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Riemann Integral of Functions from $\mathbb{R}$ into $\mathcal{R}^n$

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Summary. In this article, we define the Riemann Integral of functions from $\mathbb{R}$ into $\mathcal{R}^n$, and prove the linearity of this operator. The presented method is based on [21].

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The articles [22], [1], [23], [5], [6], [15], [20], [24], [7], [17], [16], [2], [4], [3], [8], [18], [9], [12], [10], [14], [13], [19], and [11] provide the notation and terminology for this paper.

1. Preliminaries

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{R}$, let S be a non empty Division of A, and let D be an element of S. A finite sequence of elements of $\mathbb{R}$ is said to be a middle volume of f and D if it satisfies the conditions (Def. 1).

(Def. 1)(i) $\operatorname{len} \mathrm{it} = \operatorname{len} D$, and

(ii) for every natural number i such that $i \in \operatorname{dom} D$ there exists an element r of $\mathbb{R}$ such that $r \in \operatorname{rng}(f \restriction \operatorname{divset}(D, i))$ and $\mathrm{it}(i) = r \cdot \operatorname{vol}(\operatorname{divset}(D, i))$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{R}$, let S be a non empty Division of A, let D be an element of S, and let F be a middle volume of f and D. The functor middle_sum(f, F) yielding a real number is defined as follows:

(Def. 2) $\text{middle\_sum}(f, F) = \sum F$.

We now state four propositions:

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(1) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and F be a middle volume of f and D. If $f{\restriction}A$ is lower bounded, then lower_sum$(f, D) \leq$ middle_sum(f, F).

(2) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and F be a middle volume of f and D. If $f{\restriction}A$ is upper bounded, then middle_sum$(f, F) \leq$ upper_sum(f, D).

(3) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and e be a real number. Suppose $f{\restriction}A$ is lower bounded and $0 < e$. Then there exists a middle volume F of f and D such that middle_sum$(f, F) \leq$ lower_sum$(f, D) + e$.

(4) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and e be a real number. Suppose $f{\restriction}A$ is upper bounded and $0 < e$. Then there exists a middle volume F of f and D such that upper_sum$(f, D) - e \leq$ middle_sum(f, F).

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{R}$, and let T be a DivSequence of A. A function from $\mathbb{N}$ into $\mathbb{R}^*$ is said to be a middle volume sequence of f and T if:

(Def. 3) For every element k of $\mathbb{N}$ holds it(k) is a middle volume of f and $T(k)$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{R}$, let T be a DivSequence of A, let S be a middle volume sequence of f and T, and let k be an element of $\mathbb{N}$. Then $S(k)$ is a middle volume of f and $T(k)$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{R}$, let T be a DivSequence of A, and let S be a middle volume sequence of f and T. The functor middle_sum(f, S) yields a sequence of real numbers and is defined by:

(Def. 4) For every element i of $\mathbb{N}$ holds (middle_sum$(f, S))(i) =$ middle_sum$(f, S(i))$.

We now state several propositions:

(5) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, T be a DivSequence of A, S be a middle volume sequence of f and T, and i be an element of $\mathbb{N}$. If $f{\restriction}A$ is lower bounded, then (lower_sum$(f, T))(i) \leq$ (middle_sum$(f, S))(i)$.

(6) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, T be a DivSequence of A, S be a middle volume sequence of f and T, and i be an element of $\mathbb{N}$. If $f{\restriction}A$ is upper bounded, then (middle_sum$(f, S))(i) \leq$ (upper_sum$(f, T))(i)$.

(7) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, T be a DivSequence of A, and e be an element of $\mathbb{R}$. Suppose $0 < e$ and $f{\restriction}A$ is lower bounded. Then there exists a middle volume sequence S of

f and T such that for every element i of $\mathbb{N}$ holds $(\text{middle\_sum}(f, S))(i) \leq (\text{lower\_sum}(f, T))(i) + e$.

