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Vector Functions and their Differentiation Formulas in 3-dimensional Euclidean Spaces

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Summary. In this article, we first extend several basic theorems of the operation of vector in 3-dimensional Euclidean spaces. Then three unit vectors: $e1$, $e2$, $e3$ and the definition of vector function in the same spaces are introduced. By dint of unit vector the main operation properties as well as the differentiation formulas of vector function are shown [12].

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The notation and terminology used in this paper have been introduced in the following papers: [7], [11], [2], [3], [4], [1], [5], [8], [9], [6], [10], and [13].

1. Preliminaries

For simplicity, we use the following convention: r, r_1, r_2, x, y, z, x_1, x_2, x_3, y_1, y_2, y_3 are elements of $\mathbb{R}$, p, q, p_1, p_2, p_3, q_1, q_2 are elements of $\mathcal{R}^3$, f_1, f_2, f_3, g_1, g_2, g_3, h_1, h_2, h_3 are partial functions from $\mathbb{R}$ to $\mathbb{R}$, and t, t_0, t_1, t_2 are real numbers.

Let x, y, z be real numbers. Then $[x, y, z]$ is an element of $\mathcal{R}^3$.

One can prove the following proposition

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(1) For every finite sequence f of elements of $\mathbb{R}$ such that $\operatorname{len} f = 3$ holds f is an element of $\mathcal{R}^3$.

The element e_1 of $\mathcal{R}^3$ is defined by:

(Def. 1) $e_1 = [1, 0, 0]$.

The element e_2 of $\mathcal{R}^3$ is defined as follows:

(Def. 2) $e_2 = [0, 1, 0]$.

The element e_3 of $\mathcal{R}^3$ is defined as follows:

(Def. 3) $e_3 = [0, 0, 1]$.

Let us consider p_1, p_2. The functor $p_1 \times p_2$ yielding an element of $\mathcal{R}^3$ is defined as follows:

(Def. 4) $p_1 \times p_2 = [p_1(2) \cdot p_2(3) - p_1(3) \cdot p_2(2), p_1(3) \cdot p_2(1) - p_1(1) \cdot p_2(3), p_1(1) \cdot p_2(2) - p_1(2) \cdot p_2(1)]$.

Next we state the proposition

(2) If p_1 and p_2 are linearly dependent, then $p_1 \times p_2 = 0_{\mathcal{E}^3_{\mathrm{T}}}$.

2. Vector Functions in 3-dimensional Euclidean Spaces

We now state a number of propositions:

(3) $|e_1| = 1$.

(4) $|e_2| = 1$.

(5) $|e_3| = 1$.

(6) e_1, e_2 are orthogonal.

(7) e_1, e_3 are orthogonal.

(8) e_2, e_3 are orthogonal.

(9) $|(e_1, e_1)| = 1$.

(10) $|(e_2, e_2)| = 1$.

(11) $|(e_3, e_3)| = 1$.

(12) $|(e_1, [0, 0, 0])| = 0$.

(13) $|(e_2, [0, 0, 0])| = 0$.

(14) $|(e_3, [0, 0, 0])| = 0$.

(15) $e_1 \times e_2 = e_3$.

(16) $e_2 \times e_3 = e_1$.

(17) $e_3 \times e_1 = e_2$.

(18) $e_3 \times e_2 = -e_1$.

(19) $e_1 \times e_3 = -e_2$.

(20) $e_2 \times e_1 = -e_3$.

(21) $e_1 \times [0, 0, 0] = [0, 0, 0]$.

(22) $e_2 \times [0, 0, 0] = [0, 0, 0]$.

(23) $e_3 \times [0, 0, 0] = [0, 0, 0]$.

(24) $r \cdot e_1 = [r, 0, 0]$.

(25) $r \cdot e_2 = [0, r, 0]$.

(26) $r \cdot e_3 = [0, 0, r]$.

(27) $1 \cdot e_1 = e_1$.

(28) $1 \cdot e_2 = e_2$.

(29) $1 \cdot e_3 = e_3$.

(30) $-e_1 = [-1, 0, 0]$.

(31) $-e_2 = [0, -1, 0]$.

(32) $-e_3 = [0, 0, -1]$.

(33) $0 \cdot e_1 = [0, 0, 0]$.

(34) $0 \cdot e_2 = [0, 0, 0]$.

(35) $0 \cdot e_3 = [0, 0, 0]$.

(36) $p = p(1) \cdot e_1 + p(2) \cdot e_2 + p(3) \cdot e_3$.

(37) $r \cdot p = r \cdot p(1) \cdot e_1 + r \cdot p(2) \cdot e_2 + r \cdot p(3) \cdot e_3$.

(38) $[x, y, z] = x \cdot e_1 + y \cdot e_2 + z \cdot e_3$.

(39) $r \cdot [x, y, z] = r \cdot x \cdot e_1 + r \cdot y \cdot e_2 + r \cdot z \cdot e_3$.

(40) $-p = -p(1) \cdot e_1 - p(2) \cdot e_2 - p(3) \cdot e_3$.

(41) $-[x, y, z] = -x \cdot e_1 - y \cdot e_2 - z \cdot e_3$.

(42) $p_1 + p_2 = (p_1(1) + p_2(1)) \cdot e_1 + (p_1(2) + p_2(2)) \cdot e_2 + (p_1(3) + p_2(3)) \cdot e_3$.

(43) $p_1 - p_2 = (p_1(1) - p_2(1)) \cdot e_1 + (p_1(2) - p_2(2)) \cdot e_2 + (p_1(3) - p_2(3)) \cdot e_3$.

(44) $[x_1, x_2, x_3] + [y_1, y_2, y_3] = (x_1 + y_1) \cdot e_1 + (x_2 + y_2) \cdot e_2 + (x_3 + y_3) \cdot e_3$.

(45) $[x_1, x_2, x_3] - [y_1, y_2, y_3] = (x_1 - y_1) \cdot e_1 + (x_2 - y_2) \cdot e_2 + (x_3 - y_3) \cdot e_3$.

(46) $p_1(1) \cdot e_1 + p_1(2) \cdot e_2 + p_1(3) \cdot e_3 = (p_2(1) + p_3(1)) \cdot e_1 + (p_2(2) + p_3(2)) \cdot e_2 + (p_2(3) + p_3(3)) \cdot e_3$ if and only if $p_2(1) \cdot e_1 + p_2(2) \cdot e_2 + p_2(3) \cdot e_3 = (p_1(1) - p_3(1)) \cdot e_1 + (p_1(2) - p_3(2)) \cdot e_2 + (p_1(3) - p_3(3)) \cdot e_3$.

Let f_1, f_2, f_3 be partial functions from $\mathbb{R}$ to $\mathbb{R}$. The functor $\mathrm{VFunc}(f_1, f_2, f_3)$ yielding a function from $\mathbb{R}$ into $\mathcal{R}^3$ is defined as follows:

(Def. 5) For every t holds $(\mathrm{VFunc}(f_1, f_2, f_3))(t) = [f_1(t), f_2(t), f_3(t)]$.

We now state a number of propositions:

(47) $(\mathrm{VFunc}(f_1, f_2, f_3))(t) = f_1(t) \cdot e_1 + f_2(t) \cdot e_2 + f_3(t) \cdot e_3$.

(48) $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$ iff $p(1) = f_1(t)$ and $p(2) = f_2(t)$ and $p(3) = f_3(t)$.

(49) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $\mathrm{len}\, p = 3$ and $\mathrm{dom}\, p = \mathrm{Seg}\, 3$.

(50) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $p \bullet q = \langle f_1(t_1) \cdot g_1(t_2), f_2(t_1) \cdot g_2(t_2), f_3(t_1) \cdot g_3(t_2) \rangle$.

(51) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $r \cdot p = [r \cdot f_1(t), r \cdot f_2(t), r \cdot f_3(t)]$.

(52) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $-p = [-f_1(t), -f_2(t), -f_3(t)]$.

(53) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $(-p)(1) = -f_1(t)$ and $(-p)(2) = -f_2(t)$ and $(-p)(3) = -f_3(t)$.

(54) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $\mathrm{len}(-p) = 3$.

(55) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $\mathrm{len}(-p) = \mathrm{len}\, p$.

(56) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $\mathrm{len}(p + q) = 3$.

(57) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $p + q = [f_1(t_1) + g_1(t_2), f_2(t_1) + g_2(t_2), f_3(t_1) + g_3(t_2)]$.

(58) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$ and $p = q$, then $f_1(t_1) = g_1(t_2)$ and $f_2(t_1) = g_2(t_2)$ and $f_3(t_1) = g_3(t_2)$.

(59) If $f_1(t_1) = g_1(t_2)$ and $f_2(t_1) = g_2(t_2)$ and $f_3(t_1) = g_3(t_2)$, then $(\mathrm{VFunc}(f_1, f_2, f_3))(t_1) = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$.

(60) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $p + q = [f_1(t_1) + g_1(t_2), f_2(t_1) + g_2(t_2), f_3(t_1) + g_3(t_2)]$.

(61) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $p + -q = [f_1(t_1) - g_1(t_2), f_2(t_1) - g_2(t_2), f_3(t_1) - g_3(t_2)]$.

(62) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $p - q = [f_1(t_1) - g_1(t_2), f_2(t_1) - g_2(t_2), f_3(t_1) - g_3(t_2)]$.

(63) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $\mathrm{len}(p - q) = 3$.

(64) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(p, q)| = f_1(t_1) \cdot g_1(t_2) + f_2(t_1) \cdot g_2(t_2) + f_3(t_1) \cdot g_3(t_2)$.

(65) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $|(p, p)| = f_1(t)^{\mathbf{2}} + f_2(t)^{\mathbf{2}} + f_3(t)^{\mathbf{2}}$.

(66) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $|p| = \sqrt{f_1(t)^{\mathbf{2}} + f_2(t)^{\mathbf{2}} + f_3(t)^{\mathbf{2}}}$.

(67) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $|r \cdot p| = |r| \cdot \sqrt{f_1(t)^{\mathbf{2}} + f_2(t)^{\mathbf{2}} + f_3(t)^{\mathbf{2}}}$.

(68) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $p \times q = [f_2(t_1) \cdot g_3(t_2) - f_3(t_1) \cdot g_2(t_2), f_3(t_1) \cdot g_1(t_2) - f_1(t_1) \cdot g_3(t_2), f_1(t_1) \cdot g_2(t_2) - f_2(t_1) \cdot g_1(t_2)]$.

(69) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $r_1 \cdot p + r_2 \cdot p = (r_1 + r_2) \cdot [f_1(t), f_2(t), f_3(t)]$.

(70) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $r_1 \cdot p - r_2 \cdot p = (r_1 - r_2) \cdot [f_1(t), f_2(t), f_3(t)]$.

(71) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(r \cdot p, q)| = r \cdot (f_1(t_1) \cdot g_1(t_2) + f_2(t_1) \cdot g_2(t_2) + f_3(t_1) \cdot g_3(t_2))$.

(72) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$, then $|(p, 0_{\mathcal{E}^3_{\mathrm{T}}})| = 0$.

(73) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(-p, q)| = -|(p, q)|$.

(74) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(-p, -q)| = |(p, q)|$.

