and x is differentiable on Z and y is differentiable on Z and x is continuous on Z and y is continuous on Z and $C = (x + i\, y) \restriction [a, b]$. Then $u_0\, x'_{\restriction Z} - v_0\, y'_{\restriction Z}$ is integrable on $[a, b]$ and $v_0\, x'_{\restriction Z} + u_0\, y'_{\restriction Z}$ is integrable on $[a, b]$.

Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$ and let C be a C_1-curve. We say that f is bounded on C if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let a, b be real numbers, x, y be partial functions from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and u_0, v_0 be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $a \leq b$ and $[a, b] = \operatorname{dom} C$ and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and Z is open and $[a, b] \subseteq Z$ and x is differentiable on Z and y is differentiable on Z and x is continuous on Z and y is continuous on Z and $C = (x + i\,y)\restriction[a,b]$. Then $(u_0\,x'_{\restriction Z} - v_0\,y'_{\restriction Z})\restriction[a,b]$ is bounded and $(v_0\,x'_{\restriction Z} + u_0\,y'_{\restriction Z})\restriction[a,b]$ is bounded.

2. Linearity of Complex Intergal

Next we state two propositions:

(1) Let f, g be partial functions from $\mathbb{C}$ to $\mathbb{C}$ and C be a C_1-curve. Suppose $\operatorname{rng} C \subseteq \operatorname{dom} f$ and $\operatorname{rng} C \subseteq \operatorname{dom} g$ and f is integrable on C and g is integrable on C and f is bounded on C and g is bounded on C. Then $\int\limits_C (f+g)(x)dx = \int\limits_C f(x)dx + \int\limits_C g(x)dx$.

(2) Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$ and C be a C_1-curve. Suppose $\operatorname{rng} C \subseteq \operatorname{dom} f$ and f is integrable on C and f is bounded on C. Let r be a real number. Then $\int\limits_C (r\,f)(x)dx = r \cdot \int\limits_C f(x)dx$.

3. Complex Integral of Complex Curve's Connection

We now state the proposition

(3) Let f be a partial function from $\mathbb{C}$ to $\mathbb{C}$, C, C_1, C_2 be C_1-curves, and a, b, d be real numbers. Suppose that $\operatorname{rng} C \subseteq \operatorname{dom} f$ and f is integrable on C and f is bounded on C and $a \leq b$ and $\operatorname{dom} C = [a,b]$ and $d \in [a,b]$ and $\operatorname{dom} C_1 = [a,d]$ and $\operatorname{dom} C_2 = [d,b]$ and for every t such that $t \in \operatorname{dom} C_1$ holds $C(t) = C_1(t)$ and for every t such that $t \in \operatorname{dom} C_2$ holds $C(t) = C_2(t)$. Then $\int\limits_C f(x)dx = \int\limits_{C_1} f(x)dx + \int\limits_{C_2} f(x)dx$.

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On the Lattice of Intervals and Rough Sets

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Summary. Rough sets, developed by Pawlak [6], are an important tool to describe a situation of incomplete or partially unknown information. One of the algebraic models deals with the pair of the upper and the lower approximation. Although usually the tolerance or the equivalence relation is taken into account when considering a rough set, here we rather concentrate on the model with the pair of two definable sets, hence we are close to the notion of an interval set. In this article, the lattices of rough sets and intervals are formalized. This paper, being essentially the continuation of [3], is also a step towards the formalization of the algebraic theory of rough sets, as in [4] or [9].

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The articles [2], [1], [10], [7], [3], [5], and [8] provide the terminology and notation for this paper.

1. Interval Sets

Let U be a set and let X, Y be subsets of U. The functor $[X,Y]_{\mathrm{I}}$ yielding a family of subsets of U is defined by:

(Def. 1) $[X,Y]_{\mathrm{I}} = \{A \subseteq U \colon X \subseteq A \ \wedge \ A \subseteq Y\}$.

In the sequel U denotes a set and X, Y denote subsets of U.

Next we state several propositions:

(1) For every set x holds $x \in [X,Y]_{\mathrm{I}}$ iff $X \subseteq x \subseteq Y$.

(2) If $[X,Y]_{\mathrm{I}} \neq \emptyset$, then X, $Y \in [X,Y]_{\mathrm{I}}$.

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(3) For every set U and for all subsets A, B of U such that $A \not\subseteq B$ holds $[A, B]_{\mathrm{I}} = \emptyset$.

(4) For every set U and for all subsets A, B of U such that $[A, B]_{\mathrm{I}} = \emptyset$ holds $A \not\subseteq B$.

(5) For all subsets A, B of U such that $[A, B]_{\mathrm{I}} \neq \emptyset$ holds $A \subseteq B$.

(6) For all subsets A, B, C, D of U such that $[A, B]_{\mathrm{I}} \neq \emptyset$ and $[A, B]_{\mathrm{I}} = [C, D]_{\mathrm{I}}$ holds $A = C$ and $B = D$.

(7) For every non empty set U and for every non empty subset A of U holds $[A, \emptyset_U]_{\mathrm{I}} = \emptyset$.

(8) For every subset A of U holds $[A, A]_{\mathrm{I}} = \{A\}$.

Let us consider U. A family of subsets of U is said to be an interval set of U if:

(Def. 2) There exist subsets A, B of U such that it $= [A, B]_{\mathrm{I}}$.

We now state two propositions:

(9) For every non empty set U holds $\emptyset$ is an interval set of U.

(10) For every non empty set U and for every subset A of U holds $\{A\}$ is an interval set of U.

Let us consider U and let A, B be subsets of U. Then $[A, B]_{\mathrm{I}}$ is an interval set of U.

Let U be a non empty set. Note that there exists an interval set of U which is non empty.

We now state three propositions:

(11) Let U be a non empty set and A be a set. Then A is a non empty interval set of U if and only if there exist subsets A_1, A_2 of U such that $A_1 \subseteq A_2$ and $A = [A_1, A_2]_{\mathrm{I}}$.

(12) Let U be a non empty set and A_1, A_2, B_1, B_2 be subsets of U. If $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$, then $[A_1, A_2]_{\mathrm{I}} \Cap [B_1, B_2]_{\mathrm{I}} = \{C; C$ ranges over subsets of U: $A_1 \cap B_1 \subseteq C \ \wedge \ C \subseteq A_2 \cap B_2\}$.

(13) Let U be a non empty set and A_1, A_2, B_1, B_2 be subsets of U. If $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$, then $[A_1, A_2]_{\mathrm{I}} \Cup [B_1, B_2]_{\mathrm{I}} = \{C; C$ ranges over subsets of U: $A_1 \cup B_1 \subseteq C \ \wedge \ C \subseteq A_2 \cup B_2\}$.

