

# Lebesgue's Convergence Theorem of Complex-Valued Function

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**Summary.** In this article, we formalized Lebesgue's Convergence theorem of complex-valued function. We proved Lebesgue's Convergence Theorem of real-valued function using the theorem of extensional real-valued function. Then applying the former theorem to real part and imaginary part of complex-valued functional sequences, we proved Lebesgue's Convergence Theorem of complex-valued function. We also defined partial sums of real-valued functional sequences and complex-valued functional sequences and showed their properties. In addition, we proved properties of complex-valued simple functions.

MML identifier: MESFUN9C, version: 7.11.02 4.120.1050

The articles [24], [1], [4], [12], [25], [5], [26], [6], [7], [18], [19], [2], [8], [14], [13], [20], [21], [3], [11], [22], [15], [10], [16], [9], [17], and [23] provide the notation and terminology for this paper.

## 1. PARTIAL SUMS OF REAL-VALUED FUNCTIONAL SEQUENCES

For simplicity, we use the following convention:  $X$  is a non empty set,  $S$  is a  $\sigma$ -field of subsets of  $X$ ,  $M$  is a  $\sigma$ -measure on  $S$ ,  $E$  is an element of  $S$ ,  $F$  is a sequence of partial functions from  $X$  into  $\mathbb{R}$ ,  $f$  is a partial function from  $X$  to  $\mathbb{R}$ ,  $s$  is a sequence of real numbers,  $n, m$  are natural numbers,  $x$  is an element of  $X$ , and  $z, D$  are sets.

Let  $X, Y$  be sets, let  $F$  be a sequence of partial functions from  $X$  into  $Y$ , and let  $D$  be a set. The functor  $F \upharpoonright D$  yielding a sequence of partial functions from  $X$  into  $Y$  is defined by:

(Def. 1) For every natural number  $n$  holds  $(F \upharpoonright D)(n) = F(n) \upharpoonright D$ .

One can prove the following propositions:

- (1) If  $x \in D$  and  $F \# x$  is convergent, then  $(F \upharpoonright D) \# x$  is convergent.
- (2) Let  $X, Y, D$  be sets and  $F$  be a sequence of partial functions from  $X$  into  $Y$ . If  $F$  has the same dom, then  $F \upharpoonright D$  has the same dom.
- (3) If  $D \subseteq \text{dom } F(0)$  and for every element  $x$  of  $X$  such that  $x \in D$  holds  $F \# x$  is convergent, then  $\lim F \upharpoonright D = \lim(F \upharpoonright D)$ .
- (4) Suppose  $F$  has the same dom and  $E \subseteq \text{dom } F(0)$  and for every natural number  $m$  holds  $F(m)$  is measurable on  $E$ . Then  $(F \upharpoonright E)(n)$  is measurable on  $E$ .
- (5)  $(\sum_{\alpha=0}^{\kappa} (\overline{\mathbb{R}}(s))(\alpha))_{\kappa \in \mathbb{N}} = \overline{\mathbb{R}}((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}})$ .
- (6) Suppose that for every element  $x$  of  $X$  such that  $x \in E$  holds  $F \# x$  is summable. Let  $x$  be an element of  $X$ . If  $x \in E$ , then  $(F \upharpoonright E) \# x$  is summable.

Let  $X$  be a non empty set and let  $F$  be a sequence of partial functions from  $X$  into  $\mathbb{R}$ . The functor  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$  yields a sequence of partial functions from  $X$  into  $\mathbb{R}$  and is defined by:

(Def. 2)  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(0) = F(0)$  and for every element  $n$  of  $\mathbb{N}$  holds  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) + F(n+1)$ .