(75) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(p_1 - p_2, q)| = |(p_1, q)| - |(p_2, q)|$.

(76) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(p_1 + p_2, q)| = |(p_1, q)| + |(p_2, q)|$.

(77) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(r_1 \cdot p_1 + r_2 \cdot p_2, q)| = r_1 \cdot |(p_1, q)| + r_2 \cdot |(p_2, q)|$.

(78) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q_1 = (\mathrm{VFunc}(g_1, g_2, g_3))(t_1)$ and $q_2 = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(p_1 + p_2, q_1 + q_2)| = |(p_1, q_1)| + |(p_1, q_2)| + |(p_2, q_1)| + |(p_2, q_2)|$.

(79) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q_1 = (\mathrm{VFunc}(g_1, g_2, g_3))(t_1)$ and $q_2 = (\mathrm{VFunc}(g_1, g_2, g_3))(t_2)$, then $|(p_1 - p_2, q_1 - q_2)| = (|(p_1, q_1)| - |(p_1, q_2)| - |(p_2, q_1)|) + |(p_2, q_2)|$.

(80) For every p such that $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$ holds $|(p, p)| = 0$ iff $p = 0_{\mathcal{E}^3_{\mathrm{T}}}$.

(81) For every p such that $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$ holds $|p| = 0$ iff $p = 0_{\mathcal{E}^3_{\mathrm{T}}}$.

(82) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $|(p - q, p - q)| = (|(p, p)| - 2 \cdot |(p, q)|) + |(q, q)|$.

(83) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $|(p + q, p + q)| = |(p, p)| + 2 \cdot |(p, q)| + |(q, q)|$.

(84) If $p = (\mathrm{VFunc}(f_1, f_2, f_3))(t)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $(r \cdot p) \times q = r \cdot (p \times q)$ and $(r \cdot p) \times q = p \times (r \cdot q)$.

(85) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $p_1 \times (p_2 + q) = p_1 \times p_2 + p_1 \times q$.

(86) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $(p_1 + p_2) \times q = p_1 \times q + p_2 \times q$.

Let us consider p_1, p_2, p_3. The functor $\langle|p_1, p_2, p_3|\rangle$ yields a real number and is defined as follows:

(Def. 6) $\langle|p_1, p_2, p_3|\rangle = |(p_1, p_2 \times p_3)|$.

Next we state several propositions:

(87) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$, then $\langle|p_1, p_1, p_2|\rangle = 0$.

(88) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$, then $\langle|p_2, p_1, p_2|\rangle = 0$.

(89) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$, then $\langle|p_1, p_2, p_2|\rangle = 0$.

(90) If $p_1 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\mathrm{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\mathrm{VFunc}(g_1, g_2, g_3))(t)$, then $\langle|p_1, p_2, q|\rangle = \langle|p_2, q, p_1|\rangle$.

(91) If $p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\text{VFunc}(g_1, g_2, g_3))(t)$, then $\langle|p_1, p_2, q|\rangle = |(p_1 \times p_2, q)|$.

(92) If $p_1 = (\text{VFunc}(f_1, f_2, f_3))(t_1)$ and $p_2 = (\text{VFunc}(f_1, f_2, f_3))(t_2)$ and $q = (\text{VFunc}(g_1, g_2, g_3))(t)$, then $\langle|p_1, p_2, q|\rangle = |(q \times p_1, p_2)|$.

3. The Differentiation Formulas of Vector Functions in 3-dimensional Euclidean Spaces

Let f_1, f_2, f_3 be partial functions from $\mathbb{R}$ to $\mathbb{R}$ and let t_0 be a real number. The functor $\text{VFuncdiff}(f_1, f_2, f_3, t_0)$ yielding an element of $\mathcal{R}^3$ is defined as follows:

(Def. 7) $\text{VFuncdiff}(f_1, f_2, f_3, t_0) = [f_1{}'(t_0), f_2{}'(t_0), f_3{}'(t_0)]$.

Next we state a number of propositions:

(93) Suppose f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0. Then $\text{VFuncdiff}(f_1, f_2, f_3, t_0) = f_1{}'(t_0) \cdot e_1 + f_2{}'(t_0) \cdot e_2 + f_3{}'(t_0) \cdot e_3$.

(94) Suppose that

(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in t_0,
(v) g_2 is differentiable in t_0, and
(vi) g_3 is differentiable in t_0.

Then $\text{VFuncdiff}(f_1 + g_1, f_2 + g_2, f_3 + g_3, t_0) = \text{VFuncdiff}(f_1, f_2, f_3, t_0) + \text{VFuncdiff}(g_1, g_2, g_3, t_0)$.

(95) Suppose that

(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in t_0,
(v) g_2 is differentiable in t_0, and
(vi) g_3 is differentiable in t_0.

Then $\text{VFuncdiff}(f_1 - g_1, f_2 - g_2, f_3 - g_3, t_0) = \text{VFuncdiff}(f_1, f_2, f_3, t_0) - \text{VFuncdiff}(g_1, g_2, g_3, t_0)$.

(96) If f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0, then $\text{VFuncdiff}(r\, f_1, r\, f_2, r\, f_3, t_0) = r \cdot \text{VFuncdiff}(f_1, f_2, f_3, t_0)$.

(97) Suppose that

(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,

(iv) g_1 is differentiable in t_0,
(v) g_2 is differentiable in t_0, and
(vi) g_3 is differentiable in t_0.
Then $\mathrm{VFuncdiff}(f_1\, g_1, f_2\, g_2, f_3\, g_3, t_0) = [g_1(t_0)\cdot {f_1}'(t_0), g_2(t_0)\cdot {f_2}'(t_0), g_3(t_0)\cdot {f_3}'(t_0)] + [f_1(t_0)\cdot {g_1}'(t_0), f_2(t_0)\cdot {g_2}'(t_0), f_3(t_0)\cdot {g_3}'(t_0)]$.

(98) Suppose that
(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in $f_1(t_0)$,
(v) g_2 is differentiable in $f_2(t_0)$, and
(vi) g_3 is differentiable in $f_3(t_0)$.
Then $\mathrm{VFuncdiff}(g_1\cdot f_1, g_2\cdot f_2, g_3\cdot f_3, t_0) = [{g_1}'(f_1(t_0))\cdot {f_1}'(t_0), {g_2}'(f_2(t_0))\cdot {f_2}'(t_0), {g_3}'(f_3(t_0))\cdot {f_3}'(t_0)]$.

(99) Suppose that f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and g_1 is differentiable in t_0 and g_2 is differentiable in t_0 and g_3 is differentiable in t_0 and $g_1(t_0)\neq 0$ and $g_2(t_0)\neq 0$ and $g_3(t_0)\neq 0$. Then $\mathrm{VFuncdiff}(\frac{f_1}{g_1}, \frac{f_2}{g_2}, \frac{f_3}{g_3}, t_0) = [\frac{{f_1}'(t_0)\cdot g_1(t_0) - {g_1}'(t_0)\cdot f_1(t_0)}{g_1(t_0)^{\mathbf{2}}}, \frac{{f_2}'(t_0)\cdot g_2(t_0) - {g_2}'(t_0)\cdot f_2(t_0)}{g_2(t_0)^{\mathbf{2}}}, \frac{{f_3}'(t_0)\cdot g_3(t_0) - {g_3}'(t_0)\cdot f_3(t_0)}{g_3(t_0)^{\mathbf{2}}}]$.

(100) Suppose f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and $f_1(t_0)\neq 0$ and $f_2(t_0)\neq 0$ and $f_3(t_0)\neq 0$. Then $\mathrm{VFuncdiff}(\frac{1}{f_1}, \frac{1}{f_2}, \frac{1}{f_3}, t_0) = -[\frac{{f_1}'(t_0)}{f_1(t_0)^{\mathbf{2}}}, \frac{{f_2}'(t_0)}{f_2(t_0)^{\mathbf{2}}}, \frac{{f_3}'(t_0)}{f_3(t_0)^{\mathbf{2}}}]$.

(101) Suppose f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0. Then $\mathrm{VFuncdiff}(r\, f_1, r\, f_2, r\, f_3, t_0) = r\cdot {f_1}'(t_0)\cdot e_1 + r\cdot {f_2}'(t_0)\cdot e_2 + r\cdot {f_3}'(t_0)\cdot e_3$.

(102) Suppose that
(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in t_0,
(v) g_2 is differentiable in t_0, and
(vi) g_3 is differentiable in t_0.
Then $\mathrm{VFuncdiff}(r\,(f_1+g_1), r\,(f_2+g_2), r\,(f_3+g_3), t_0) =$
$r\cdot \mathrm{VFuncdiff}(f_1, f_2, f_3, t_0) + r\cdot \mathrm{VFuncdiff}(g_1, g_2, g_3, t_0)$.

(103) Suppose that
(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in t_0,
(v) g_2 is differentiable in t_0, and

(vi) g_3 is differentiable in t_0.
Then VFuncdiff$(r\,(f_1 - g_1), r\,(f_2 - g_2), r\,(f_3 - g_3), t_0) =$
$r \cdot$ VFuncdiff$(f_1, f_2, f_3, t_0) - r \cdot$ VFuncdiff(g_1, g_2, g_3, t_0).

(104) Suppose that
(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in t_0,
(v) g_2 is differentiable in t_0, and
(vi) g_3 is differentiable in t_0.
Then VFuncdiff$(r\,f_1\,g_1, r\,f_2\,g_2, r\,f_3\,g_3, t_0) = r\cdot[g_1(t_0)\cdot {f_1}'(t_0), g_2(t_0)\cdot {f_2}'(t_0), g_3(t_0) \cdot {f_3}'(t_0)] + r \cdot [f_1(t_0) \cdot {g_1}'(t_0), f_2(t_0) \cdot {g_2}'(t_0), f_3(t_0) \cdot {g_3}'(t_0)]$.

(105) Suppose that
(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in $f_1(t_0)$,
(v) g_2 is differentiable in $f_2(t_0)$, and
(vi) g_3 is differentiable in $f_3(t_0)$.
Then VFuncdiff$((r\,g_1) \cdot f_1, (r\,g_2) \cdot f_2, (r\,g_3) \cdot f_3, t_0) = r \cdot [{g_1}'(f_1(t_0)) \cdot {f_1}'(t_0), {g_2}'(f_2(t_0)) \cdot {f_2}'(t_0), {g_3}'(f_3(t_0)) \cdot {f_3}'(t_0)]$.

(106) Suppose that f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and g_1 is differentiable in t_0 and g_2 is differentiable in t_0 and g_3 is differentiable in t_0 and $g_1(t_0) \neq 0$ and $g_2(t_0) \neq 0$ and $g_3(t_0) \neq 0$. Then VFuncdiff$(\frac{r\,f_1}{g_1}, \frac{r\,f_2}{g_2}, \frac{r\,f_3}{g_3}, t_0) = r \cdot [\frac{{f_1}'(t_0)\cdot g_1(t_0) - {g_1}'(t_0)\cdot f_1(t_0)}{g_1(t_0)^{\mathbf{2}}}, \frac{{f_2}'(t_0)\cdot g_2(t_0) - {g_2}'(t_0)\cdot f_2(t_0)}{g_2(t_0)^{\mathbf{2}}}, \frac{{f_3}'(t_0)\cdot g_3(t_0) - {g_3}'(t_0)\cdot f_3(t_0)}{g_3(t_0)^{\mathbf{2}}}]$.