Let U be a non empty set and let A, B be non empty interval sets of U. The functor $A \cap_{\mathrm{I}} B$ yielding an interval set of U is defined by:

(Def. 3) $A \cap_{\mathrm{I}} B = A \Cap B$.

The functor $A \cup_{\mathrm{I}} B$ yields an interval set of U and is defined by:

(Def. 4) $A \cup_{\mathrm{I}} B = A \Cup B$.

Let U be a non empty set and let A, B be non empty interval sets of U. Note that $A \cap_{\mathrm{I}} B$ is non empty and $A \cup_{\mathrm{I}} B$ is non empty.

In the sequel U denotes a non empty set and A, B, C denote non empty interval sets of U.

Let us consider U, A. The functor $A_{\mathbf{1}}$ yielding a subset of U is defined by:

(Def. 5) There exists a subset B of U such that $A = [A_{\mathbf{1}}, B]_{\mathrm{I}}$.

The functor $A_{\mathbf{2}}$ yielding a subset of U is defined as follows:

(Def. 6) There exists a subset B of U such that $A = [B, A_{\mathbf{2}}]_{\mathrm{I}}$.

We now state several propositions:

(14) For every set X holds $X \in A$ iff $A_{\mathbf{1}} \subseteq X \subseteq A_{\mathbf{2}}$.

(15) $A = [A_{\mathbf{1}}, A_{\mathbf{2}}]_{\mathrm{I}}$.

(16) $A_{\mathbf{1}} \subseteq A_{\mathbf{2}}$.

(17) $A \cup_{\mathrm{I}} B = [A_{\mathbf{1}} \cup B_{\mathbf{1}}, A_{\mathbf{2}} \cup B_{\mathbf{2}}]_{\mathrm{I}}$.

(18) $A \cap_{\mathrm{I}} B = [A_{\mathbf{1}} \cap B_{\mathbf{1}}, A_{\mathbf{2}} \cap B_{\mathbf{2}}]_{\mathrm{I}}$.

Let us consider U and let us consider A, B. Let us observe that $A = B$ if and only if:

(Def. 7) $A_{\mathbf{1}} = B_{\mathbf{1}}$ and $A_{\mathbf{2}} = B_{\mathbf{2}}$.

The following propositions are true:

(19) $A \cup_{\mathrm{I}} A = A$.

(20) $A \cap_{\mathrm{I}} A = A$.

(21) $A \cup_{\mathrm{I}} B = B \cup_{\mathrm{I}} A$.

(22) $A \cap_{\mathrm{I}} B = B \cap_{\mathrm{I}} A$.

(23) $(A \cup_{\mathrm{I}} B) \cup_{\mathrm{I}} C = A \cup_{\mathrm{I}} (B \cup_{\mathrm{I}} C)$.

(24) $(A \cap_{\mathrm{I}} B) \cap_{\mathrm{I}} C = A \cap_{\mathrm{I}} (B \cap_{\mathrm{I}} C)$.

Let X be a set and let F be a family of subsets of X. We say that F is ordered if and only if:

(Def. 8) There exist sets A, B such that A, $B \in F$ and for every set Y holds $Y \in F$ iff $A \subseteq Y \subseteq B$.

Let X be a set. Observe that there exists a family of subsets of X which is non empty and ordered.

Next we state two propositions:

(25) For all subsets A, B of U such that $A \subseteq B$ holds $[A, B]_{\mathrm{I}}$ is a non empty ordered family of subsets of U.

(26) Every non empty interval set of U is a non empty ordered family of subsets of U.

Let X be a set. We introduce $\min X$ as a synonym of $\bigcap X$. We introduce $\max X$ as a synonym of $\bigcup X$.

Let X be a set and let F be a non empty ordered family of subsets of X. Then $\min F$ is an element of F. Then $\max F$ is an element of F.

We now state a number of propositions:

(27) Let A, B be subsets of U and F be an ordered non empty family of subsets of U. If $F = [A, B]_{\mathrm{I}}$, then $\min F = A$ and $\max F = B$.

(28) For all sets X, Y and for every non empty ordered family A of subsets of X holds $Y \in A$ iff $\min A \subseteq Y \subseteq \max A$.

(29) For every set X and for all non empty ordered families A, B, C of subsets of X holds $A \Cup B \Cap C = (A \Cup B) \Cap (A \Cup C)$.

(30) For every set X and for all non empty ordered families A, B, C of subsets of X holds $A \Cap (B \Cup C) = A \Cap B \Cup A \Cap C$.

(31) $A \cup_{\mathrm{I}} B \cap_{\mathrm{I}} C = (A \cup_{\mathrm{I}} B) \cap_{\mathrm{I}} (A \cup_{\mathrm{I}} C)$.

(32) $A \cap_{\mathrm{I}} (B \cup_{\mathrm{I}} C) = A \cap_{\mathrm{I}} B \cup_{\mathrm{I}} A \cap_{\mathrm{I}} C$.

(33) For every set X and for all non empty ordered families A, B of subsets of X holds $A \Cap (A \Cup B) = A$.

(34) For every set X and for all non empty ordered families A, B of subsets of X holds $A \Cap B \Cup B = B$.

(35) $A \cap_{\mathrm{I}} (A \cup_{\mathrm{I}} B) = A$.

(36) $A \cap_{\mathrm{I}} B \cup_{\mathrm{I}} B = B$.

2. Families of Subsets

One can prove the following propositions:

(37) For every non empty set U and for all families A, B of subsets of U holds $A \setminus\setminus B$ is a family of subsets of U.

(38) Let U be a non empty set and A, B be non empty ordered families of subsets of U. Then $A \setminus\setminus B = \{C \subseteq U\colon \min A \setminus \max B \subseteq C \ \wedge\ C \subseteq \max A \setminus \min B\}$.

(39) Let U be a non empty set and A_1, A_2, B_1, B_2 be subsets of U. If $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$, then $[A_1, A_2]_{\mathrm{I}} \setminus\setminus [B_1, B_2]_{\mathrm{I}} = \{C \subseteq U\colon A_1 \setminus B_2 \subseteq C \ \wedge\ C \subseteq A_2 \setminus B_1\}$.

Let U be a non empty set and let A, B be non empty interval sets of U. The functor $A \setminus_{\mathrm{I}} B$ yields an interval set of U and is defined as follows:

(Def. 9) $A \setminus_{\mathrm{I}} B = A \setminus\setminus B$.

Let U be a non empty set and let A, B be non empty interval sets of U. Observe that $A \setminus_{\mathrm{I}} B$ is non empty.