(8) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, T be a DivSequence of A, and e be an element of $\mathbb{R}$. Suppose $0 < e$ and $f \restriction A$ is upper bounded. Then there exists a middle volume sequence S of f and T such that for every element i of $\mathbb{N}$ holds $(\text{upper\_sum}(f, T))(i) - e \leq (\text{middle\_sum}(f, S))(i)$.

(9) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{R}$, T be a DivSequence of A, and S be a middle volume sequence of f and T. Suppose f is bounded and f is integrable on A and δ_T is convergent and $\lim(\delta_T) = 0$. Then $\text{middle\_sum}(f, S)$ is convergent and $\lim \text{middle\_sum}(f, S) = \text{integral}\, f$.

(10) Let A be a closed-interval subset of $\mathbb{R}$ and f be a function from A into $\mathbb{R}$. Suppose f is bounded. Then f is integrable on A if and only if there exists a real number I such that for every DivSequence T of A and for every middle volume sequence S of f and T such that δ_T is convergent and $\lim(\delta_T) = 0$ holds $\text{middle\_sum}(f, S)$ is convergent and $\lim \text{middle\_sum}(f, S) = I$.

Let n be an element of $\mathbb{N}$, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathcal{R}^n$, let S be a non empty Division of A, and let D be an element of S. A finite sequence of elements of $\mathcal{R}^n$ is said to be a middle volume of f and D if it satisfies the conditions (Def. 5).

(Def. 5)(i) $\text{len}\, \text{it} = \text{len}\, D$, and

(ii) for every natural number i such that $i \in \text{dom}\, D$ there exists an element r of $\mathcal{R}^n$ such that $r \in \text{rng}(f \restriction \text{divset}(D, i))$ and $\text{it}(i) = \text{vol}(\text{divset}(D, i)) \cdot r$.

Let n be an element of $\mathbb{N}$, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathcal{R}^n$, let S be a non empty Division of A, let D be an element of S, and let F be a middle volume of f and D. The functor $\text{middle\_sum}(f, F)$ yielding an element of $\mathcal{R}^n$ is defined by the condition (Def. 6).

(Def. 6) Let i be an element of $\mathbb{N}$. Suppose $i \in \text{Seg}\, n$. Then there exists a finite sequence F_1 of elements of $\mathbb{R}$ such that $F_1 = \text{proj}(i, n) \cdot F$ and $(\text{middle\_sum}(f, F))(i) = \sum F_1$.

Let n be an element of $\mathbb{N}$, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathcal{R}^n$, and let T be a DivSequence of A. A function from $\mathbb{N}$ into $(\mathcal{R}^n)^*$ is said to be a middle volume sequence of f and T if:

(Def. 7) For every element k of $\mathbb{N}$ holds $\text{it}(k)$ is a middle volume of f and $T(k)$.

Let n be an element of $\mathbb{N}$, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathcal{R}^n$, let T be a DivSequence of A, let S be a middle volume sequence of f and T, and let k be an element of $\mathbb{N}$. Then $S(k)$ is a middle volume of f and $T(k)$.

Let n be an element of $\mathbb{N}$, let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathcal{R}^n$, let T be a DivSequence of A, and let S be a middle volume sequence of f and T. The functor middle_sum(f, S) yields a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and is defined as follows:

(Def. 8) For every element i of $\mathbb{N}$ holds $(\text{middle\_sum}(f, S))(i) = \text{middle\_sum}(f, S(i))$.

Let n be an element of $\mathbb{N}$, let Z be a non empty set, and let f, g be partial functions from Z to $\mathcal{R}^n$. The functor $f + g$ yielding a partial function from Z to $\mathcal{R}^n$ is defined by:

(Def. 9) $\text{dom}(f + g) = \text{dom}\, f \cap \text{dom}\, g$ and for every element c of Z such that $c \in \text{dom}(f + g)$ holds $(f + g)_c = f_c + g_c$.