(107) Suppose that f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and $f_1(t_0) \neq 0$ and $f_2(t_0) \neq 0$ and $f_3(t_0) \neq 0$ and $r \neq 0$. Then VFuncdiff$(\frac{1}{r\,f_1}, \frac{1}{r\,f_2}, \frac{1}{r\,f_3}, t_0) = -\frac{1}{r} \cdot [\frac{{f_1}'(t_0)}{f_1(t_0)^{\mathbf{2}}}, \frac{{f_2}'(t_0)}{f_2(t_0)^{\mathbf{2}}}, \frac{{f_3}'(t_0)}{f_3(t_0)^{\mathbf{2}}}]$.

(108) Suppose that
(i) f_1 is differentiable in t_0,
(ii) f_2 is differentiable in t_0,
(iii) f_3 is differentiable in t_0,
(iv) g_1 is differentiable in t_0,
(v) g_2 is differentiable in t_0, and
(vi) g_3 is differentiable in t_0.
Then VFuncdiff$(f_2\,g_3 - f_3\,g_2, f_3\,g_1 - f_1\,g_3, f_1\,g_2 - f_2\,g_1, t_0) = [f_2(t_0) \cdot {g_3}'(t_0) - f_3(t_0)\cdot {g_2}'(t_0), f_3(t_0)\cdot {g_1}'(t_0) - f_1(t_0)\cdot {g_3}'(t_0), f_1(t_0)\cdot {g_2}'(t_0) - f_2(t_0)\cdot {g_1}'(t_0)] + [{f_2}'(t_0) \cdot g_3(t_0) - {f_3}'(t_0) \cdot g_2(t_0), {f_3}'(t_0) \cdot g_1(t_0) - {f_1}'(t_0) \cdot g_3(t_0), {f_1}'(t_0) \cdot g_2(t_0) - {f_2}'(t_0) \cdot g_1(t_0)]$.

(109) Suppose that f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and g_1 is differentiable in t_0 and g_2 is differentiable in t_0 and g_3 is differentiable in t_0 and h_1 is differentiable in t_0 and h_2 is differentiable in t_0 and h_3 is differentiable in t_0. Then $\mathrm{VFuncdiff}(h_1\,(f_2\,g_3 - f_3\,g_2), h_2\,(f_3\,g_1 - f_1\,g_3), h_3\,(f_1\,g_2 - f_2\,g_1), t_0) = [{h_1}'(t_0) \cdot (f_2(t_0) \cdot g_3(t_0) - f_3(t_0) \cdot g_2(t_0)), {h_2}'(t_0) \cdot (f_3(t_0) \cdot g_1(t_0) - f_1(t_0) \cdot g_3(t_0)), {h_3}'(t_0) \cdot (f_1(t_0) \cdot g_2(t_0) - f_2(t_0) \cdot g_1(t_0))] + [h_1(t_0) \cdot ({f_2}'(t_0) \cdot g_3(t_0) - {f_3}'(t_0) \cdot g_2(t_0)), h_2(t_0) \cdot ({f_3}'(t_0) \cdot g_1(t_0) - {f_1}'(t_0) \cdot g_3(t_0)), h_3(t_0) \cdot ({f_1}'(t_0) \cdot g_2(t_0) - {f_2}'(t_0) \cdot g_1(t_0))] + [h_1(t_0) \cdot (f_2(t_0) \cdot {g_3}'(t_0) - f_3(t_0) \cdot {g_2}'(t_0)), h_2(t_0) \cdot (f_3(t_0) \cdot {g_1}'(t_0) - f_1(t_0) \cdot {g_3}'(t_0)), h_3(t_0) \cdot (f_1(t_0) \cdot {g_2}'(t_0) - f_2(t_0) \cdot {g_1}'(t_0))]$.

(110) Suppose that f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and g_1 is differentiable in t_0 and g_2 is differentiable in t_0 and g_3 is differentiable in t_0 and h_1 is differentiable in t_0 and h_2 is differentiable in t_0 and h_3 is differentiable in t_0. Then $\mathrm{VFuncdiff}(h_2\,f_2\,g_3 - h_3\,f_3\,g_2, h_3\,f_3\,g_1 - h_1\,f_1\,g_3, h_1\,f_1\,g_2 - h_2\,f_2\,g_1, t_0) = [h_2(t_0) \cdot f_2(t_0) \cdot {g_3}'(t_0) - h_3(t_0) \cdot f_3(t_0) \cdot {g_2}'(t_0), h_3(t_0) \cdot f_3(t_0) \cdot {g_1}'(t_0) - h_1(t_0) \cdot f_1(t_0) \cdot {g_3}'(t_0), h_1(t_0) \cdot f_1(t_0) \cdot {g_2}'(t_0) - h_2(t_0) \cdot f_2(t_0) \cdot {g_1}'(t_0)] + [h_2(t_0) \cdot {f_2}'(t_0) \cdot g_3(t_0) - h_3(t_0) \cdot {f_3}'(t_0) \cdot g_2(t_0), h_3(t_0) \cdot {f_3}'(t_0) \cdot g_1(t_0) - h_1(t_0) \cdot {f_1}'(t_0) \cdot g_3(t_0), h_1(t_0) \cdot {f_1}'(t_0) \cdot g_2(t_0) - h_2(t_0) \cdot {f_2}'(t_0) \cdot g_1(t_0)] + [{h_2}'(t_0) \cdot f_2(t_0) \cdot g_3(t_0) - {h_3}'(t_0) \cdot f_3(t_0) \cdot g_2(t_0), {h_3}'(t_0) \cdot f_3(t_0) \cdot g_1(t_0) - {h_1}'(t_0) \cdot f_1(t_0) \cdot g_3(t_0), {h_1}'(t_0) \cdot f_1(t_0) \cdot g_2(t_0) - {h_2}'(t_0) \cdot f_2(t_0) \cdot g_1(t_0)]$.

(111) Suppose that f_1 is differentiable in t_0 and f_2 is differentiable in t_0 and f_3 is differentiable in t_0 and g_1 is differentiable in t_0 and g_2 is differentiable in t_0 and g_3 is differentiable in t_0 and h_1 is differentiable in t_0 and h_2 is differentiable in t_0 and h_3 is differentiable in t_0. Then $\mathrm{VFuncdiff}(h_2\,(f_1\,g_2 - f_2\,g_1) - h_3\,(f_3\,g_1 - f_1\,g_3), h_3\,(f_2\,g_3 - f_3\,g_2) - h_1\,(f_1\,g_2 - f_2\,g_1), h_1\,(f_3\,g_1 - f_1\,g_3) - h_2\,(f_2\,g_3 - f_3\,g_2), t_0) = [h_2(t_0) \cdot (f_1(t_0) \cdot {g_2}'(t_0) - f_2(t_0) \cdot {g_1}'(t_0)) - h_3(t_0) \cdot (f_3(t_0) \cdot {g_1}'(t_0) - f_1(t_0) \cdot {g_3}'(t_0)), h_3(t_0) \cdot (f_2(t_0) \cdot {g_3}'(t_0) - f_3(t_0) \cdot {g_2}'(t_0)) - h_1(t_0) \cdot (f_1(t_0) \cdot {g_2}'(t_0) - f_2(t_0) \cdot {g_1}'(t_0)), h_1(t_0) \cdot (f_3(t_0) \cdot {g_1}'(t_0) - f_1(t_0) \cdot {g_3}'(t_0)) - h_2(t_0) \cdot (f_2(t_0) \cdot {g_3}'(t_0) - f_3(t_0) \cdot {g_2}'(t_0))] + [h_2(t_0) \cdot ({f_1}'(t_0) \cdot g_2(t_0) - {f_2}'(t_0) \cdot g_1(t_0)) - h_3(t_0) \cdot ({f_3}'(t_0) \cdot g_1(t_0) - {f_1}'(t_0) \cdot g_3(t_0)), h_3(t_0) \cdot ({f_2}'(t_0) \cdot g_3(t_0) - {f_3}'(t_0) \cdot g_2(t_0)) - h_1(t_0) \cdot ({f_1}'(t_0) \cdot g_2(t_0) - {f_2}'(t_0) \cdot g_1(t_0)), h_1(t_0) \cdot ({f_3}'(t_0) \cdot g_1(t_0) - {f_1}'(t_0) \cdot g_3(t_0)) - h_2(t_0) \cdot ({f_2}'(t_0) \cdot g_3(t_0) - {f_3}'(t_0) \cdot g_2(t_0))] + [{h_2}'(t_0) \cdot (f_1(t_0) \cdot g_2(t_0) - f_2(t_0) \cdot g_1(t_0)) - {h_3}'(t_0) \cdot (f_3(t_0) \cdot g_1(t_0) - f_1(t_0) \cdot g_3(t_0)), {h_3}'(t_0) \cdot (f_2(t_0) \cdot g_3(t_0) - f_3(t_0) \cdot g_2(t_0)) - {h_1}'(t_0) \cdot (f_1(t_0) \cdot g_2(t_0) - f_2(t_0) \cdot g_1(t_0)), {h_1}'(t_0) \cdot (f_3(t_0) \cdot g_1(t_0) - f_1(t_0) \cdot g_3(t_0)) - {h_2}'(t_0) \cdot (f_2(t_0) \cdot g_3(t_0) - f_3(t_0) \cdot g_2(t_0))]$.

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Banach Algebra of Continuous Functionals and the Space of Real-Valued Continuous Functionals with Bounded Support[1]

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Summary. In this article, we give a definition of a functional space which is constructed from all continuous functions defined on a compact topological space. We prove that this functional space is a Banach algebra. Next, we give a definition of a function space which is constructed from all real-valued continuous functions with bounded support. We prove that this function space is a real normed space.

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The notation and terminology used here have been introduced in the following papers: [2], [15], [7], [17], [16], [10], [3], [18], [14], [13], [12], [1], [4], [11], [6], [8], [19], [20], [9], and [5].

1. Banach Algebra of Continuous Functionals

Let X be a 1-sorted structure and let y be a real number. The functor $X \longmapsto y$ yielding a real map of X is defined as follows:

(Def. 1) $X \longmapsto y = (\text{the carrier of } X) \longmapsto y$.

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Let X be a topological space and let y be a real number. Note that $X \longmapsto y$ is continuous.

Next we state the proposition

(1) Let X be a non empty topological space and f be a real map of X. Then f is continuous if and only if for every point x of X and for every subset V of $\mathbb{R}$ such that $f(x) \in V$ and V is open there exists a subset W of X such that $x \in W$ and W is open and $f^\circ W \subseteq V$.

In the sequel X denotes a non empty topological space.

Let us consider X. The functor $\mathrm{C}(X;\mathbb{R})$ yielding a subset of RAlgebra (the carrier of X) is defined by:

(Def. 2) $\mathrm{C}(X;\mathbb{R}) = \{f : f$ ranges over continuous real maps of $X\}$.