Next we state several propositions:

(40) $A \setminus_{\mathrm{I}} B = [A_{\mathbf{1}} \setminus B_{\mathbf{2}}, A_{\mathbf{2}} \setminus B_{\mathbf{1}}]_{\mathrm{I}}$.

(41) For all subsets X, Y of U such that $A = [X, Y]_{\mathrm{I}}$ holds $A \setminus_{\mathrm{I}} C = [X \setminus C_{\mathbf{2}}, Y \setminus C_{\mathbf{1}}]_{\mathrm{I}}$.

(42) For all subsets X, Y, W, Z of U such that $A = [X, Y]_{\mathrm{I}}$ and $C = [W, Z]_{\mathrm{I}}$ holds $A \setminus_{\mathrm{I}} C = [X \setminus Z, Y \setminus W]_{\mathrm{I}}$.

(43) For every non empty set U holds $[\Omega_U, \Omega_U]_{\mathrm{I}}$ is a non empty interval set of U.

(44) For every non empty set U holds $[\emptyset_U, \emptyset_U]_{\mathrm{I}}$ is a non empty interval set of U.

Let U be a non empty set. Note that $[\Omega_U, \Omega_U]_{\mathrm{I}}$ is non empty and $[\emptyset_U, \emptyset_U]_{\mathrm{I}}$ is non empty.

Let U be a non empty set and let A be a non empty interval set of U. The functor $-A$ yielding a non empty interval set of U is defined as follows:

(Def. 10) $-A = [\Omega_U, \Omega_U]_{\mathrm{I}} \setminus_{\mathrm{I}} A$.

We now state four propositions:

(45) For every non empty set U and for every non empty interval set A of U holds $-A = [(A_{\mathbf{2}})^{\mathrm{c}}, (A_{\mathbf{1}})^{\mathrm{c}}]_{\mathrm{I}}$.

(46) For all subsets X, Y of U such that $A = [X, Y]_{\mathrm{I}}$ and $X \subseteq Y$ holds $-A = [Y^{\mathrm{c}}, X^{\mathrm{c}}]_{\mathrm{I}}$.

(47) $-[\emptyset_U, \emptyset_U]_{\mathrm{I}} = [\Omega_U, \Omega_U]_{\mathrm{I}}$.

(48) $-[\Omega_U, \Omega_U]_{\mathrm{I}} = [\emptyset_U, \emptyset_U]_{\mathrm{I}}$.

3. Counterexamples

Next we state several propositions:

(49) There exists a non empty interval set A of U such that $A \cap_{\mathrm{I}} -A \neq [\emptyset_U, \emptyset_U]_{\mathrm{I}}$.

(50) There exists a non empty interval set A of U such that $A \cup_{\mathrm{I}} -A \neq [\Omega_U, \Omega_U]_{\mathrm{I}}$.

(51) There exists a non empty interval set A of U such that $A \setminus_{\mathrm{I}} A \neq [\emptyset_U, \emptyset_U]_{\mathrm{I}}$.

(52) For every non empty interval set A of U holds $U \in A \cup_{\mathrm{I}} -A$.

(53) For every non empty interval set A of U holds $\emptyset \in A \cap_{\mathrm{I}} -A$.

(54) For every non empty interval set A of U holds $\emptyset \in A \setminus_{\mathrm{I}} A$.

4. Lattice of Interval Sets

Let U be a non empty set. The functor $\mathrm{I}(2^U)$ yielding a non empty set is defined by:

(Def. 11) For every set x holds $x \in \mathrm{I}(2^U)$ iff x is a non empty interval set of U.

Let U be a non empty set. The functor $\mathrm{InterLatt}\, U$ yields a strict non empty lattice structure and is defined by the conditions (Def. 12).

(Def. 12)(i) The carrier of $\mathrm{InterLatt}\, U = \mathrm{I}(2^U)$, and

(ii) for all elements a, b of the carrier of $\mathrm{InterLatt}\, U$ and for all non empty interval sets a', b' of U such that $a' = a$ and $b' = b$ holds (the join operation of InterLatt U)$(a, b) = a' \cup_{\mathrm{I}} b'$ and (the meet operation of InterLatt U)$(a, b) = a' \cap_{\mathrm{I}} b'$.

Let U be a non empty set. Observe that InterLatt U is lattice-like.

Let X be a tolerance space.

(Def. 13) An element of $2^{\text{the carrier of } X} \times 2^{\text{the carrier of } X}$ is said to be a rough set of X.

One can prove the following proposition

(55) For every tolerance space X and for every rough set A of X there exist subsets B, C of X such that $A = \langle B, C \rangle$.

Let X be a tolerance space and let A be a subset of X. The functor RS A yielding a rough set of X is defined by:

(Def. 14) $\mathrm{RS}\, A = \langle \mathrm{LAp}(A), \mathrm{UAp}(A) \rangle$.

Let X be a tolerance space and let A be a rough set of X. The functor $\mathrm{LAp}(A)$ yielding a subset of X is defined as follows:

(Def. 15) $\mathrm{LAp}(A) = A_{\mathbf{1}}$.

The functor $\mathrm{UAp}(A)$ yielding a subset of X is defined by:

(Def. 16) $\mathrm{UAp}(A) = A_{\mathbf{2}}$.

Let X be a tolerance space and let A, B be rough sets of X. Let us observe that $A = B$ if and only if:

(Def. 17) $\mathrm{LAp}(A) = \mathrm{LAp}(B)$ and $\mathrm{UAp}(A) = \mathrm{UAp}(B)$.

Let X be a tolerance space and let A, B be rough sets of X. The functor $A \cup_{\mathrm{I}} B$ yields a rough set of X and is defined by:

(Def. 18) $A \cup_{\mathrm{I}} B = \langle \mathrm{LAp}(A) \cup \mathrm{LAp}(B), \mathrm{UAp}(A) \cup \mathrm{UAp}(B) \rangle$.

The functor $A \cap_{\mathrm{I}} B$ yielding a rough set of X is defined as follows:

(Def. 19) $A \cap_{\mathrm{I}} B = \langle \mathrm{LAp}(A) \cap \mathrm{LAp}(B), \mathrm{UAp}(A) \cap \mathrm{UAp}(B) \rangle$.

In the sequel X denotes a tolerance space and A, B, C denote rough sets of X.

Next we state a number of propositions:

(56) $\mathrm{LAp}(A \cup_{\mathrm{I}} B) = \mathrm{LAp}(A) \cup \mathrm{LAp}(B)$.

