

Kolmogorov's Zero-One Law

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Summary. This article presents the proof of Kolmogorov's zero-one law in probability theory. The independence of a family of σ -fields is defined and basic theorems on it are given.

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The articles [8], [19], [2], [10], [12], [18], [20], [1], [15], [5], [21], [11], [3], [9], [7], [6], [17], [4], [16], [14], and [13] provide the terminology and notation for this paper.

For simplicity, we adopt the following convention: Ω, I are non empty sets, \mathcal{F} is a σ -field of subsets of Ω , P is a probability on \mathcal{F} , D, E, F are families of subsets of Ω , A, B, s are non empty subsets of \mathcal{F} , b is an element of B , a is an element of \mathcal{F} , p, q, u, v are events of \mathcal{F} , n is an element of \mathbb{N} , and i is a set.

Next we state three propositions:

- (1) For every function f and for every set X such that $X \subseteq \text{dom } f$ holds if $X \neq \emptyset$, then $\text{rng}(f \upharpoonright X) \neq \emptyset$.
- (2) For every real number r such that $r \cdot r = r$ holds $r = 0$ or $r = 1$.
- (3) For every family X of subsets of Ω such that $X = \emptyset$ holds $\sigma(X) = \{\emptyset, \Omega\}$.

Let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , let B be a subset of \mathcal{F} , and let P be a probability on \mathcal{F} . The functor $\text{Indep}(B, P)$ yielding a subset of \mathcal{F} is defined as follows:

- (Def. 1) For every element a of \mathcal{F} holds $a \in \text{Indep}(B, P)$ iff for every element b of B holds $P(a \cap b) = P(a) \cdot P(b)$.

Next we state several propositions:

- (4) Let f be a sequence of subsets of \mathcal{F} . Suppose for all n, b holds $P(f(n) \cap b) = P(f(n)) \cdot P(b)$ and f is disjoint valued. Then $P(b \cap \bigcup f) = P(b) \cdot P(\bigcup f)$.

- (5) $\text{Indep}(B, P)$ is a Dynkin system of Ω .
- (6) For every family A of subsets of Ω such that A is intersection stable and $A \subseteq \text{Indep}(B, P)$ holds $\sigma(A) \subseteq \text{Indep}(B, P)$.
- (7) Let A, B be non empty subsets of \mathcal{F} . Then $A \subseteq \text{Indep}(B, P)$ if and only if for all p, q such that $p \in A$ and $q \in B$ holds p and q are independent w.r.t. P .
- (8) For all non empty subsets A, B of \mathcal{F} such that $A \subseteq \text{Indep}(B, P)$ holds $B \subseteq \text{Indep}(A, P)$.
- (9) Let A be a family of subsets of Ω . Suppose A is a non empty subset of \mathcal{F} and intersection stable. Let B be a non empty subset of \mathcal{F} . Suppose B is intersection stable. If $A \subseteq \text{Indep}(B, P)$, then for all D, s such that $D = B$ and $\sigma(D) = s$ holds $\sigma(A) \subseteq \text{Indep}(s, P)$.
- (10) Let given E, F . Suppose that
 - (i) E is a non empty subset of \mathcal{F} and intersection stable, and
 - (ii) F is a non empty subset of \mathcal{F} and intersection stable.
Suppose that for all p, q such that $p \in E$ and $q \in F$ holds p and q are independent w.r.t. P . Let given u, v . If $u \in \sigma(E)$ and $v \in \sigma(F)$, then u and v are independent w.r.t. P .

Let I be a set, let Ω be a non empty set, and let \mathcal{F} be a σ -field of subsets of Ω . A function from I into $2^{\mathcal{F}}$ is said to be a many sorted σ -field over I and \mathcal{F} if:

- (Def. 2) For every i such that $i \in I$ holds $\text{it}(i)$ is a σ -field of subsets of Ω .

Let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , let P be a probability on \mathcal{F} , let I be a set, and let A be a function from I into \mathcal{F} . We say that A is independent w.r.t. P if and only if:

- (Def. 3) For every one-to-one finite sequence e of elements of I such that $e \neq \emptyset$ holds $\prod(P \cdot A \cdot e) = P(\bigcap \text{rng}(A \cdot e))$.

Let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , let I be a set, let J be a subset of I , and let F be a many sorted σ -field over I and \mathcal{F} . A function from J into \mathcal{F} is said to be a σ -section over J and F if:

- (Def. 4) For every i such that $i \in J$ holds $\text{it}(i) \in F(i)$.