The functor $f - g$ yielding a partial function from Z to $\mathcal{R}^n$ is defined as follows:

(Def. 10) $\text{dom}(f - g) = \text{dom}\, f \cap \text{dom}\, g$ and for every element c of Z such that $c \in \text{dom}(f - g)$ holds $(f - g)_c = f_c - g_c$.

Let n be an element of $\mathbb{N}$, let r be a real number, let Z be a non empty set, and let f be a partial function from Z to $\mathcal{R}^n$. The functor $r\, f$ yielding a partial function from Z to $\mathcal{R}^n$ is defined as follows:

(Def. 11) $\text{dom}(r\, f) = \text{dom}\, f$ and for every element c of Z such that $c \in \text{dom}(r\, f)$ holds $(r\, f)_c = r \cdot f_c$.

2. Definition of Riemann Integral of Functions from $\mathbb{R}$ into $\mathcal{R}^n$

Let n be an element of $\mathbb{N}$, let A be a closed-interval subset of $\mathbb{R}$, and let f be a function from A into $\mathcal{R}^n$. We say that f is bounded if and only if:

(Def. 12) For every element i of $\mathbb{N}$ such that $i \in \text{Seg}\, n$ holds $\text{proj}(i, n) \cdot f$ is bounded.

Let n be an element of $\mathbb{N}$, let A be a closed-interval subset of $\mathbb{R}$, and let f be a function from A into $\mathcal{R}^n$. We say that f is integrable if and only if:

(Def. 13) For every element i of $\mathbb{N}$ such that $i \in \text{Seg}\, n$ holds $\text{proj}(i, n) \cdot f$ is integrable on A.

Let n be an element of $\mathbb{N}$, let A be a closed-interval subset of $\mathbb{R}$, and let f be a function from A into $\mathcal{R}^n$. The functor integral f yielding an element of $\mathcal{R}^n$ is defined by:

(Def. 14) $\text{dom}\, \text{integral}\, f = \text{Seg}\, n$ and for every element i of $\mathbb{N}$ such that $i \in \text{Seg}\, n$ holds $(\text{integral}\, f)(i) = \text{integral}\, \text{proj}(i, n) \cdot f$.

One can prove the following two propositions:

(11) Let n be an element of $\mathbb{N}$, A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathcal{R}^n$, T be a DivSequence of A, and S be a middle volume sequence of f and T. Suppose f is bounded and integrable and δ_T is convergent and $\lim(\delta_T) = 0$. Then middle_sum(f, S) is convergent and $\lim \text{middle\_sum}(f, S) = \text{integral}\, f$.

(12) Let n be an element of $\mathbb{N}$, A be a closed-interval subset of $\mathbb{R}$, and f be a function from A into $\mathcal{R}^n$. Suppose f is bounded. Then f is integrable if and only if there exists an element I of $\mathcal{R}^n$ such that for every DivSequence T of A and for every middle volume sequence S of f and T such that δ_T is convergent and $\lim(\delta_T) = 0$ holds $\mathrm{middle\_sum}(f, S)$ is convergent and $\lim \mathrm{middle\_sum}(f, S) = I$.

Let n be an element of $\mathbb{N}$ and let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. We say that f is bounded if and only if:

(Def. 15) For every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{proj}(i, n) \cdot f$ is bounded.

Let n be an element of $\mathbb{N}$, let A be a closed-interval subset of $\mathbb{R}$, and let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. We say that f is integrable on A if and only if:

(Def. 16) For every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $\mathrm{proj}(i, n) \cdot f$ is integrable on A.

Let n be an element of $\mathbb{N}$, let A be a closed-interval subset of $\mathbb{R}$, and let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. The functor $\int\limits_A f(x)dx$ yields an element of $\mathcal{R}^n$ and is defined by:

(Def. 17) $\mathrm{dom} \int\limits_A f(x)dx = \mathrm{Seg}\, n$ and for every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $(\int\limits_A f(x)dx)(i) = \int\limits_A (\mathrm{proj}(i, n) \cdot f)(x)dx$.