One can prove the following propositions:

- (7)  $(\sum_{\alpha=0}^{\kappa} (\overline{\mathbb{R}}(F))(\alpha))_{\kappa \in \mathbb{N}} = \overline{\mathbb{R}}((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})$ .
- (8) If  $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$  and  $m \leq n$ , then  $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$  and  $z \in \text{dom } F(m)$ .
- (9)  $\overline{\mathbb{R}}(F)$  is additive.
- (10)  $\text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) = \bigcap \{ \text{dom } F(k); k \text{ ranges over elements of } \mathbb{N}: k \leq n \}$ .
- (11) If  $F$  has the same dom, then  $\text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) = \text{dom } F(0)$ .
- (12) If  $F$  has the same dom and  $D \subseteq \text{dom } F(0)$  and  $x \in D$ , then  $(\sum_{\alpha=0}^{\kappa} (F \# x)(\alpha))_{\kappa \in \mathbb{N}}(n) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x)(n)$ .
- (13) If  $F$  has the same dom and  $D \subseteq \text{dom } F(0)$  and  $x \in D$ , then  $(\sum_{\alpha=0}^{\kappa} (F \# x)(\alpha))_{\kappa \in \mathbb{N}}$  is convergent iff  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x$  is convergent.
- (14) If  $F$  has the same dom and  $\text{dom } f \subseteq \text{dom } F(0)$  and  $x \in \text{dom } f$  and  $f(x) = \sum(F \# x)$ , then  $f(x) = \lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x)$ .
- (15) If for every natural number  $m$  holds  $F(m)$  is simple function in  $S$ , then  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$  is simple function in  $S$ .

- (16) If for every natural number  $n$  holds  $F(n)$  is measurable on  $E$ , then  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$  is measurable on  $E$ .
- (17) Let  $X$  be a non empty set and  $F$  be a sequence of partial functions from  $X$  into  $\mathbb{R}$ . If  $F$  has the same dom, then  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$  has the same dom.
- (18) Suppose that
- (i)  $\text{dom } F(0) = E$ ,
  - (ii)  $F$  has the same dom,
  - (iii) for every natural number  $n$  holds  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$  is measurable on  $E$ , and
  - (iv) for every element  $x$  of  $X$  such that  $x \in E$  holds  $F\#x$  is summable. Then  $\lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})$  is measurable on  $E$ .
- (19) Suppose that for every natural number  $n$  holds  $F(n)$  is integrable on  $M$ . Let  $m$  be a natural number. Then  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$  is integrable on  $M$ .

## 2. PARTIAL SUMS OF COMPLEX-VALUED FUNCTIONAL SEQUENCES

In the sequel  $F$  denotes a sequence of partial functions from  $X$  into  $\mathbb{C}$ ,  $f$  denotes a partial function from  $X$  to  $\mathbb{C}$ , and  $A$  denotes a set.

We now state several propositions:

- (20)  $\Re(f) \upharpoonright A = \Re(f \upharpoonright A)$  and  $\Im(f) \upharpoonright A = \Im(f \upharpoonright A)$ .
- (21)  $\Re(F \upharpoonright D) = \Re(F) \upharpoonright D$ .
- (22)  $\Im(F \upharpoonright D) = \Im(F) \upharpoonright D$ .
- (23) If  $F$  has the same dom and  $D \subseteq \text{dom } F(0)$  and  $x \in D$ , then if  $F\#x$  is convergent, then  $(F \upharpoonright D)\#x$  is convergent.
- (24)  $F$  has the same dom iff  $\Re(F)$  has the same dom.
- (25)  $\Re(F)$  has the same dom iff  $\Im(F)$  has the same dom.
- (26) If  $F$  has the same dom and  $D = \text{dom } F(0)$  and for every element  $x$  of  $X$  such that  $x \in D$  holds  $F\#x$  is convergent, then  $\lim F \upharpoonright D = \lim(F \upharpoonright D)$ .
- (27) Suppose  $F$  has the same dom and  $E \subseteq \text{dom } F(0)$  and for every natural number  $m$  holds  $F(m)$  is measurable on  $E$ . Then  $(F \upharpoonright E)(n)$  is measurable on  $E$ .
- (28) Suppose  $E \subseteq \text{dom } F(0)$  and  $F$  has the same dom and for every element  $x$  of  $X$  such that  $x \in E$  holds  $F\#x$  is summable. Let  $x$  be an element of  $X$ . If  $x \in E$ , then  $(F \upharpoonright E)\#x$  is summable.

Let  $X$  be a non empty set and let  $F$  be a sequence of partial functions from  $X$  into  $\mathbb{C}$ . The functor  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$  yielding a sequence of partial functions from  $X$  into  $\mathbb{C}$  is defined by:

(Def. 3)  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(0) = F(0)$  and for every natural number  $n$  holds  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) + F(n+1)$ .