(57) $\mathrm{UAp}(A \cup_{\mathrm{I}} B) = \mathrm{UAp}(A) \cup \mathrm{UAp}(B)$.

(58) $\mathrm{LAp}(A \cap_{\mathrm{I}} B) = \mathrm{LAp}(A) \cap \mathrm{LAp}(B)$.

(59) $\mathrm{UAp}(A \cap_{\mathrm{I}} B) = \mathrm{UAp}(A) \cap \mathrm{UAp}(B)$.

(60) $A \cup_{\mathrm{I}} A = A$.

(61) $A \cap_{\mathrm{I}} A = A$.

(62) $A \cup_{\mathrm{I}} B = B \cup_{\mathrm{I}} A$.

(63) $A \cap_{\mathrm{I}} B = B \cap_{\mathrm{I}} A$.

(64) $(A \cup_{\mathrm{I}} B) \cup_{\mathrm{I}} C = A \cup_{\mathrm{I}} (B \cup_{\mathrm{I}} C)$.

(65) $(A \cap_{\mathrm{I}} B) \cap_{\mathrm{I}} C = A \cap_{\mathrm{I}} (B \cap_{\mathrm{I}} C)$.

(66) $A \cap_{\mathrm{I}} (B \cup_{\mathrm{I}} C) = A \cap_{\mathrm{I}} B \cup_{\mathrm{I}} A \cap_{\mathrm{I}} C$.

(67) $A \cup_{\mathrm{I}} A \cap_{\mathrm{I}} B = A$.

(68) $A \cap_{\mathrm{I}} (A \cup_{\mathrm{I}} B) = A$.

5. Lattice of Rough Sets

Let us consider X. The functor RoughSets X is defined as follows:

(Def. 20) For every set x holds $x \in$ RoughSets X iff x is a rough set of X.

Let us consider X. One can check that RoughSets X is non empty.

Let us consider X and let R be an element of RoughSets X. The functor $^{@}R$ yielding a rough set of X is defined by:

(Def. 21) $^{@}R = R$.

Let us consider X and let R be a rough set of X. The functor $^{@}R$ yielding an element of RoughSets X is defined as follows:

(Def. 22) $^{@}R = R$.

Let us consider X. The functor RSLattice X yields a strict lattice structure and is defined by the conditions (Def. 23).

(Def. 23)(i) The carrier of RSLattice $X =$ RoughSets X, and

(ii) for all elements A, B of RoughSets X and for all rough sets A', B' of X such that $A = A'$ and $B = B'$ holds (the join operation of RSLattice X)(A, B) $= A' \cup_{\mathrm{I}} B'$ and (the meet operation of RSLattice X)(A, B) $= A' \cap_{\mathrm{I}} B'$.

Let us consider X. Observe that RSLattice X is non empty.

Let us consider X. Observe that RSLattice X is lattice-like.

Let us consider X. Note that RSLattice X is distributive.

Let us consider X. The functor ERS X yields a rough set of X and is defined by:

(Def. 24) ERS $X = \langle \emptyset, \emptyset \rangle$.

One can prove the following proposition

(69) For every rough set A of X holds ERS $X \cup_{\mathrm{I}} A = A$.

Let us consider X. The functor TRS(X) is a rough set of X and is defined as follows:

(Def. 25) TRS(X) $= \langle \Omega_X, \Omega_X \rangle$.

One can prove the following proposition

(70) For every rough set A of X holds TRS(X) $\cap_{\mathrm{I}} A = A$.

Let us consider X. Note that RSLattice X is bounded.

We now state the proposition

(71) Let X be a tolerance space, A, B be elements of RSLattice X, and A', B' be rough sets of X. If $A = A'$ and $B = B'$, then $A \sqsubseteq B$ iff LAp(A') $\subseteq$ LAp(B') and UAp(A') $\subseteq$ UAp(B').

Let us consider X. Observe that $\mathrm{RSLattice}\, X$ is complete.

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Basic Properties of Periodic Functions

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Summary. In this article we present definitions, basic properties and some examples of periodic functions according to [5].

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The papers [2], [6], [3], [10], [11], [9], [8], [1], [4], and [7] provide the terminology and notation for this paper.

1. Basic Properties of a Period of a Function

We use the following convention: x, t, t_1, t_2, r, a, b are real numbers and F, G are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let t be a real number. We say that t is a period of F if and only if:

(Def. 1) $t \neq 0$ and for every x holds $x \in \operatorname{dom} F$ iff $x+t \in \operatorname{dom} F$ and if $x \in \operatorname{dom} F$, then $F(x) = F(x+t)$.

Let F be a partial function from $\mathbb{R}$ to $\mathbb{R}$. We say that F is periodic if and only if:

(Def. 2) There exists t which is a period of F.

We now state a number of propositions:

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(1) t is a period of F iff $t \neq 0$ and for every x such that $x \in \operatorname{dom} F$ holds $x + t$, $x - t \in \operatorname{dom} F$ and $F(x) = F(x + t)$.
(2) If t is a period of F and a period of G, then t is a period of $F + G$.
(3) If t is a period of F and a period of G, then t is a period of $F - G$.
(4) If t is a period of F and a period of G, then t is a period of $F\,G$.
(5) If t is a period of F and a period of G, then t is a period of F/G.
(6) If t is a period of F, then t is a period of $-F$.
(7) If t is a period of F, then t is a period of $r\,F$.
(8) If t is a period of F, then t is a period of $r + F$.
(9) If t is a period of F, then t is a period of $F - r$.
(10) If t is a period of F, then t is a period of $|F|$.
(11) If t is a period of F, then t is a period of F^{-1}.
(12) If t is a period of F, then t is a period of $F^{\mathbf{2}}$.
(13) If t is a period of F, then for every x such that $x \in \operatorname{dom} F$ holds $F(x) = F(x - t)$.
(14) If t is a period of F, then $-t$ is a period of F.
(15) If t_1 is a period of F and t_2 is a period of F and $t_1 + t_2 \neq 0$, then $t_1 + t_2$ is a period of F.
(16) If t_1 is a period of F and t_2 is a period of F and $t_1 - t_2 \neq 0$, then $t_1 - t_2$ is a period of F.
(17) Suppose $t \neq 0$ and for every x such that $x \in \operatorname{dom} F$ holds $x + t$, $x - t \in \operatorname{dom} F$ and $F(x + t) = F(x - t)$. Then $2 \cdot t$ is a period of F and F is periodic.
(18) Suppose $t_1 + t_2 \neq 0$ and for every x such that $x \in \operatorname{dom} F$ holds $x + t_1$, $x - t_1$, $x + t_2$, $x - t_2 \in \operatorname{dom} F$ and $F(x + t_1) = F(x - t_2)$. Then $t_1 + t_2$ is a period of F and F is periodic.
(19) Suppose $t_1 - t_2 \neq 0$ and for every x such that $x \in \operatorname{dom} F$ holds $x + t_1$, $x - t_1$, $x + t_2$, $x - t_2 \in \operatorname{dom} F$ and $F(x + t_1) = F(x + t_2)$. Then $t_1 - t_2$ is a period of F and F is periodic.
(20) Suppose $t \neq 0$ and for every x such that $x \in \operatorname{dom} F$ holds $x + t$, $x - t \in \operatorname{dom} F$ and $F(x+t) = F(x)^{-1}$. Then $2 \cdot t$ is a period of F and F is periodic.