The following two propositions are true:

(13) Let n be an element of $\mathbb{N}$, A be a closed-interval subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and g be a function from A into $\mathcal{R}^n$. Suppose $f \restriction A = g$. Then f is integrable on A if and only if g is integrable.

(14) Let n be an element of $\mathbb{N}$, A be a closed-interval subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and g be a function from A into $\mathcal{R}^n$. If $f \restriction A = g$, then $\int\limits_A f(x)dx = \mathrm{integral}\, g$.

Let a, b be real numbers, let n be an element of $\mathbb{N}$, and let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. The functor $\int\limits_a^b f(x)dx$ yielding an element of $\mathcal{R}^n$ is defined as follows:

(Def. 18) $\mathrm{dom} \int\limits_a^b f(x)dx = \mathrm{Seg}\, n$ and for every element i of $\mathbb{N}$ such that $i \in \mathrm{Seg}\, n$ holds $(\int\limits_a^b f(x)dx)(i) = \int\limits_a^b (\mathrm{proj}(i, n) \cdot f)(x)dx$.

3. Linearity of Integration Operator

We now state several propositions:

(15) Let n be an element of $\mathbb{N}$, f_1, f_2 be partial functions from $\mathbb{R}$ to $\mathcal{R}^n$, and i be an element of $\mathbb{N}$. If $i \in \mathrm{Seg}\, n$, then $\mathrm{proj}(i,n) \cdot (f_1 + f_2) = \mathrm{proj}(i,n) \cdot f_1 + \mathrm{proj}(i,n) \cdot f_2$ and $\mathrm{proj}(i,n) \cdot (f_1 - f_2) = \mathrm{proj}(i,n) \cdot f_1 - \mathrm{proj}(i,n) \cdot f_2$.

(16) Let n be an element of $\mathbb{N}$, r be a real number, f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and i be an element of $\mathbb{N}$. If $i \in \mathrm{Seg}\, n$, then $\mathrm{proj}(i,n) \cdot (r\, f) = r\, (\mathrm{proj}(i,n) \cdot f)$.

(17) Let n be an element of $\mathbb{N}$, A be a closed-interval subset of $\mathbb{R}$, and f_1, f_2 be partial functions from $\mathbb{R}$ to $\mathcal{R}^n$. Suppose f_1 is integrable on A and f_2 is integrable on A and $A \subseteq \mathrm{dom}\, f_1$ and $A \subseteq \mathrm{dom}\, f_2$ and $f_1{\restriction}A$ is bounded and $f_2{\restriction}A$ is bounded. Then $f_1 + f_2$ is integrable on A and $f_1 - f_2$ is integrable on A and $\int\limits_A (f_1 + f_2)(x)dx = \int\limits_A f_1(x)dx + \int\limits_A f_2(x)dx$ and $\int\limits_A (f_1 - f_2)(x)dx = \int\limits_A f_1(x)dx - \int\limits_A f_2(x)dx$.

(18) Let n be an element of $\mathbb{N}$, r be a real number, A be a closed-interval subset of $\mathbb{R}$, and f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Suppose $A \subseteq \mathrm{dom}\, f$ and f is integrable on A and $f{\restriction}A$ is bounded. Then $r\, f$ is integrable on A and $\int\limits_A (r\, f)(x)dx = r \cdot \int\limits_A f(x)dx$.

(19) Let n be an element of $\mathbb{N}$, f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, A be a closed-interval subset of $\mathbb{R}$, and a, b be real numbers. If $A = [a,b]$, then $\int\limits_A f(x)dx = \int\limits_a^b f(x)dx$.

(20) Let n be an element of $\mathbb{N}$, f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, A be a closed-interval subset of $\mathbb{R}$, and a, b be real numbers. If $A = [b,a]$, then $-\int\limits_A f(x)dx = \int\limits_a^b f(x)dx$.

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