The following propositions are true:

$$(29) \quad (\sum_{\alpha=0}^{\kappa} \Re(F)(\alpha))_{\kappa \in \mathbb{N}} = \Re((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa}), \quad (\sum_{\alpha=0}^{\kappa} \Im(F)(\alpha))_{\kappa \in \mathbb{N}} = \Im((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}).$$

(30) If  $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$  and  $m \leq n$ , then  $z \in \text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$  and  $z \in \text{dom} F(m)$ .

(31)  $\text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) = \bigcap \{\text{dom} F(k); k \text{ ranges over elements of } \mathbb{N}; k \leq n\}$ .

(32) If  $F$  has the same dom, then  $\text{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n) = \text{dom} F(0)$ .

(33) If  $F$  has the same dom and  $D \subseteq \text{dom} F(0)$  and  $x \in D$ , then  $(\sum_{\alpha=0}^{\kappa} (F \# x)(\alpha))_{\kappa \in \mathbb{N}}(n) = ((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x)(n)$ .

(34) If  $F$  has the same dom, then  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}$  has the same dom.

(35) If  $F$  has the same dom and  $D \subseteq \text{dom} F(0)$  and  $x \in D$ , then  $(\sum_{\alpha=0}^{\kappa} (F \# x)(\alpha))_{\kappa \in \mathbb{N}}$  is convergent iff  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x$  is convergent.

(36) If  $F$  has the same dom and  $\text{dom} f \subseteq \text{dom} F(0)$  and  $x \in \text{dom} f$  and  $F \# x$  is summable and  $f(x) = \sum(F \# x)$ , then  $f(x) = \lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \# x)$ .

(37) If for every natural number  $m$  holds  $F(m)$  is simple function in  $S$ , then  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$  is simple function in  $S$ .

(38) If for every natural number  $n$  holds  $F(n)$  is measurable on  $E$ , then  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$  is measurable on  $E$ .

(39) Suppose that

(i)  $\text{dom} F(0) = E$ ,

(ii)  $F$  has the same dom,

(iii) for every natural number  $n$  holds  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)$  is measurable on  $E$ , and

(iv) for every element  $x$  of  $X$  such that  $x \in E$  holds  $F \# x$  is summable.

Then  $\lim((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})$  is measurable on  $E$ .

(40) Suppose that for every natural number  $n$  holds  $F(n)$  is integrable on  $M$ . Let  $m$  be a natural number. Then  $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$  is integrable on  $M$ .

### 3. SELECTED PROPERTIES OF COMPLEX-VALUED SIMPLE FUNCTIONS

In the sequel  $f, g$  denote partial functions from  $X$  to  $\mathbb{C}$  and  $A$  denotes an element of  $S$ .

Next we state several propositions:

(41) If  $f$  is simple function in  $S$ , then  $f$  is measurable on  $A$ .

(42) If  $f$  is simple function in  $S$ , then  $f|A$  is simple function in  $S$ .

- (43) If  $f$  is simple function in  $S$ , then  $\text{dom } f$  is an element of  $S$ .
- (44) If  $f$  is simple function in  $S$  and  $g$  is simple function in  $S$ , then  $f + g$  is simple function in  $S$ .
- (45) For every complex number  $c$  such that  $f$  is simple function in  $S$  holds  $cf$  is simple function in  $S$ .

#### 4. LEBESGUE'S CONVERGENCE THEOREM OF COMPLEX-VALUED FUNCTION

In the sequel  $F$  denotes a sequence of partial functions from  $X$  into  $\overline{\mathbb{R}}$  with the same dom and  $P$  denotes a partial function from  $X$  to  $\overline{\mathbb{R}}$ .

The following proposition is true

- (46) Suppose that
  - (i)  $E = \text{dom } F(0)$ ,
  - (ii)  $E = \text{dom } P$ ,
  - (iii) for every natural number  $n$  holds  $F(n)$  is measurable on  $E$ ,
  - (iv)  $P$  is integrable on  $M$ ,
  - (v) for every element  $x$  of  $X$  and for every natural number  $n$  such that  $x \in E$  holds  $|F(n)|(x) \leq P(x)$ , and
  - (vi) for every element  $x$  of  $X$  such that  $x \in E$  holds  $F \# x$  is convergent.
 Then  $\lim F$  is integrable on  $M$ .