Let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , let P be a probability on \mathcal{F} , let I be a set, and let F be a many sorted σ -field over I and \mathcal{F} . We say that F is independent w.r.t. P if and only if:

- (Def. 5) For every finite subset E of I holds every σ -section over E and F is independent w.r.t. P .

Let I be a set, let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , let F be a many sorted σ -field over I and \mathcal{F} , and let J be a subset of I . Then $F \upharpoonright J$ is a function from J into $2^{\mathcal{F}}$.

Let I be a set, let J be a subset of I , let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , and let F be a function from J into $2^{\mathcal{F}}$. Then $\bigcup F$ is a family of subsets of Ω .

Let I be a set, let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , let F be a many sorted σ -field over I and \mathcal{F} , and let J be a subset of I . The functor $\text{sigUn}(F, J)$ yields a σ -field of subsets of Ω and is defined as follows:

(Def. 6) $\text{sigUn}(F, J) = \sigma(\bigcup(F \upharpoonright J))$.

Let I be a set, let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , and let F be a many sorted σ -field over I and \mathcal{F} . The functor $\text{futSigmaFields}(F, I)$ yielding a family of subsets of 2^{Ω} is defined as follows:

(Def. 7) For every family S of subsets of Ω holds $S \in \text{futSigmaFields}(F, I)$ iff there exists a finite subset E of I such that $S = \text{sigUn}(F, I \setminus E)$.

Let I be a set, let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , and let F be a many sorted σ -field over I and \mathcal{F} . Note that $\text{futSigmaFields}(F, I)$ is non empty.

Let I be a set, let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , and let F be a many sorted σ -field over I and \mathcal{F} . The functor $\text{tailSigmaField}(F, I)$ yielding a family of subsets of Ω is defined as follows:

(Def. 8) $\text{tailSigmaField}(F, I) = \bigcap \text{futSigmaFields}(F, I)$.

Let I be a set, let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , and let F be a many sorted σ -field over I and \mathcal{F} . Note that $\text{tailSigmaField}(F, I)$ is non empty.

Let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , let I be a non empty set, let J be a non empty subset of I , and let F be a many sorted σ -field over I and \mathcal{F} . The functor $\text{MeetSections}(J, F)$ yields a family of subsets of Ω and is defined by the condition (Def. 9).

(Def. 9) Let x be a subset of Ω . Then $x \in \text{MeetSections}(J, F)$ if and only if there exists a non empty finite subset E of I and there exists a σ -section f over E and F such that $E \subseteq J$ and $x = \bigcap \text{rng } f$.

One can prove the following propositions:

(11) For every many sorted σ -field F over I and \mathcal{F} and for every non empty subset J of I holds $\sigma(\text{MeetSections}(J, F)) = \text{sigUn}(F, J)$.

(12) Let F be a many sorted σ -field over I and \mathcal{F} and J, K be non empty subsets of I . Suppose F is independent w.r.t. P and J misses K . Let a, c be subsets of Ω . If $a \in \text{MeetSections}(J, F)$ and $c \in \text{MeetSections}(K, F)$, then $P(a \cap c) = P(a) \cdot P(c)$.

(13) Let F be a many sorted σ -field over I and \mathcal{F} and J be a non empty subset of I . Then $\text{MeetSections}(J, F)$ is a non empty subset of \mathcal{F} .

Let us consider I, Ω, \mathcal{F} , let F be a many sorted σ -field over I and \mathcal{F} , and let J be a non empty subset of I . Observe that $\text{MeetSections}(J, F)$ is intersection

stable.

The following proposition is true

- (14) Let F be a many sorted σ -field over I and \mathcal{F} and J, K be non empty subsets of I . Suppose F is independent w.r.t. P and J misses K . Let given u, v . If $u \in \text{sigUn}(F, J)$ and $v \in \text{sigUn}(F, K)$, then $P(u \cap v) = P(u) \cdot P(v)$.

Let I be a set, let Ω be a non empty set, let \mathcal{F} be a σ -field of subsets of Ω , and let F be a many sorted σ -field over I and \mathcal{F} . The functor $\text{finSigmaFields}(F, I)$ yielding a family of subsets of Ω is defined as follows:

- (Def. 10) For every subset S of Ω holds $S \in \text{finSigmaFields}(F, I)$ iff there exists a finite subset E of I such that $S \in \text{sigUn}(F, E)$.

One can prove the following propositions:

- (15) For every many sorted σ -field F over I and \mathcal{F} holds $\text{tailSigmaField}(F, I)$ is a σ -field of subsets of Ω .
- (16) Let F be a many sorted σ -field over I and \mathcal{F} . If F is independent w.r.t. P and $a \in \text{tailSigmaField}(F, I)$, then $P(a) = 0$ or $P(a) = 1$.

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