Let us observe that there exists a partial function from $\mathbb{R}$ to $\mathbb{R}$ which is periodic.

Let F be a periodic partial function from $\mathbb{R}$ to $\mathbb{R}$. One can check that $-F$ is periodic.

Let F be a periodic partial function from $\mathbb{R}$ to $\mathbb{R}$ and let r be a real number. One can check the following observations:

* $r\,F$ is periodic,
* $r + F$ is periodic, and

* $F - r$ is periodic.

Let F be a periodic partial function from $\mathbb{R}$ to $\mathbb{R}$. One can check the following observations:

* $|F|$ is periodic,
* F^{-1} is periodic, and
* $F^{\mathbf{2}}$ is periodic.

2. Some Examples

Let us note that the function sin is periodic and the function cos is periodic.

We now state two propositions:

(21) For every element k of $\mathbb{N}$ holds $2 \cdot \pi \cdot (k+1)$ is a period of the function sin.

(22) For every element k of $\mathbb{N}$ holds $2 \cdot \pi \cdot (k+1)$ is a period of the function cos.

Let us observe that the function cosec is periodic and the function sec is periodic.

We now state two propositions:

(23) For every element k of $\mathbb{N}$ holds $2 \cdot \pi \cdot (k+1)$ is a period of the function sec.

(24) For every element k of $\mathbb{N}$ holds $2 \cdot \pi \cdot (k+1)$ is a period of the function cosec.

Let us mention that the function tan is periodic and the function cot is periodic.

Next we state a number of propositions:

(25) For every element k of $\mathbb{N}$ holds $\pi \cdot (k+1)$ is a period of the function tan.

(26) For every element k of $\mathbb{N}$ holds $\pi \cdot (k+1)$ is a period of the function cot.

(27) For every element k of $\mathbb{N}$ holds $\pi \cdot (k+1)$ is a period of |the function sin|.

(28) For every element k of $\mathbb{N}$ holds $\pi \cdot (k+1)$ is a period of |the function cos|.

(29) For every element k of $\mathbb{N}$ holds $\frac{\pi}{2} \cdot (k+1)$ is a period of |the function sin| + |the function cos|.

(30) For every element k of $\mathbb{N}$ holds $\pi \cdot (k+1)$ is a period of (the function sin)$^{\mathbf{2}}$.

(31) For every element k of $\mathbb{N}$ holds $\pi \cdot (k+1)$ is a period of (the function cos)$^{\mathbf{2}}$.

(32) For every element k of $\mathbb{N}$ holds $\pi \cdot (k+1)$ is a period of (the function sin) (the function cos).

(33) For every element k of $\mathbb{N}$ holds $\pi \cdot (k+1)$ is a period of (the function cos) (the function sin).

(34) For every element k of $\mathbb{N}$ holds $2 \cdot \pi \cdot (k+1)$ is a period of $b + a$ (the function sin).

(35) For every element k of $\mathbb{N}$ holds $2 \cdot \pi \cdot (k+1)$ is a period of a (the function sin)$-b$.

(36) For every element k of $\mathbb{N}$ holds $2 \cdot \pi \cdot (k+1)$ is a period of $b + a$ (the function cos).

(37) For every element k of $\mathbb{N}$ holds $2 \cdot \pi \cdot (k+1)$ is a period of a (the function cos)$-b$.

(38) If $\operatorname{dom} F = \mathbb{R}$ and for every real number x holds $F(x) = a$, then for every element k of $\mathbb{N}$ holds $k+1$ is a period of F.

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Epsilon Numbers and Cantor Normal Form

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Summary. An epsilon number is a transfinite number which is a fixed point of an exponential map: $\omega^{\varepsilon} = \varepsilon$. The formalization of the concept is done with use of the tetration of ordinals (Knuth's arrow notation, $\uparrow\uparrow$). Namely, the ordinal indexing of epsilon numbers is defined as follows:

$$\varepsilon_0 = \omega \uparrow\uparrow \omega, \quad \varepsilon_{\alpha+1} = \varepsilon_\alpha \uparrow\uparrow \omega,$$

and for limit ordinal λ:

$$\varepsilon_\lambda = \lim_{\alpha<\lambda} \varepsilon_\alpha = \bigcup_{\alpha<\lambda} \varepsilon_\alpha.$$

Tetration stabilizes at ω:

$$\alpha \uparrow\uparrow \beta = \alpha \uparrow\uparrow \omega \quad \text{for} \quad \alpha \neq 0 \quad \text{and} \quad \beta \geq \omega.$$

Every ordinal number α can be uniquely written as

$$n_1\omega^{\beta_1} + n_2\omega^{\beta_2} + \dots + n_k\omega^{\beta_k},$$

where k is a natural number, n_1, n_2, …, n_k are positive integers, and $\beta_1 > \beta_2 > \ldots > \beta_k$ are ordinal numbers ($\beta_k = 0$). This decomposition of α is called the Cantor Normal Form of α.

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The notation and terminology used here are introduced in the following papers: [9], [3], [8], [7], [1], [5], [6], [4], [2], and [10].

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1. Preliminaries

For simplicity, we follow the rules: α, β, γ denote ordinal numbers, m, n denote natural numbers, f denotes a sequence of ordinal numbers, and x denotes a set.

One can prove the following proposition

(1) If $\alpha \subseteq \operatorname{succ}\beta$, then $\alpha \subseteq \beta$ or $\alpha = \operatorname{succ}\beta$.

Let us note that ω is limit ordinal and every empty set is ordinal yielding.

One can verify that there exists a transfinite sequence which is non empty and finite.

Let f be a transfinite sequence and let g be a non empty transfinite sequence. One can check that $f \frown g$ is non empty and $g \frown f$ is non empty.

In the sequel ψ, ψ_1, ψ_2 denote transfinite sequences.