Let us consider X. Observe that $\mathrm{C}(X;\mathbb{R})$ is non empty.

Let us consider X. One can verify that $\mathrm{C}(X;\mathbb{R})$ is additively-linearly-closed and multiplicatively-closed.

Let X be a non empty topological space. The functor $\mathrm{C_A}(X;\mathbb{R})$ yielding an algebra structure is defined by the condition (Def. 3).

(Def. 3) $\mathrm{C_A}(X;\mathbb{R}) = \langle \mathrm{C}(X;\mathbb{R}), \mathrm{mult}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)),$ $\mathrm{Add}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{Mult}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{One}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{Zero}(\mathrm{C}(X;\mathbb{R}),$ $\mathrm{RAlgebra}\,(\text{the carrier of } X))\rangle$.

One can prove the following proposition

(2) $\mathrm{C_A}(X;\mathbb{R})$ is a subalgebra of RAlgebra (the carrier of X).

Let us consider X. Note that $\mathrm{C_A}(X;\mathbb{R})$ is strict and non empty.

Let us consider X. Observe that $\mathrm{C_A}(X;\mathbb{R})$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, scalar unital, commutative, associative, right unital, right distributive, vector distributive, scalar distributive, scalar associative, and vector associative.

We use the following convention: F, G, H denote vectors of $\mathrm{C_A}(X;\mathbb{R})$, g, h denote real maps of X, and a denotes a real number.

One can prove the following propositions:

(3) Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element x of the carrier of X holds $h(x) = f(x) + g(x)$.

(4) If $f = F$ and $g = G$, then $G = a \cdot F$ iff for every element x of X holds $g(x) = a \cdot f(x)$.

(5) Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element x of the carrier of X holds $h(x) = f(x) \cdot g(x)$.

(6) $0_{\mathrm{C_A}(X;\mathbb{R})} = X \longmapsto 0$.

(7) $\mathbf{1}_{\mathrm{C_A}(X;\mathbb{R})} = X \longmapsto 1$.

In the sequel X denotes a compact non empty topological space and f, g, h denote real maps of X.

We now state two propositions:

(8) Let A be an algebra and A_1, A_2 be subalgebras of A. Suppose the carrier of $A_1 \subseteq$ the carrier of A_2. Then A_1 is a subalgebra of A_2.

(9) $\mathrm{C_A}(X;\mathbb{R})$ is a subalgebra of the $\mathbb{R}$-algebra of bounded functions on the carrier of X.

Let us consider X. The functor $||\cdot||_{C(X;\mathbb{R})}$ yielding a function from $\mathrm{C}(X;\mathbb{R})$ into $\mathbb{R}$ is defined as follows:

(Def. 4) $||\cdot||_{C(X;\mathbb{R})} = \mathrm{BoundedFunctionsNorm}\,(\text{the carrier of } X) \restriction \mathrm{C}(X;\mathbb{R})$.

Let us consider X. The functor $\mathrm{C_{NA}}(X;\mathbb{R})$ yielding a normed algebra structure is defined by the condition (Def. 5).

(Def. 5) $\mathrm{C_{NA}}(X;\mathbb{R}) = \langle \mathrm{C}(X;\mathbb{R}), \mathrm{mult}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{Add}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{Mult}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{One}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), \mathrm{Zero}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)), ||\cdot||_{C(X;\mathbb{R})}\rangle$.

Let us consider X. Observe that $\mathrm{C_{NA}}(X;\mathbb{R})$ is strict and non empty.

Let us consider X. Note that $\mathrm{C_{NA}}(X;\mathbb{R})$ is unital.

Next we state the proposition

(10) Let W be a normed algebra structure and V be an algebra. If the algebra structure of $W = V$, then W is an algebra.

In the sequel F, G, H denote points of $\mathrm{C_{NA}}(X;\mathbb{R})$.

Let us consider X. Note that $\mathrm{C_{NA}}(X;\mathbb{R})$ is Abelian, add-associative, right zeroed, right complementable, commutative, associative, right unital, right distributive, vector distributive, scalar distributive, scalar associative, and vector associative.

We now state the proposition

(11) $(\mathrm{Mult}(\mathrm{C}(X;\mathbb{R}), \mathrm{RAlgebra}\,(\text{the carrier of } X)))(1, F) = F$.

Let us consider X. Note that $\mathrm{C_{NA}}(X;\mathbb{R})$ is vector distributive, scalar distributive, scalar associative, and scalar unital.

We now state several propositions:

(12) $X \longmapsto 0 = 0_{\mathrm{C_{NA}}(X;\mathbb{R})}$.

(13) $0 \leq \|F\|$.

(14) $0 = \|(0_{\mathrm{C_{NA}}(X;\mathbb{R})})\|$.

(15) If $f = F$ and $g = G$ and $h = H$, then $H = F + G$ iff for every element x of X holds $h(x) = f(x) + g(x)$.

(16) If $f = F$ and $g = G$, then $G = a \cdot F$ iff for every element x of X holds $g(x) = a \cdot f(x)$.

(17) If $f = F$ and $g = G$ and $h = H$, then $H = F \cdot G$ iff for every element x of X holds $h(x) = f(x) \cdot g(x)$.

(18) $\|F\| = 0$ iff $F = 0_{\mathrm{C_{NA}}(X;\mathbb{R})}$ and $\|a \cdot F\| = |a| \cdot \|F\|$ and $\|F + G\| \leq \|F\| + \|G\|$.

Let us consider X. One can check that $\mathrm{C_{NA}}(X;\mathbb{R})$ is reflexive, discernible, and real normed space-like.

Next we state four propositions:

(19) If $f = F$ and $g = G$ and $h = H$, then $H = F - G$ iff for every element x of X holds $h(x) = f(x) - g(x)$.

(20) Let X be a real Banach space, Y be a subset of X, and s_1 be a sequence of X. Suppose Y is closed and $\operatorname{rng} s_1 \subseteq Y$ and s_1 is Cauchy sequence by norm. Then s_1 is convergent and $\lim s_1 \in Y$.

(21) Let Y be a subset of the $\mathbb{R}$-normed algebra of bounded functions on the carrier of X. If $Y = \mathrm{C}(X;\mathbb{R})$, then Y is closed.

(22) For every sequence s_1 of $\mathrm{C_{NA}}(X;\mathbb{R})$ such that s_1 is Cauchy sequence by norm holds s_1 is convergent.

Let us consider X. One can verify that $\mathrm{C_{NA}}(X;\mathbb{R})$ is complete.

Let us consider X. Observe that $\mathrm{C_{NA}}(X;\mathbb{R})$ is Banach Algebra-like.

2. Some Properties of Support

Next we state three propositions:

(23) For every non empty topological space X and for all real maps f, g of X holds $\operatorname{support}(f + g) \subseteq \operatorname{support} f \cup \operatorname{support} g$.

(24) For every non empty topological space X and for every real number a and for every real map f of X holds $\operatorname{support}(a\, f) \subseteq \operatorname{support} f$.

(25) For every non empty topological space X and for all real maps f, g of X holds $\operatorname{support}(f\, g) \subseteq \operatorname{support} f \cup \operatorname{support} g$.

3. The Space of Real-valued Continuous Functionals with Bounded Support

Let X be a non empty topological space. The functor $\mathrm{C_0}(X)$ yielding a non empty subset of $\mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}$ is defined by the condition (Def. 6).

(Def. 6) $\mathrm{C_0}(X) = \{f; f$ ranges over real maps of X: f is continuous $\wedge$ $\bigvee_{Y:\text{ non empty subset of } X} (Y$ is compact $\wedge$ $\bigwedge_{A:\text{subset of } X} (A = \operatorname{support} f \;\Rightarrow\; \overline{A}$ is a subset of $Y))\}$.

The following propositions are true:

(26) For every non empty topological space X holds $\mathrm{C_0}(X)$ is a non empty non empty subset of RAlgebra (the carrier of X).

(27) Let X be a non empty topological space and W be a non empty subset of RAlgebra (the carrier of X). If $W = \mathrm{C}_0(X)$, then W is additively-linearly-closed.

(28) For every non empty topological space X holds $\mathrm{C}_0(X)$ is linearly closed.

Let X be a non empty topological space. Note that $\mathrm{C}_0(X)$ is non empty and linearly closed.

Let X be a non empty topological space. The functor $\mathrm{C}_0^{\mathrm{VS}}(X)$ yielding a real linear space is defined by:

(Def. 7) $\mathrm{C}_0^{\mathrm{VS}}(X) = \langle \mathrm{C}_0(X), \mathrm{Zero}(\mathrm{C}_0(X), \mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}), \mathrm{Add}(\mathrm{C}_0(X), \mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}), \mathrm{Mult}(\mathrm{C}_0(X), \mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X})\rangle$.

The following two propositions are true:

(29) For every non empty topological space X holds $\mathrm{C}_0^{\mathrm{VS}}(X)$ is a subspace of $\mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}$.

(30) For every non empty topological space X and for every set x such that $x \in \mathrm{C}_0(X)$ holds $x \in$ BoundedFunctions (the carrier of X).

Let X be a non empty topological space. The functor $||\cdot||_{\mathrm{C}_0(X)}$ yielding a function from $\mathrm{C}_0(X)$ into $\mathbb{R}$ is defined by:

(Def. 8) $||\cdot||_{\mathrm{C}_0(X)} =$ BoundedFunctionsNorm (the carrier of X) $\restriction \mathrm{C}_0(X)$.

Let X be a non empty topological space. The functor $\mathrm{C}_0^{\mathrm{NS}}(X)$ yields a non empty normed structure and is defined as follows:

(Def. 9) $\mathrm{C}_0^{\mathrm{NS}}(X) = \langle \mathrm{C}_0(X), \mathrm{Zero}(\mathrm{C}_0(X), \mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}), \mathrm{Add}(\mathrm{C}_0(X), \mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}), \mathrm{Mult}(\mathrm{C}_0(X), \mathbb{R}_{\mathbb{R}}^{\text{the carrier of } X}), ||\cdot||_{\mathrm{C}_0(X)}\rangle$.

Let X be a non empty topological space. One can verify that $\mathrm{C}_0^{\mathrm{NS}}(X)$ is strict and non empty.

Next we state several propositions:

(31) For every non empty topological space X and for every set x such that $x \in \mathrm{C}_0(X)$ holds $x \in \mathrm{C}(X;\mathbb{R})$.

(32) For every non empty topological space X holds $0_{\mathrm{C}_0^{\mathrm{VS}}(X)} = X \longmapsto 0$.

(33) For every non empty topological space X holds $0_{\mathrm{C}_0^{\mathrm{NS}}(X)} = X \longmapsto 0$.

(34) Let a be a real number, X be a non empty topological space, and F, G be points of $\mathrm{C}_0^{\mathrm{NS}}(X)$. Then $\|F\| = 0$ iff $F = 0_{\mathrm{C}_0^{\mathrm{NS}}(X)}$ and $\|a \cdot F\| = |a| \cdot \|F\|$ and $\|F + G\| \leq \|F\| + \|G\|$.

(35) For every non empty topological space X holds $\mathrm{C}_0^{\mathrm{NS}}(X)$ is real normed space-like.