In the sequel  $F$  denotes a sequence of partial functions from  $X$  into  $\mathbb{R}$  with the same dom and  $f, P$  denote partial functions from  $X$  to  $\mathbb{R}$ .

One can prove the following propositions:

- (47) Suppose that
  - (i)  $E = \text{dom } F(0)$ ,
  - (ii)  $E = \text{dom } P$ ,
  - (iii) for every natural number  $n$  holds  $F(n)$  is measurable on  $E$ ,
  - (iv)  $P$  is integrable on  $M$ ,
  - (v) for every element  $x$  of  $X$  and for every natural number  $n$  such that  $x \in E$  holds  $|F(n)|(x) \leq P(x)$ , and
  - (vi) for every element  $x$  of  $X$  such that  $x \in E$  holds  $F \# x$  is convergent.
 Then  $\lim F$  is integrable on  $M$ .

- (48) Suppose that
  - (i)  $E = \text{dom } F(0)$ ,
  - (ii)  $E = \text{dom } P$ ,
  - (iii) for every natural number  $n$  holds  $F(n)$  is measurable on  $E$ ,
  - (iv)  $P$  is integrable on  $M$ , and
  - (v) for every element  $x$  of  $X$  and for every natural number  $n$  such that  $x \in E$  holds  $|F(n)|(x) \leq P(x)$ .
 Then there exists a sequence  $I$  of real numbers such that

- (vi) for every natural number  $n$  holds  $I(n) = \int F(n) \, dM$ , and
- (vii) if for every element  $x$  of  $X$  such that  $x \in E$  holds  $F \# x$  is convergent, then  $I$  is convergent and  $\lim I = \int \lim F \, dM$ .

Let  $X$  be a set and let  $F$  be a sequence of partial functions from  $X$  into  $\mathbb{R}$ . We say that  $F$  is uniformly bounded if and only if the condition (Def. 4) is satisfied.

- (Def. 4) There exists a real number  $K$  such that for every natural number  $n$  and for every element  $x$  of  $X$  if  $x \in \text{dom } F(0)$ , then  $|F(n)(x)| \leq K$ .

Next we state the proposition

- (49) Suppose that
- (i)  $M(E) < +\infty$ ,
  - (ii)  $E = \text{dom } F(0)$ ,
  - (iii) for every natural number  $n$  holds  $F(n)$  is measurable on  $E$ ,
  - (iv)  $F$  is uniformly bounded, and
  - (v) for every element  $x$  of  $X$  such that  $x \in E$  holds  $F \# x$  is convergent.

Then

- (vi) for every natural number  $n$  holds  $F(n)$  is integrable on  $M$ ,
- (vii)  $\lim F$  is integrable on  $M$ , and
- (viii) there exists a sequence  $I$  of extended reals such that for every natural number  $n$  holds  $I(n) = \int F(n) \, dM$  and  $I$  is convergent and  $\lim I = \int \lim F \, dM$ .

Let  $X$  be a set, let  $F$  be a sequence of partial functions from  $X$  into  $\mathbb{R}$ , and let  $f$  be a partial function from  $X$  to  $\mathbb{R}$ . We say that  $F$  is uniformly convergent to  $f$  if and only if the conditions (Def. 5) are satisfied.

- (Def. 5)(i)  $F$  has the same dom,
- (ii)  $\text{dom } F(0) = \text{dom } f$ , and
  - (iii) for every real number  $e$  such that  $e > 0$  there exists a natural number  $N$  such that for every natural number  $n$  and for every element  $x$  of  $X$  such that  $n \geq N$  and  $x \in \text{dom } F(0)$  holds  $|F(n)(x) - f(x)| < e$ .

We now state the proposition

- (50) Suppose that
- (i)  $M(E) < +\infty$ ,
  - (ii)  $E = \text{dom } F(0)$ ,
  - (iii) for every natural number  $n$  holds  $F(n)$  is integrable on  $M$ , and
  - (iv)  $F$  is uniformly convergent to  $f$ .

Then

- (v)  $f$  is integrable on  $M$ , and
- (vi) there exists a sequence  $I$  of extended reals such that for every natural number  $n$  holds  $I(n) = \int F(n) \, dM$  and  $I$  is convergent and  $\lim I = \int f \, dM$ .