One can prove the following three propositions:

(2) If $\operatorname{dom}\psi = \alpha + \beta$, then there exist ψ_1, ψ_2 such that $\psi = \psi_1 \frown \psi_2$ and $\operatorname{dom}\psi_1 = \alpha$ and $\operatorname{dom}\psi_2 = \beta$.

(3) $\operatorname{rng}\psi_1 \subseteq \operatorname{rng}(\psi_1 \frown \psi_2)$ and $\operatorname{rng}\psi_2 \subseteq \operatorname{rng}(\psi_1 \frown \psi_2)$.

(4) If $\psi_1 \frown \psi_2$ is ordinal yielding, then ψ_1 is ordinal yielding and ψ_2 is ordinal yielding.

Let f be a transfinite sequence. We say that f is decreasing if and only if:

(Def. 1) For all α, β such that $\alpha \in \beta$ and $\beta \in \operatorname{dom} f$ holds $f(\beta) \in f(\alpha)$.

We say that f is non-decreasing if and only if:

(Def. 2) For all α, β such that $\alpha \in \beta$ and $\beta \in \operatorname{dom} f$ holds $f(\alpha) \subseteq f(\beta)$.

We say that f is non-increasing if and only if:

(Def. 3) For all α, β such that $\alpha \in \beta$ and $\beta \in \operatorname{dom} f$ holds $f(\beta) \subseteq f(\alpha)$.

Let us observe that every sequence of ordinal numbers which is increasing is also non-decreasing and every sequence of ordinal numbers which is decreasing is also non-increasing.

We now state the proposition

(5) For every transfinite sequence f holds f is infinite iff $\omega \subseteq \operatorname{dom} f$.

Let us note that every transfinite sequence which is decreasing is also finite and every sequence of ordinal numbers which is empty is also decreasing and increasing.

Let us consider α. Observe that $\langle\alpha\rangle$ is ordinal yielding.

Let us consider α. One can check that $\langle\alpha\rangle$ is decreasing and increasing.

Let us observe that there exists a sequence of ordinal numbers which is decreasing, increasing, non-decreasing, non-increasing, finite, and non empty.

The following propositions are true:

(6) For every non-decreasing sequence f of ordinal numbers such that $\operatorname{dom} f$ is non empty holds $\bigcup f$ is the limit of f.

(7) If $\alpha \in \beta$, then $n \cdot \omega^{\alpha} \in \omega^{\beta}$.

(8) If $0 \in \alpha$ and for every β such that $\beta \in \operatorname{dom} f$ holds $f(\beta) = \alpha^{\beta}$, then f is non-decreasing.

(9) If α is a limit ordinal number and $0 \in \beta$, then α^{β} is a limit ordinal number.

(10) If $1 \in \alpha$ and for every β such that $\beta \in \operatorname{dom} f$ holds $f(\beta) = \alpha^{\beta}$, then f is increasing.

(11) If $0 \in \alpha$ and β is a non empty limit ordinal number, then $x \in \alpha^{\beta}$ iff there exists γ such that $\gamma \in \beta$ and $x \in \alpha^{\gamma}$.

(12) If $0 \in \alpha$ and $\alpha^{\beta} \in \alpha^{\gamma}$, then $\beta \in \gamma$.

2. Tetration (Knuth's Arrow Notation) of Ordinals[1]

Let α, β be ordinal numbers. The functor $\alpha \uparrow\uparrow \beta$ yields an ordinal number and is defined by the condition (Def. 4).

(Def. 4) There exists a sequence φ of ordinal numbers such that

(i) $\alpha \uparrow\uparrow \beta = \operatorname{last} \varphi$,

(ii) $\operatorname{dom} \varphi = \operatorname{succ} \beta$,

(iii) $\varphi(\emptyset) = 1$,

(iv) for every ordinal number γ such that $\operatorname{succ} \gamma \in \operatorname{succ} \beta$ holds $\varphi(\operatorname{succ} \gamma) = \alpha^{\varphi(\gamma)}$, and

(v) for every ordinal number γ such that $\gamma \in \operatorname{succ} \beta$ and $\gamma \neq \emptyset$ and γ is a limit ordinal number holds $\varphi(\gamma) = \lim(\varphi \restriction \gamma)$.

We now state a number of propositions:

(13) $\alpha \uparrow\uparrow 0 = 1$.

(14) $\alpha \uparrow\uparrow \operatorname{succ} \beta = \alpha^{\alpha \uparrow\uparrow \beta}$.

(15) Suppose $\beta \neq \emptyset$ and β is a limit ordinal number. Let φ be a sequence of ordinal numbers. If $\operatorname{dom} \varphi = \beta$ and for every γ such that $\gamma \in \beta$ holds $\varphi(\gamma) = \alpha \uparrow\uparrow \gamma$, then $\alpha \uparrow\uparrow \beta = \lim \varphi$.

(16) $\alpha \uparrow\uparrow 1 = \alpha$.

(17) $1 \uparrow\uparrow \alpha = 1$.

(18) $\alpha \uparrow\uparrow 2 = \alpha^{\alpha}$.

(19) $\alpha \uparrow\uparrow 3 = \alpha^{\alpha^{\alpha}}$.

(20) For every natural number n holds $0 \uparrow\uparrow (2 \cdot n) = 1$ and $0 \uparrow\uparrow (2 \cdot n + 1) = 0$.

(21) If $\alpha \subseteq \beta$ and $0 \in \gamma$, then $\gamma \uparrow\uparrow \alpha \subseteq \gamma \uparrow\uparrow \beta$.

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[1] Important fact (32)

$$\alpha \uparrow\uparrow \beta = \alpha \uparrow\uparrow \omega \quad \text{for} \quad \beta \geq \omega \quad \text{and} \quad \alpha > 0$$

(22) If $0 \in \alpha$ and for every β such that $\beta \in \operatorname{dom} f$ holds $f(\beta) = \alpha \uparrow\uparrow \beta$, then f is non-decreasing.

(23) If $0 \in \alpha$ and $0 \in \beta$, then $\alpha \subseteq \alpha \uparrow\uparrow \beta$.

(24) If $1 \in \alpha$ and $m < n$, then $\alpha \uparrow\uparrow m \in \alpha \uparrow\uparrow n$.

(25) If $1 \in \alpha$ and $\operatorname{dom} f \subseteq \omega$ and for every β such that $\beta \in \operatorname{dom} f$ holds $f(\beta) = \alpha \uparrow\uparrow \beta$, then f is increasing.

(26) If $1 \in \alpha$ and $1 \in \beta$, then $\alpha \in \alpha \uparrow\uparrow \beta$.

(27) For all natural numbers n, k holds $n^k = n^k$.