Let X be a non empty topological space. Note that $\mathrm{C}_0^{\mathrm{NS}}(X)$ is reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

Next we state the proposition

(36) For every non empty topological space X holds $\mathrm{C}_0^{\mathrm{NS}}(X)$ is a real normed space.

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Free Magmas

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Summary. This article introduces the free magma $M(X)$ constructed on a set X [6]. Then, we formalize some theorems about $M(X)$: if f is a function from the set X to a magma N, the free magma $M(X)$ has a unique extension of f to a morphism of $M(X)$ into N and every magma is isomorphic to a magma generated by a set X under a set of relators on $M(X)$. In doing it, the article defines the stable subset under the law of composition of a magma, the submagma, the equivalence relation compatible with the law of composition and the equivalence kernel of a function. We also introduce some schemes on the recursive function.

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The terminology and notation used here have been introduced in the following articles: [19], [12], [7], [2], [14], [4], [8], [9], [17], [15], [1], [3], [10], [5], [20], [21], [13], [18], [16], and [11].

1. Preliminaries

Let X be a set, let f be a function from $\mathbb{N}$ into X, and let n be a natural number. Observe that $f{\restriction}n$ is transfinite sequence-like.

Let X, x be sets. The 0-sequence $_X(x)$ yielding a finite 0-sequence of X is defined as follows:

(Def. 1) The 0-sequence $_X(x) = \begin{cases} x, & \text{if } x \text{ is a finite 0-sequence of } X, \\ \langle\rangle_X, & \text{otherwise.} \end{cases}$

Let X be a non empty set, let f be a function from X^ω into X, and let c be a finite 0-sequence of X. Then $f(c)$ is an element of X.

One can prove the following proposition

(1) For all sets X, Y, Z such that $Y \subseteq$ the universe of X and $Z \subseteq$ the universe of X holds $Y \times Z \subseteq$ the universe of X.

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In this article we present several logical schemes. The scheme *FuncRecursiveUniq* deals with a unary functor $\mathcal{F}$ yielding a set and functions $\mathcal{A}$, $\mathcal{B}$, and states that:

$\mathcal{A} = \mathcal{B}$

provided the parameters satisfy the following conditions:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$ and for every natural number n holds $\mathcal{A}(n) = \mathcal{F}(\mathcal{A} \restriction n)$, and
- $\operatorname{dom} \mathcal{B} = \mathbb{N}$ and for every natural number n holds $\mathcal{B}(n) = \mathcal{F}(\mathcal{B} \restriction n)$.

The scheme *FuncRecursiveExist* deals with a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and for every natural number n holds $f(n) = \mathcal{F}(f \restriction n)$

for all values of the parameter.

The scheme *FuncRecursiveUniqu2* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and functions $\mathcal{B}$, $\mathcal{C}$ from $\mathbb{N}$ into $\mathcal{A}$, and states that:

$\mathcal{B} = \mathcal{C}$

provided the parameters meet the following requirements:

- For every element n of $\mathbb{N}$ holds $\mathcal{B}(n) = \mathcal{F}(\mathcal{B} \restriction n)$, and
- For every element n of $\mathbb{N}$ holds $\mathcal{C}(n) = \mathcal{F}(\mathcal{C} \restriction n)$.

The scheme *FuncRecursiveExist2* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that for every natural number n holds $f(n) = \mathcal{F}(f \restriction n)$

for all values of the parameters.

Let f, g be functions. We say that f extends g if and only if:

(Def. 2) $\operatorname{dom} g \subseteq \operatorname{dom} f$ and $f \approx g$.

Let us note that there exists a multiplicative magma which is empty.

2. Equivalence Relations and Relators

Let M be a multiplicative magma and let R be an equivalence relation of M. We say that R is compatible if and only if:

(Def. 3) For all elements v, v', w, w' of M such that $v \in [v']_R$ and $w \in [w']_R$ holds $v \cdot w \in [v' \cdot w']_R$.

Let M be a multiplicative magma. Observe that $\nabla_{\text{the carrier of } M}$ is compatible.

Let M be a multiplicative magma. Observe that there exists an equivalence relation of M which is compatible.

One can prove the following proposition

(2) Let M be a multiplicative magma and R be an equivalence relation of M. Then R is compatible if and only if for all elements v, v', w, w' of M such that $[v]_R = [v']_R$ and $[w]_R = [w']_R$ holds $[v \cdot w]_R = [v' \cdot w']_R$.

Let M be a multiplicative magma and let R be a compatible equivalence relation of M. The functor $\circ_R$ yielding a binary operation on Classes R is defined as follows:

(Def. 4)(i) For all elements x, y of Classes R and for all elements v, w of M such that $x = [v]_R$ and $y = [w]_R$ holds $(\circ_R)(x, y) = [v \cdot w]_R$ if M is non empty,

(ii) $\circ_R = \emptyset$, otherwise.

Let M be a multiplicative magma and let R be a compatible equivalence relation of M. The functor ${}^{M}/_{R}$ yielding a multiplicative magma is defined as follows:

(Def. 5) ${}^{M}/_{R} = \langle \text{Classes}\, R, \circ_R \rangle$.

Let M be a multiplicative magma and let R be a compatible equivalence relation of M. Observe that ${}^{M}/_{R}$ is strict.

Let M be a non empty multiplicative magma and let R be a compatible equivalence relation of M. One can check that ${}^{M}/_{R}$ is non empty.

Let M be a non empty multiplicative magma and let R be a compatible equivalence relation of M. The canonical homomorphism onto cosets of R yields a function from M into ${}^{M}/_{R}$ and is defined by:

(Def. 6) For every element v of M holds (the canonical homomorphism onto cosets of R)$(v) = [v]_R$.

Let M be a non empty multiplicative magma and let R be a compatible equivalence relation of M. Note that the canonical homomorphism onto cosets of R is multiplicative.

Let M be a non empty multiplicative magma and let R be a compatible equivalence relation of M. Note that the canonical homomorphism onto cosets of R is onto.

Let M be a multiplicative magma. A function is called a relators of M if:

(Def. 7) $\text{rng}\, \text{it} \subseteq (\text{the carrier of } M) \times (\text{the carrier of } M)$.

Let M be a multiplicative magma and let r be a relators of M. The equivalence relation of r yielding an equivalence relation of M is defined by the condition (Def. 8).

(Def. 8) The equivalence relation of $r = \bigcap\{R; R$ ranges over compatible equivalence relations of M: $\bigwedge_{i\,:\,\text{set}} \bigwedge_{v,w\,:\,\text{element of } M} (i \in \text{dom}\, r \ \wedge \ r(i) = \langle v, w \rangle \ \Rightarrow \ v \in [w]_R)\}$.

Next we state the proposition

(3) Let M be a multiplicative magma, r be a relators of M, and R be a compatible equivalence relation of M. Suppose that for every set i and

for all elements v, w of M such that $i \in \operatorname{dom} r$ and $r(i) = \langle v, w \rangle$ holds $v \in [w]_R$. Then the equivalence relation of $r \subseteq R$.

Let M be a multiplicative magma and let r be a relators of M. Note that the equivalence relation of r is compatible.

Let X, Y be sets and let f be a function from X into Y. The equivalence kernel of f yielding an equivalence relation of X is defined as follows:

(Def. 9) For all sets x, y holds $\langle x, y \rangle \in$ the equivalence kernel of f iff $x, y \in X$ and $f(x) = f(y)$.

In the sequel M, N are non empty multiplicative magmas and f is a function from M into N.

The following propositions are true:

(4) If f is multiplicative, then the equivalence kernel of f is compatible.

(5) Suppose f is multiplicative. Then there exists a relators r of M such that the equivalence kernel of $f =$ the equivalence relation of r.

3. Submagmas and Stable Subsets

Let M be a multiplicative magma. A multiplicative magma is said to be a submagma of M if it satisfies the conditions (Def. 10).

(Def. 10)(i) The carrier of it $\subseteq$ the carrier of M, and

(ii) the multiplication of it $=$ (the multiplication of M) $\restriction$ (the carrier of it).

Let M be a multiplicative magma. One can check that there exists a submagma of M which is strict.

Let M be a non empty multiplicative magma. Note that there exists a submagma of M which is non empty.

In the sequel M denotes a multiplicative magma and N, K denote submagmas of M.

One can prove the following propositions:

(6) Suppose N is a submagma of K and K is a submagma of N. Then the multiplicative magma of $N =$ the multiplicative magma of K.

(7) Suppose the carrier of $N =$ the carrier of M. Then the multiplicative magma of $N =$ the multiplicative magma of M.

Let M be a multiplicative magma and let A be a subset of M. We say that A is stable if and only if:

(Def. 11) For all elements v, w of M such that $v, w \in A$ holds $v \cdot w \in A$.

Let M be a multiplicative magma. One can check that there exists a subset of M which is stable.

We now state the proposition

(8) The carrier of N is a stable subset of M.

Let M be a multiplicative magma and let N be a submagma of M. Note that the carrier of N is stable.

We now state the proposition

(9) Let F be a function. Suppose that for every set i such that $i \in \operatorname{dom} F$ holds $F(i)$ is a stable subset of M. Then $\bigcap F$ is a stable subset of M.

For simplicity, we adopt the following convention: M, N are non empty multiplicative magmas, A is a subset of M, f, g are functions from M into N, X is a stable subset of M, and Y is a stable subset of N.

Next we state four propositions:

(10) A is stable iff $A \cdot A \subseteq A$.

(11) If f is multiplicative, then $f^\circ X$ is a stable subset of N.

(12) If f is multiplicative, then $f^{-1}(Y)$ is a stable subset of M.

(13) If f is multiplicative and g is multiplicative, then $\{v \in M \colon f(v) = g(v)\}$ is a stable subset of M.

Let M be a multiplicative magma and let A be a stable subset of M. The multiplication induced by A yields a binary operation on A and is defined by:

(Def. 12) The multiplication induced by $A =$ (the multiplication of M) $\restriction A$.

Let M be a multiplicative magma and let A be a subset of M. The submagma generated by A yielding a strict submagma of M is defined by the conditions (Def. 13).

(Def. 13)(i) $A \subseteq$ the carrier of the submagma generated by A, and

(ii) for every strict submagma N of M such that $A \subseteq$ the carrier of N holds the submagma generated by A is a submagma of N.

We now state the proposition

(14) Let M be a multiplicative magma and A be a subset of M. Then A is empty if and only if the submagma generated by A is empty.

Let M be a multiplicative magma and let A be an empty subset of M. Note that the submagma generated by A is empty.

The following proposition is true

(15) Let M, N be non empty multiplicative magmas, f be a function from M into N, and X be a subset of M. Suppose f is multiplicative. Then f°(the carrier of the submagma generated by X) $=$ the carrier of the submagma generated by $f^\circ X$.

4. Free Magmas

Let X be a set. The free magma sequence of X yielding a function from $\mathbb{N}$ into $2^{\text{the universe of } X \cup \mathbb{N}}$ is defined by the conditions (Def. 14).