In the sequel  $F$  denotes a sequence of partial functions from  $X$  into  $\mathbb{C}$  with the same dom and  $f$  denotes a partial function from  $X$  to  $\mathbb{C}$ .

The following two propositions are true:

- (51) Suppose that
- (i)  $E = \text{dom } F(0)$ ,
  - (ii)  $E = \text{dom } P$ ,
  - (iii) for every natural number  $n$  holds  $F(n)$  is measurable on  $E$ ,
  - (iv)  $P$  is integrable on  $M$ ,
  - (v) for every element  $x$  of  $X$  and for every natural number  $n$  such that  $x \in E$  holds  $|F(n)|(x) \leq P(x)$ , and
  - (vi) for every element  $x$  of  $X$  such that  $x \in E$  holds  $F\#x$  is convergent.

Then  $\lim F$  is integrable on  $M$ .

- (52) Suppose that
- (i)  $E = \text{dom } F(0)$ ,
  - (ii)  $E = \text{dom } P$ ,
  - (iii) for every natural number  $n$  holds  $F(n)$  is measurable on  $E$ ,
  - (iv)  $P$  is integrable on  $M$ , and
  - (v) for every element  $x$  of  $X$  and for every natural number  $n$  such that  $x \in E$  holds  $|F(n)|(x) \leq P(x)$ .

Then there exists a complex sequence  $I$  such that

- (vi) for every natural number  $n$  holds  $I(n) = \int F(n) \, dM$ , and
- (vii) if for every element  $x$  of  $X$  such that  $x \in E$  holds  $F\#x$  is convergent, then  $I$  is convergent and  $\lim I = \int \lim F \, dM$ .

Let  $X$  be a set and let  $F$  be a sequence of partial functions from  $X$  into  $\mathbb{C}$ . We say that  $F$  is uniformly bounded if and only if the condition (Def. 6) is satisfied.

- (Def. 6) There exists a real number  $K$  such that for every natural number  $n$  and for every element  $x$  of  $X$  if  $x \in \text{dom } F(0)$ , then  $|F(n)(x)| \leq K$ .

The following proposition is true

- (53) Suppose that
- (i)  $M(E) < +\infty$ ,
  - (ii)  $E = \text{dom } F(0)$ ,
  - (iii) for every natural number  $n$  holds  $F(n)$  is measurable on  $E$ ,
  - (iv)  $F$  is uniformly bounded, and
  - (v) for every element  $x$  of  $X$  such that  $x \in E$  holds  $F\#x$  is convergent.

Then

- (vi) for every natural number  $n$  holds  $F(n)$  is integrable on  $M$ ,
- (vii)  $\lim F$  is integrable on  $M$ , and
- (viii) there exists a complex sequence  $I$  such that for every natural number  $n$  holds  $I(n) = \int F(n) \, dM$  and  $I$  is convergent and  $\lim I = \int \lim F \, dM$ .

Let  $X$  be a set, let  $F$  be a sequence of partial functions from  $X$  into  $\mathbb{C}$ , and let  $f$  be a partial function from  $X$  to  $\mathbb{C}$ . We say that  $F$  is uniformly convergent to  $f$  if and only if the conditions (Def. 7) are satisfied.

- (Def. 7)(i)  $F$  has the same dom,  
(ii)  $\text{dom } F(0) = \text{dom } f$ , and  
(iii) for every real number  $e$  such that  $e > 0$  there exists a natural number  $N$  such that for every natural number  $n$  and for every element  $x$  of  $X$  such that  $n \geq N$  and  $x \in \text{dom } F(0)$  holds  $|F(n)(x) - f(x)| < e$ .

We now state the proposition

- (54) Suppose that  
(i)  $M(E) < +\infty$ ,  
(ii)  $E = \text{dom } F(0)$ ,  
(iii) for every natural number  $n$  holds  $F(n)$  is integrable on  $M$ , and  
(iv)  $F$  is uniformly convergent to  $f$ .

Then

- (v)  $f$  is integrable on  $M$ , and  
(vi) there exists a complex sequence  $I$  such that for every natural number  $n$  holds  $I(n) = \int F(n) \, dM$  and  $I$  is convergent and  $\lim I = \int f \, dM$ .

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*Received March 17, 2009*

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