Let n, k be natural numbers. Observe that n^k is natural.

Let n, k be natural numbers. One can check that $n \uparrow\uparrow k$ is natural.

Next we state several propositions:

(28) For all natural numbers n, k such that $n > 1$ holds $n \uparrow\uparrow k > k$.

(29) For every natural number n such that $n > 1$ holds $n \uparrow\uparrow \omega = \omega$.

(30) If $1 \in \alpha$, then $\alpha \uparrow\uparrow \omega$ is a limit ordinal number.

(31) If $0 \in \alpha$, then $\alpha^{\alpha \uparrow\uparrow \omega} = \alpha \uparrow\uparrow \omega$.

(32) If $0 \in \alpha$ and $\omega \subseteq \beta$, then $\alpha \uparrow\uparrow \beta = \alpha \uparrow\uparrow \omega$.

3. Critical Numbers[2]

In this article we present several logical schemes. The scheme *CriticalNumber2* deals with an ordinal number $\mathcal{A}$, a sequence $\mathcal{B}$ of ordinal numbers, and a unary functor $\mathcal{F}$ yielding an ordinal number, and states that:

$\mathcal{A} \subseteq \bigcup \mathcal{B}$ and $\mathcal{F}(\bigcup \mathcal{B}) = \bigcup \mathcal{B}$ and for every β such that $\mathcal{A} \subseteq \beta$ and $\mathcal{F}(\beta) = \beta$ holds $\bigcup \mathcal{B} \subseteq \beta$

provided the following requirements are met:

- For all α, β such that $\alpha \in \beta$ holds $\mathcal{F}(\alpha) \in \mathcal{F}(\beta)$,
- Let given α. Suppose α is a non empty limit ordinal number. Let φ be a sequence of ordinal numbers. If $\operatorname{dom} \varphi = \alpha$ and for every β such that $\beta \in \alpha$ holds $\varphi(\beta) = \mathcal{F}(\beta)$, then $\mathcal{F}(\alpha)$ is the limit of φ,
- $\operatorname{dom} \mathcal{B} = \omega$ and $\mathcal{B}(0) = \mathcal{A}$, and
- For every α such that $\alpha \in \omega$ holds $\mathcal{B}(\operatorname{succ} \alpha) = \mathcal{F}(\mathcal{B}(\alpha))$.

The scheme *CriticalNumber3* deals with an ordinal number $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an ordinal number, and states that:

There exists α such that $\mathcal{A} \in \alpha$ and $\mathcal{F}(\alpha) = \alpha$

provided the following requirements are met:

- For all α, β such that $\alpha \in \beta$ holds $\mathcal{F}(\alpha) \in \mathcal{F}(\beta)$, and

[2] $\mathcal{F}$ is increasing continuous map of ordinals and $\alpha = \mathcal{F}(\alpha)$ is a critical number of $\mathcal{F}$

- Let given α. Suppose α is a non empty limit ordinal number. Let φ be a sequence of ordinal numbers. If $\operatorname{dom}\varphi = \alpha$ and for every β such that $\beta \in \alpha$ holds $\varphi(\beta) = \mathcal{F}(\beta)$, then $\mathcal{F}(\alpha)$ is the limit of φ.

4. Epsilon Numbers[3]

Let α be an ordinal number. We say that α is epsilon if and only if:

(Def. 5) $\omega^{\alpha} = \alpha$.

One can prove the following proposition

(33) There exists β such that $\alpha \in \beta$ and β is epsilon.

Let us note that there exists an ordinal number which is epsilon.

Let α be an ordinal number. The first ε greater than α yielding an epsilon number is defined by the conditions (Def. 6).

(Def. 6)(i) $\alpha \in$ the first ε greater than α, and

(ii) for every epsilon number β such that $\alpha \in \beta$ holds the first ε greater than $\alpha \subseteq \beta$.

One can prove the following four propositions:

(34) If $\alpha \subseteq \beta$, then the first ε greater than $\alpha \subseteq$ the first ε greater than β.

(35) Suppose $\alpha \in \beta$ and $\beta \in$ the first ε greater than α. Then the first ε greater than $\beta =$ the first ε greater than α.

(36) The first ε greater than $0 = \omega \uparrow\uparrow \omega$.

(37) For every epsilon number e holds $\omega \in e$.

One can check that every ordinal number which is epsilon is also non empty limit ordinal.

One can prove the following propositions:

(38) For every epsilon number e holds $\omega^{e^{e^{\omega}}} = e^{e^{\omega}}$.

(39) For every epsilon number e such that $0 \in n$ holds $e \uparrow\uparrow (n+2) = \omega^{e \uparrow\uparrow (n+1)}$.

(40) For every epsilon number e holds the first ε greater than $e = e \uparrow\uparrow \omega$.

Let α be an ordinal number. The functor ε_{α} yields an ordinal number and is defined by the condition (Def. 7).

(Def. 7) There exists a sequence φ of ordinal numbers such that

(i) $\varepsilon_{\alpha} = \operatorname{last}\varphi$,

(ii) $\operatorname{dom}\varphi = \operatorname{succ}\alpha$,

(iii) $\varphi(\emptyset) = \omega \uparrow\uparrow \omega$,

(iv) for every ordinal number β such that $\operatorname{succ}\beta \in \operatorname{succ}\alpha$ holds $\varphi(\operatorname{succ}\beta) = \varphi(\beta) \uparrow\uparrow \omega$, and

[3]An ordinal number α is epsilon iff it is a critical number of exponential map: $\alpha \mapsto \omega^{\alpha}$

(v) for every ordinal number γ such that $\gamma \in \operatorname{succ} \alpha$ and $\gamma \neq \emptyset$ and γ is a limit ordinal number holds $\varphi(\gamma) = \lim(\varphi \restriction \gamma)$.

The following propositions are true:

(41) $\varepsilon_0 = \omega \uparrow\uparrow \omega$.

(42) $\varepsilon_{\operatorname{succ} \alpha} = \varepsilon_\alpha \uparrow\uparrow \omega$.

(43) Suppose $\beta \neq \emptyset$ and β is a limit ordinal number. Let φ be a sequence of ordinal numbers. If $\operatorname{dom} \varphi = \beta$ and for every γ such that $\gamma \in \beta$ holds $\varphi(\gamma) = \varepsilon_\gamma$, then $\varepsilon_\beta = \lim \varphi$.

(44) If $\alpha \in \beta$, then $\varepsilon_\alpha \in \varepsilon_\beta$.

(45) For every sequence φ of ordinal numbers such that for every γ such that $\gamma \in \operatorname{dom} \varphi$ holds $\varphi(\gamma) = \varepsilon_\gamma$ holds φ is increasing.