(Def. 14)(i) (The free magma sequence of X)$(0) = \emptyset$,

(ii) (the free magma sequence of X)$(1) = X$, and

(iii) for every natural number n such that $n \geq 2$ there exists a finite sequence f_1 such that $\operatorname{len} f_1 = n - 1$ and for every natural number p such that $p \geq 1$ and $p \leq n - 1$ holds $f_1(p) =$ (the free magma sequence of X)$(p) \times$ (the free magma sequence of X)$(n - p)$ and (the free magma sequence of X)$(n) = \bigcup \text{disjoint}\, f_1$.

Let X be a set and let n be a natural number. The functor $\mathrm{M_n}(X)$ is defined by:

(Def. 15) $\mathrm{M_n}(X) =$ (the free magma sequence of X)(n).

Let X be a non empty set and let n be a non zero natural number. Observe that $\mathrm{M_n}(X)$ is non empty.

In the sequel X is a set.

We now state four propositions:

(16) $\mathrm{M_0}(X) = \emptyset$.

(17) $\mathrm{M_1}(X) = X$.

(18) $\mathrm{M_2}(X) = X \times X \times \{1\}$.

(19) $\mathrm{M_3}(X) = X \times (X \times X \times \{1\}) \times \{1\} \cup X \times X \times \{1\} \times X \times \{2\}$.

We adopt the following convention: x, y, Y are sets and n, m, p are elements of $\mathbb{N}$.

One can prove the following propositions:

(20) Suppose $n \geq 2$. Then there exists a finite sequence f_1 such that $\operatorname{len} f_1 = n-1$ and for every p such that $p \geq 1$ and $p \leq n-1$ holds $f_1(p) = \mathrm{M_p}(X) \times \mathrm{M_{n-'p}}(X)$ and $\mathrm{M_n}(X) = \bigcup \text{disjoint}\, f_1$.

(21) Suppose $n \geq 2$ and $x \in \mathrm{M_n}(X)$. Then there exist p, m such that $x_{\mathbf{2}} = p$ and $1 \leq p \leq n-1$ and $(x_{\mathbf{1}})_{\mathbf{1}} \in \mathrm{M_p}(X)$ and $(x_{\mathbf{1}})_{\mathbf{2}} \in \mathrm{M_m}(X)$ and $n = p+m$ and $x \in \mathrm{M_p}(X) \times \mathrm{M_m}(X) \times \{p\}$.

(22) If $x \in \mathrm{M_n}(X)$ and $y \in \mathrm{M_m}(X)$, then $\langle\langle x, y\rangle, n\rangle \in \mathrm{M_{n+m}}(X)$.

(23) If $X \subseteq Y$, then $\mathrm{M_n}(X) \subseteq \mathrm{M_n}(Y)$.

Let X be a set. The carrier of free magma on X is defined as follows:

(Def. 16) The carrier of free magma on $X = \bigcup \text{disjoint}(($the free magma sequence of $X) \restriction \mathbb{N}^+)$.

One can prove the following proposition

(24) $X = \emptyset$ iff the carrier of free magma on $X = \emptyset$.

Let X be an empty set. Observe that the carrier of free magma on X is empty.

Let X be a non empty set. One can verify that the carrier of free magma on X is non empty. Let w be an element of the carrier of free magma on X. Observe that $w_{\mathbf{2}}$ is non zero and natural.

We now state four propositions:

(25) For every non empty set X and for every element w of the carrier of free magma on X holds $w \in \mathrm{M}_{\mathrm{w}_{\mathbf{2}}}(X) \times \{w_{\mathbf{2}}\}$.

(26) Let X be a non empty set and v, w be elements of the carrier of free magma on X. Then $\langle\langle\langle v_{\mathbf{1}}, w_{\mathbf{1}}\rangle, v_{\mathbf{2}}\rangle, v_{\mathbf{2}} + w_{\mathbf{2}}\rangle$ is an element of the carrier of free magma on X.

(27) If $X \subseteq Y$, then the carrier of free magma on $X \subseteq$ the carrier of free magma on Y.

(28) If $n > 0$, then $\mathrm{M}_{\mathrm{n}}(X) \times \{n\} \subseteq$ the carrier of free magma on X.

Let X be a set. The multiplication of free magma on X yields a binary operation on the carrier of free magma on X and is defined as follows:

(Def. 17)(i) For all elements v, w of the carrier of free magma on X and for all n, m such that $n = v_{\mathbf{2}}$ and $m = w_{\mathbf{2}}$ holds (the multiplication of free magma on $X)(v, w) = \langle\langle\langle v_{\mathbf{1}}, w_{\mathbf{1}}\rangle, v_{\mathbf{2}}\rangle, n + m\rangle$ if X is non empty,

(ii) the multiplication of free magma on $X = \emptyset$, otherwise.

Let X be a set. The functor $\mathrm{M}(X)$ yields a multiplicative magma and is defined by:

(Def. 18) $\mathrm{M}(X) = \langle$the carrier of free magma on X, the multiplication of free magma on $X\rangle$.

Let X be a set. Note that $\mathrm{M}(X)$ is strict.

Let X be an empty set. One can verify that $\mathrm{M}(X)$ is empty.

Let X be a non empty set. Note that $\mathrm{M}(X)$ is non empty. Let w be an element of $\mathrm{M}(X)$. One can verify that $w_{\mathbf{2}}$ is non zero and natural.

The following proposition is true

(29) For every set X and for every subset S of X holds $\mathrm{M}(S)$ is a submagma of $\mathrm{M}(X)$.

Let X be a set and let w be an element of $\mathrm{M}(X)$. The functor length w yields a natural number and is defined by:

(Def. 19) $\operatorname{length} w = \begin{cases} w_{\mathbf{2}}, & \text{if } X \text{ is non empty,} \\ 0, & \text{otherwise.} \end{cases}$

One can prove the following proposition

(30) $X = \{w_{\mathbf{1}}; w$ ranges over elements of $\mathrm{M}(X)$: $\operatorname{length} w = 1\}$.

In the sequel v, v_1, v_2, w, w_1, w_2 denote elements of $\mathrm{M}(X)$.

One can prove the following propositions:

(31) If X is non empty, then $v \cdot w = \langle\langle\langle v_{\mathbf{1}}, w_{\mathbf{1}}\rangle, v_{\mathbf{2}}\rangle, \operatorname{length} v + \operatorname{length} w\rangle$.

(32) If X is non empty, then $v = \langle v_{\mathbf{1}}, v_{\mathbf{2}}\rangle$ and $\operatorname{length} v \geq 1$.

(33) $\operatorname{length}(v \cdot w) = \operatorname{length} v + \operatorname{length} w$.

(34) If $\operatorname{length} w \geq 2$, then there exist w_1, w_2 such that $w = w_1 \cdot w_2$ and $\operatorname{length} w_1 < \operatorname{length} w$ and $\operatorname{length} w_2 < \operatorname{length} w$.

(35) If $v_1 \cdot v_2 = w_1 \cdot w_2$, then $v_1 = w_1$ and $v_2 = w_2$.

Let X be a set and let n be a natural number. The n-canonical image of X yields a function from $\mathrm{M_n}(X)$ into $\mathrm{M}(X)$ and is defined as follows:

(Def. 20)(i) For every x such that $x \in \operatorname{dom}$(the n-canonical image of X) holds (the n-canonical image of X)$(x) = \langle x, n\rangle$ if $n > 0$,

(ii) the n-canonical image of $X = \emptyset$, otherwise.

Let X be a set and let n be a natural number. Observe that the n-canonical image of X is one-to-one.

Let X be a non empty set. Observe that the 1-canonical image of X

In the sequel X, Y, Z are non empty sets.

Next we state three propositions:

(36) For every subset A of $\mathrm{M}(X)$ such that $A =$ (the 1-canonical image of $X)^\circ X$ holds $\mathrm{M}(X) =$ the submagma generated by A.

(37) Let R be a compatible equivalence relation of $\mathrm{M}(X)$. Then ${}^{\mathrm{M}(X)}/_{R} =$ the submagma generated by (the canonical homomorphism onto cosets of $R)^\circ$(the 1-canonical image of $X)^\circ X$.

(38) For every function f from X into Y holds (the 1-canonical image of $Y) \cdot f$ is a function from X into $\mathrm{M}(Y)$.

Let X be a non empty set, let M be a non empty multiplicative magma, let n, m be non zero natural numbers, let f be a function from $\mathrm{M_n}(X)$ into M, and let g be a function from $\mathrm{M_m}(X)$ into M. The functor $f \times g$ yielding a function from $\mathrm{M_n}(X) \times \mathrm{M_m}(X) \times \{n\}$ into M is defined by the condition (Def. 21).

(Def. 21) Let x be an element of $\mathrm{M_n}(X) \times \mathrm{M_m}(X) \times \{n\}$, y be an element of $\mathrm{M_n}(X)$, and z be an element of $\mathrm{M_m}(X)$. If $y = (x_{\mathbf{1}})_{\mathbf{1}}$ and $z = (x_{\mathbf{1}})_{\mathbf{2}}$, then $(f \times g)(x) = f(y) \cdot g(z)$.

In the sequel M is a non empty multiplicative magma.

One can prove the following propositions:

(39) Let f be a function from X into M. Then there exists a function h from $\mathrm{M}(X)$ into M such that h is multiplicative and h extends $f \cdot$ (the 1-canonical image of $X)^{-1}$.

(40) Let f be a function from X into M and h, g be functions from $\mathrm{M}(X)$ into M. Suppose that

(i) h is multiplicative,

(ii) h extends $f \cdot$ (the 1-canonical image of $X)^{-1}$,

(iii) g is multiplicative, and

(iv) g extends $f \cdot (\text{the 1-canonical image of } X)^{-1}$.
Then $h = g$.

For simplicity, we adopt the following rules: M, N are non empty multiplicative magmas, f is a function from M into N, H is a non empty submagma of N, and R is a compatible equivalence relation of M.

We now state three propositions:

(41) Suppose f is multiplicative and the carrier of $H = \operatorname{rng} f$ and $R =$ the equivalence kernel of f. Then there exists a function g from ${}^{M}/_{R}$ into H such that $f = g \cdot$ the canonical homomorphism onto cosets of R and g is bijective and multiplicative.

(42) Let g_1, g_2 be functions from ${}^{M}/_{R}$ into N. Suppose $g_1 \cdot$ the canonical homomorphism onto cosets of $R = g_2 \cdot$ the canonical homomorphism onto cosets of R. Then $g_1 = g_2$.

(43) Let M be a non empty multiplicative magma. Then there exists a non empty set X and there exists a relators r of $\mathrm{M}(X)$ such that there exists a function from ${}^{\mathrm{M}(X)}/_{\text{the equivalence relation of } r}$ into M which is bijective and multiplicative.

Let X, Y be non empty sets and let f be a function from X into Y. The functor $\mathbf{M}(f)$ yields a function from $\mathrm{M}(X)$ into $\mathrm{M}(Y)$ and is defined by:

(Def. 22) $\mathbf{M}(f)$ is multiplicative and $\mathbf{M}(f)$ extends $(\text{the 1-canonical image of } Y) \cdot f \cdot (\text{the 1-canonical image of } X)^{-1}$.

Let X, Y be non empty sets and let f be a function from X into Y. One can verify that $\mathbf{M}(f)$ is multiplicative.

In the sequel f denotes a function from X into Y and g denotes a function from Y into Z.

Next we state several propositions:

(44) $\mathbf{M}(g \cdot f) = \mathbf{M}(g) \cdot \mathbf{M}(f)$.

(45) For every function g from X into Z such that $Y \subseteq Z$ and $f = g$ holds $\mathbf{M}(f) = \mathbf{M}(g)$.

(46) $\mathbf{M}(\mathrm{id}_X) = \mathrm{id}_{\operatorname{dom} \mathbf{M}(f)}$.

(47) If f is one-to-one, then $\mathbf{M}(f)$ is one-to-one.

(48) If f is onto, then $\mathbf{M}(f)$ is onto.

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Integrability Formulas. Part I

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Summary. In this article, we give several differentiation and integrability formulas of special and composite functions including the trigonometric function, and the polynomial function.

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The papers [12], [2], [3], [1], [7], [11], [13], [4], [17], [8], [9], [6], [18], [5], [10], [15], [16], and [14] provide the terminology and notation for this paper.

One can check that there exists a subset of $\mathbb{R}$ which is closed-interval.

For simplicity, we use the following convention: a, b, x, r are real numbers, n is an element of $\mathbb{N}$, A is a closed-interval subset of $\mathbb{R}$, f, g, f_1, f_2, g_1, g_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$, and Z is an open subset of $\mathbb{R}$.

We now state a number of propositions:

(1) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{f_1+f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f_2 = \square^2$. Then $\frac{1}{f_1+f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f_1+f_2})'_{\restriction Z}(x) = -\frac{2 \cdot x}{(1+x^2)^{\mathbf{2}}}$.

(2) Suppose that $A \subseteq Z$ and $f = \frac{\frac{1}{g_1+g_2}}{f_2}$ and $f_2 =$ the function arccot and $Z \subseteq]-1, 1[$ and $g_2 = \square^2$ and for every x such that $x \in Z$ holds $g_1(x) = 1$ and $f_2(x) > 0$ and $Z = \mathrm{dom}\, f$. Then $\int_A f(x)dx = (-(\text{the function ln}) \cdot (\text{the function arccot}))(\sup A) - (-(\text{the function ln}) \cdot (\text{the function arccot}))(\inf A)$.

(3) Suppose that

(i) $A \subseteq Z$,

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(ii) for every x such that $x \in Z$ holds (the function exp)$(x) < 1$ and $f_1(x) = 1$,
(iii) $Z \subseteq$ dom((the function arctan) $\cdot$(the function exp)),
(iv) $Z = \operatorname{dom} f$, and
(v) $f = \frac{\text{the function exp}}{f_1 + (\text{the function exp})^{\mathbf{2}}}$.

Then $\int\limits_A f(x)dx =$ ((the function arctan) $\cdot$(the function exp))$(\sup A) -$ ((the function arctan) $\cdot$(the function exp))$(\inf A)$.

(4) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds (the function exp)$(x) < 1$ and $f_1(x) = 1$,
(iii) $Z \subseteq$ dom((the function arccot) $\cdot$(the function exp)),
(iv) $Z = \operatorname{dom} f$, and
(v) $f = \frac{-\text{the function exp}}{f_1 + (\text{the function exp})^{\mathbf{2}}}$.

Then $\int\limits_A f(x)dx =$ ((the function arccot) $\cdot$(the function exp))$(\sup A) -$ ((the function arccot) $\cdot$(the function exp))$(\inf A)$.

(5) Suppose that
(i) $A \subseteq Z$,
(ii) $Z = \operatorname{dom} f$, and
(iii) $f =$ (the function exp) $\left(\frac{\text{the function sin}}{\text{the function cos}} + \frac{\text{the function exp}}{(\text{the function cos})^{\mathbf{2}}}\right)$.

Then $\int\limits_A f(x)dx =$ ((the function exp) (the function tan))$(\sup A) -$ ((the function exp) (the function tan))$(\inf A)$.

(6) Suppose that
(i) $A \subseteq Z$,
(ii) $Z = \operatorname{dom} f$, and
(iii) $f =$ (the function exp) $\left(\frac{\text{the function cos}}{\text{the function sin}} - \frac{\text{the function exp}}{(\text{the function sin})^{\mathbf{2}}}\right)$.

Then $\int\limits_A f(x)dx =$ ((the function exp) (the function cot))$(\sup A) -$ ((the function exp) (the function cot))$(\inf A)$.

(7) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f_1(x) = 1$,
(iii) $Z \subseteq]-1, 1[$,
(iv) $Z = \operatorname{dom} f$, and
(v) $f =$ (the function exp) (the function arctan)$+\frac{\text{the function exp}}{f_1 + \square^2}$.

Then $\int_A f(x)dx = (\text{(the function exp)} \ \text{(the function arctan)})(\sup A) - (\text{(the function exp)} \ \text{(the function arctan)})(\inf A)$.

(8) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $f_1(x) = 1$,

(iii) $Z \subseteq]-1, 1[$,

(iv) $Z = \operatorname{dom} f$, and

(v) $f = \text{(the function exp)} \ \text{(the function arccot)} - \frac{\text{the function exp}}{f_1 + \square^2}$.

Then $\int_A f(x)dx = (\text{(the function exp)} \ \text{(the function arccot)})(\sup A) - (\text{(the function exp)} \ \text{(the function arccot)})(\inf A)$.

(9) Suppose $A \subseteq Z = \operatorname{dom} f$ and $f = (\text{(the function exp)} \cdot \text{(the function sin)}) \ \text{(the function cos)}$. Then $\int_A f(x)dx = (\text{(the function exp)} \cdot \text{(the function sin)})(\sup A) - (\text{(the function exp)} \cdot \text{(the function sin)})(\inf A)$.

(10) Suppose $A \subseteq Z = \operatorname{dom} f$ and $f = (\text{(the function exp)} \cdot \text{(the function cos)}) \ \text{(the function sin)}$.
Then $\int_A f(x)dx = (-\text{(the function exp)} \cdot \text{(the function cos)})(\sup A) - (-\text{(the function exp)} \cdot \text{(the function cos)})(\inf A)$.

(11) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $x > 0$ and $Z = \operatorname{dom} f$ and $f = (\text{(the function cos)} \cdot \text{(the function ln)}) \ \frac{1}{\mathrm{id}_Z}$. Then $\int_A f(x)dx = (\text{(the function sin)} \cdot \text{(the function ln)})(\sup A) - (\text{(the function sin)} \cdot \text{(the function ln)})(\inf A)$.

(12) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $x > 0$ and $Z = \operatorname{dom} f$ and $f = (\text{(the function sin)} \cdot \text{(the function ln)}) \ \frac{1}{\mathrm{id}_Z}$. Then $\int_A f(x)dx = (-\text{(the function cos)} \cdot \text{(the function ln)})(\sup A) - (-\text{(the function cos)} \cdot \text{(the function ln)})(\inf A)$.

(13) Suppose $A \subseteq Z = \operatorname{dom} f$ and $f = \text{(the function exp)} \ (\text{(the function cos)} \cdot \text{(the function exp)})$. Then $\int_A f(x)dx = (\text{(the function sin)} \cdot \text{(the function exp)})(\sup A) - (\text{(the function sin)} \cdot \text{(the function exp)})(\inf A)$.

(14) Suppose $A \subseteq Z = \operatorname{dom} f$ and $f = \text{(the function exp)} \ (\text{(the function sin)} \cdot \text{(the function exp)})$.
Then $\int_A f(x)dx = (-\text{(the function cos)} \cdot \text{(the function exp)})(\sup A) - (-\text{(the function cos)} \cdot \text{(the function exp)})(\inf A)$.

(15) Suppose that $A \subseteq Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (f_1+f_2))$ and $r \neq 0$ and for every x such that $x \in Z$ holds $g(x) = \frac{x}{r}$ and $g(x) > -1$ and $g(x) < 1$ and $f_1(x) = 1$ and $f_2 = (\square^2) \cdot g$ and $Z = \operatorname{dom} f$ and $f = (\text{the function arctan}) \cdot g$. Then $\int\limits_A f(x)dx = (\mathrm{id}_Z\, ((\text{the function arctan}) \cdot g) - \frac{r}{2}\, ((\text{the function ln}) \cdot (f_1 + f_2)))(\sup A) - (\mathrm{id}_Z\, ((\text{the function arctan}) \cdot g) - \frac{r}{2}\, ((\text{the function ln}) \cdot (f_1 + f_2)))(\inf A)$.

(16) Suppose that $A \subseteq Z \subseteq \operatorname{dom}((\text{the function ln}) \cdot (f_1+f_2))$ and $r \neq 0$ and for every x such that $x \in Z$ holds $g(x) = \frac{x}{r}$ and $g(x) > -1$ and $g(x) < 1$ and $f_1(x) = 1$ and $f_2 = (\square^2) \cdot g$ and $Z = \operatorname{dom} f$ and $f = (\text{the function arccot}) \cdot g$. Then $\int\limits_A f(x)dx = (\mathrm{id}_Z\, ((\text{the function arccot}) \cdot g) + \frac{r}{2}\, ((\text{the function ln}) \cdot (f_1 + f_2)))(\sup A) - (\mathrm{id}_Z\, ((\text{the function arccot}) \cdot g) + \frac{r}{2}\, ((\text{the function ln}) \cdot (f_1 + f_2)))(\inf A)$.

(17) Suppose that
- (i) $A \subseteq Z$,
- (ii) $f = (\text{the function arctan}) \cdot f_1 + \frac{\mathrm{id}_Z}{r\,(g+{f_1}^{\mathbf{2}})}$,
- (iii) for every x such that $x \in Z$ holds $g(x) = 1$ and $f_1(x) = \frac{x}{r}$ and $f_1(x) > -1$ and $f_1(x) < 1$,
- (iv) $Z = \operatorname{dom} f$, and
- (v) f is continuous on A.

Then $\int\limits_A f(x)dx = (\mathrm{id}_Z\, ((\text{the function arctan}) \cdot f_1))(\sup A) - (\mathrm{id}_Z\, ((\text{the function arctan}) \cdot f_1))(\inf A)$.

(18) Suppose that
- (i) $A \subseteq Z$,
- (ii) $f = (\text{the function arccot}) \cdot f_1 - \frac{\mathrm{id}_Z}{r\,(g+{f_1}^{\mathbf{2}})}$,
- (iii) for every x such that $x \in Z$ holds $g(x) = 1$ and $f_1(x) = \frac{x}{r}$ and $f_1(x) > -1$ and $f_1(x) < 1$,
- (iv) $Z = \operatorname{dom} f$, and
- (v) f is continuous on A.

Then $\int\limits_A f(x)dx = (\mathrm{id}_Z\, ((\text{the function arccot}) \cdot f_1))(\sup A) - (\mathrm{id}_Z\, ((\text{the function arccot}) \cdot f_1))(\inf A)$.

(19) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $Z = \operatorname{dom} f$ and $Z \subseteq \operatorname{dom}((\square^n) \cdot (\text{the function arcsin}))$ and $1 < n$ and $f = \frac{n\,((\square^{n-1}) \cdot (\text{the function arcsin}))}{(\square^{\frac{1}{2}}) \cdot (f_1 - \square^2)}$. Then $\int\limits_A f(x)dx = ((\square^n) \cdot (\text{the function arcsin}))(\sup A) - ((\square^n) \cdot (\text{the function arcsin}))(\inf A)$.