(46) Suppose $\beta \neq \emptyset$ and β is a limit ordinal number. Let φ be a sequence of ordinal numbers. If $\operatorname{dom} \varphi = \beta$ and for every γ such that $\gamma \in \beta$ holds $\varphi(\gamma) = \varepsilon_\gamma$, then $\varepsilon_\beta = \bigcup \varphi$.

(47) If β is a non empty limit ordinal number, then $x \in \varepsilon_\beta$ iff there exists γ such that $\gamma \in \beta$ and $x \in \varepsilon_\gamma$.

(48)[4] $\alpha \subseteq \varepsilon_\alpha$.

(49) Let X be a non empty set. Suppose that for every x such that $x \in X$ holds x is an epsilon number and there exists an epsilon number e such that $x \in e$ and $e \in X$. Then $\bigcup X$ is an epsilon number.

(50) Let X be a non empty set. Suppose that
 (i) for every x such that $x \in X$ holds x is an epsilon number, and
 (ii) for every α such that $\alpha \in X$ holds the first ε greater than $\alpha \in X$.

Then $\bigcup X$ is an epsilon number.

Let us consider α. Observe that ε_α is epsilon.

The following proposition is true

(51) If α is epsilon, then there exists β such that $\alpha = \varepsilon_\beta$.

5. Cantor Normal Form

Let A be a finite sequence of ordinal numbers. The functor $\sum A$ yielding an ordinal number is defined by the condition (Def. 8).

(Def. 8) There exists a sequence f of ordinal numbers such that $\sum A = \operatorname{last} f$ and $\operatorname{dom} f = \operatorname{succ} \operatorname{dom} A$ and $f(0) = 0$ and for every natural number n such that $n \in \operatorname{dom} A$ holds $f(n+1) = f(n) + A(n)$.

One can prove the following propositions:

(52) $\sum \emptyset = 0$.

[4] Of course not always $\alpha \in \varepsilon_\alpha$ because there are critical α's such that $\alpha = \varepsilon_\alpha$

(53) $\sum\langle\alpha\rangle = \alpha$.

(54) For every finite sequence A of ordinal numbers holds $\sum A ^\frown \langle\alpha\rangle = \sum A + \alpha$.

(55) For every finite sequence A of ordinal numbers holds $\sum\langle\alpha\rangle ^\frown A = \alpha + \sum A$.

(56) For every finite sequence A of ordinal numbers holds $A(0) \subseteq \sum A$.

Let us consider α. We say that α is Cantor component if and only if:

(Def. 9) There exist β, n such that $0 \in n$ and $\alpha = n \cdot \omega^{\beta}$.

Let us note that every ordinal number which is Cantor component is also non empty.

Let us note that there exists an ordinal number which is Cantor component.

Let us consider α, β. The functor β-exponent(α) yields an ordinal number and is defined by:

(Def. 10)(i) $\beta^{\beta\text{-exponent}(\alpha)} \subseteq \alpha$ and for every γ such that $\beta^{\gamma} \subseteq \alpha$ holds $\gamma \subseteq \beta$-exponent(α) if $1 \in \beta$ and $0 \in \alpha$,

(ii) β-exponent$(\alpha) = 0$, otherwise.

The following propositions are true:

(57) $\alpha \in \omega^{\mathrm{succ}(\omega\text{-exponent}(\alpha))}$.

(58)[5] If $0 \in n$, then ω-exponent$(n \cdot \omega^{\alpha}) = \alpha$.

(59) If $0 \in \alpha$ and $\gamma = \omega$-exponent(α), then $\alpha \div \omega^{\gamma} \in \omega$.

(60) If $0 \in \alpha$ and $\gamma = \omega$-exponent(α), then $0 \in \alpha \div \omega^{\gamma}$.

(61) If $0 \in \alpha$ and $\gamma = \omega$-exponent(α), then $\alpha \bmod \omega^{\gamma} \in \omega^{\gamma}$.

(62) If $0 \in \alpha$, then there exist n, β such that $\alpha = n \cdot \omega^{\omega\text{-exponent}(\alpha)} + \beta$ and $0 \in n$ and $\beta \in \omega^{\omega\text{-exponent}(\alpha)}$.

(63) If ω-exponent$(\beta) \in \omega$-exponent(α), then $\beta \in \alpha$.

Let A be a sequence of ordinal numbers. We say that A is Cantor normal form if and only if:

(Def. 11) For every α such that $\alpha \in \operatorname{dom} A$ holds $A(\alpha)$ is Cantor component and for all α, β such that $\alpha \in \beta$ and $\beta \in \operatorname{dom} A$ holds ω-exponent$(A(\beta)) \in \omega$-exponent$(A(\alpha))$.

Let us note that every sequence of ordinal numbers which is empty is also Cantor normal form and every sequence of ordinal numbers which is Cantor normal form is also decreasing and finite.

In the sequel $\mathcal{C}$, $\mathcal{B}$ are Cantor normal form sequences of ordinal numbers.

One can prove the following propositions:

(64) If $\sum \mathcal{C} = 0$, then $\mathcal{C} = \emptyset$.

(65) If $0 \in n$, then $\langle n \cdot \omega^{\beta}\rangle$ is Cantor normal form.

Let us note that there exists a sequence of ordinal numbers which is non empty and Cantor normal form.

[5] α-exponent(β) is the entier of the logarithm

The following four propositions are true:

(66) Let $\mathcal{C}$, $\mathcal{B}$ be sequences of ordinal numbers. Suppose $\mathcal{C}^\frown\mathcal{B}$ is Cantor normal form. Then $\mathcal{C}$ is Cantor normal form and $\mathcal{B}$ is Cantor normal form.

(67) If $\mathcal{C} \neq \emptyset$, then there exists a Cantor component ordinal number γ and there exists $\mathcal{B}$ such that $\mathcal{C} = \langle\gamma\rangle ^\frown \mathcal{B}$.

(68) Let $\mathcal{C}$ be a non empty Cantor normal form sequence of ordinal numbers and γ be a Cantor component ordinal number. If ω-exponent($\mathcal{C}(0)$) $\in$ ω-exponent(γ), then $\langle\gamma\rangle ^\frown \mathcal{C}$ is Cantor normal form.

(69)[6] For every ordinal number α there exists a Cantor normal form sequence $\mathcal{C}$ of ordinal numbers such that $\alpha = \sum\mathcal{C}$.

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[6] $\alpha = n_1\omega^{\beta_1} + n_2\omega^{\beta_2} + \cdots + n_k\omega^{\beta_k}$