(20) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds

$f_1(x) = 1$ and $Z \subseteq \mathrm{dom}((\square^n) \cdot (\text{the function arccos}))$ and $Z = \mathrm{dom}\, f$ and $1 < n$ and $f = \frac{n\,((\square^{n-1})\cdot(\text{the function arccos}))}{(\square^{\frac{1}{2}})\cdot(f_1 - \square^2)}$. Then $\int\limits_A f(x)dx = (-(\square^n) \cdot (\text{the function arccos}))(\sup A) - (-(\square^n) \cdot (\text{the function arccos}))(\inf A)$.

(21) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $Z \subseteq]-1,1[$ and $Z = \mathrm{dom}\, f$ and $f = (\text{the function arcsin}) + \frac{\mathrm{id}_Z}{(\square^{\frac{1}{2}})\cdot(f_1 - \square^2)}$. Then $\int\limits_A f(x)dx = (\mathrm{id}_Z\,(\text{the function arcsin}))(\sup A) - (\mathrm{id}_Z\,(\text{the function arcsin}))(\inf A)$.

(22) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $Z \subseteq]-1,1[$ and $Z = \mathrm{dom}\, f$ and $f = (\text{the function arccos}) - \frac{\mathrm{id}_Z}{(\square^{\frac{1}{2}})\cdot(f_1 - \square^2)}$. Then $\int\limits_A f(x)dx = (\mathrm{id}_Z\,(\text{the function arccos}))(\sup A) - (\mathrm{id}_Z\,(\text{the function arccos}))(\inf A)$.

(23) Suppose that
- (i) $A \subseteq Z$,
- (ii) $Z \subseteq]-1,1[$,
- (iii) for every x such that $x \in Z$ holds $f_1(x) = a \cdot x + b$ and $f_2(x) = 1$,
- (iv) $Z = \mathrm{dom}\, f$, and
- (v) $f = a\,(\text{the function arcsin}) + \frac{f_1}{(\square^{\frac{1}{2}})\cdot(f_2 - \square^2)}$.

Then $\int\limits_A f(x)dx = (f_1\,(\text{the function arcsin}))(\sup A) - (f_1\,(\text{the function arcsin}))(\inf A)$.

(24) Suppose that
- (i) $A \subseteq Z$,
- (ii) $Z \subseteq]-1,1[$,
- (iii) for every x such that $x \in Z$ holds $f_1(x) = a \cdot x + b$ and $f_2(x) = 1$,
- (iv) $Z = \mathrm{dom}\, f$, and
- (v) $f = a\,(\text{the function arccos}) - \frac{f_1}{(\square^{\frac{1}{2}})\cdot(f_2 - \square^2)}$.

Then $\int\limits_A f(x)dx = (f_1\,(\text{the function arccos}))(\sup A) - (f_1\,(\text{the function arccos}))(\inf A)$.

(25) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $g(x) = 1$ and $f_1(x) = \frac{x}{a}$ and $f_1(x) > -1$ and $f_1(x) < 1$,
- (iii) $Z = \mathrm{dom}\, f$,
- (iv) f is continuous on A, and

(v) $f = (\text{the function arcsin}) \cdot f_1 + \frac{\mathrm{id}_Z}{a\,((\Box^{\frac{1}{2}})\cdot(g - f_1{}^{\mathbf{2}}))}$.

Then $\int\limits_A f(x)dx = (\mathrm{id}_Z\,((\text{the function arcsin})\ \cdot f_1))(\sup A) - (\mathrm{id}_Z\,((\text{the function arcsin})\ \cdot f_1))(\inf A)$.

(26) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $g(x) = 1$ and $f_1(x) = \frac{x}{a}$ and $f_1(x) > -1$ and $f_1(x) < 1$,

(iii) $Z = \operatorname{dom} f$,

(iv) f is continuous on A, and

(v) $f = (\text{the function arccos}) \cdot f_1 - \frac{\mathrm{id}_Z}{a\,((\Box^{\frac{1}{2}})\cdot(g - f_1{}^{\mathbf{2}}))}$.

Then $\int\limits_A f(x)dx = (\mathrm{id}_Z\,((\text{the function arccos})\ \cdot f_1))(\sup A) - (\mathrm{id}_Z\,((\text{the function arccos})\ \cdot f_1))(\inf A)$.

(27) Suppose $A \subseteq Z$ and $f = \frac{n\,((\Box^{n-1})\cdot(\text{the function sin}))}{(\Box^{n+1})\cdot(\text{the function cos})}$ and $1 \leq n$ and $Z \subseteq \operatorname{dom}((\Box^n) \cdot (\text{the function tan}))$ and $Z = \operatorname{dom} f$. Then $\int\limits_A f(x)dx = ((\Box^n) \cdot (\text{the function tan}))(\sup A) - ((\Box^n) \cdot (\text{the function tan}))(\inf A)$.

(28) Suppose $A \subseteq Z$ and $f = \frac{n\,((\Box^{n-1})\cdot(\text{the function cos}))}{(\Box^{n+1})\cdot(\text{the function sin})}$ and $1 \leq n$ and $Z \subseteq \operatorname{dom}((\Box^n) \cdot (\text{the function cot}))$ and $Z = \operatorname{dom} f$. Then $\int\limits_A f(x)dx = (-(\Box^n) \cdot (\text{the function cot}))(\sup A) - (-(\Box^n) \cdot (\text{the function cot}))(\inf A)$.

(29) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq \operatorname{dom}((\text{the function tan})\ \cdot f_1)$,

(iii) $f = \frac{((\text{the function sin})\cdot f_1)^{\mathbf{2}}}{((\text{the function cos})\cdot f_1)^{\mathbf{2}}}$,

(iv) for every x such that $x \in Z$ holds $f_1(x) = a \cdot x$ and $a \neq 0$, and

(v) $Z = \operatorname{dom} f$.

Then $\int\limits_A f(x)dx = (\frac{1}{a}\,((\text{the function tan})\ \cdot f_1) - \mathrm{id}_Z)(\sup A) - (\frac{1}{a}\,((\text{the function tan})\ \cdot f_1) - \mathrm{id}_Z)(\inf A)$.

(30) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq \operatorname{dom}((\text{the function cot})\ \cdot f_1)$,

(iii) $f = \frac{((\text{the function cos})\cdot f_1)^{\mathbf{2}}}{((\text{the function sin})\cdot f_1)^{\mathbf{2}}}$,

(iv) for every x such that $x \in Z$ holds $f_1(x) = a \cdot x$ and $a \neq 0$, and

(v) $Z = \operatorname{dom} f$.

Then $\int\limits_A f(x)dx = ((-\frac{1}{a})((\text{the function cot}) \cdot f_1) - \mathrm{id}_Z)(\sup A) - ((-\frac{1}{a})((\text{the function cot}) \cdot f_1) - \mathrm{id}_Z)(\inf A)$.

(31) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f_1(x) = a \cdot x + b$,
(iii) $Z = \mathrm{dom}\, f$, and
(iv) $f = a \frac{\text{the function sin}}{\text{the function cos}} + \frac{f_1}{(\text{the function cos})^{\mathbf{2}}}$.

Then $\int\limits_A f(x)dx = (f_1\,(\text{the function tan}))(\sup A) - (f_1\,(\text{the function tan}))(\inf A)$.

(32) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f_1(x) = a \cdot x + b$,
(iii) $Z = \mathrm{dom}\, f$, and
(iv) $f = a \frac{\text{the function cos}}{\text{the function sin}} - \frac{f_1}{(\text{the function sin})^{\mathbf{2}}}$.

Then $\int\limits_A f(x)dx = (f_1\,(\text{the function cot}))(\sup A) - (f_1\,(\text{the function cot}))(\inf A)$.

(33) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function sin})(x)^{\mathbf{2}}}{(\text{the function cos})(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq \mathrm{dom}((\text{the function tan}) - \mathrm{id}_Z)$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx = ((\text{the function tan}) - \mathrm{id}_Z)(\sup A) - ((\text{the function tan}) - \mathrm{id}_Z)(\inf A)$.

(34) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function cos})(x)^{\mathbf{2}}}{(\text{the function sin})(x)^{\mathbf{2}}}$,
(iii) $Z \subseteq \mathrm{dom}(-\text{the function cot} - \mathrm{id}_Z)$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.

Then $\int\limits_A f(x)dx = (-\text{the function cot} - \mathrm{id}_Z)(\sup A) - (-\text{the function cot} - \mathrm{id}_Z)(\inf A)$.

(35) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{x \cdot (1 + (\text{the function ln})(x)^{\mathbf{2}})}$ and $(\text{the function ln})(x) > -1$ and $(\text{the function ln})(x) < 1$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function arctan}) \cdot (\text{the function ln}))$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.
Then $\int_A f(x)dx = ((\text{the function arctan}) \cdot (\text{the function ln}))(\sup A) - ((\text{the function arctan}) \cdot (\text{the function ln}))(\inf A)$.

(36) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = -\frac{1}{x \cdot (1 + (\text{the function ln})(x)^{2})}$ and (the function ln)$(x) > -1$ and (the function ln)$(x) < 1$,
(iii) $Z \subseteq \mathrm{dom}((\text{the function arccot}) \cdot (\text{the function ln}))$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.
Then $\int_A f(x)dx = ((\text{the function arccot}) \cdot (\text{the function ln}))(\sup A) - ((\text{the function arccot}) \cdot (\text{the function ln}))(\inf A)$.

(37) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{a}{\sqrt{1-(a \cdot x + b)^{2}}}$ and $f_1(x) = a \cdot x + b$ and $f_1(x) > -1$ and $f_1(x) < 1$,
(iii) $Z \subseteq \mathrm{dom}((\text{the function arcsin}) \cdot f_1)$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.
Then $\int_A f(x)dx = ((\text{the function arcsin}) \cdot f_1)(\sup A) - ((\text{the function arcsin}) \cdot f_1)(\inf A)$.

(38) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $f(x) = \frac{a}{\sqrt{1-(a \cdot x + b)^{2}}}$ and $f_1(x) = a \cdot x + b$ and $f_1(x) > -1$ and $f_1(x) < 1$,
(iii) $Z \subseteq \mathrm{dom}((\text{the function arccos}) \cdot f_1)$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) f is continuous on A.
Then $\int_A f(x)dx = (-(\text{the function arccos}) \cdot f_1)(\sup A) - (-(\text{the function arccos}) \cdot f_1)(\inf A)$.

(39) Suppose that $A \subseteq Z$ and $f_1 = g - f_2$ and $f_2 = \square^{2}$ and for every x such that $x \in Z$ holds $f(x) = x \cdot (1 - x^{2})^{-\frac{1}{2}}$ and $g(x) = 1$ and $f_1(x) > 0$ and $Z \subseteq \mathrm{dom}((\square^{\frac{1}{2}}) \cdot f_1)$ and $Z = \mathrm{dom}\, f$ and f is continuous on A. Then $\int_A f(x)dx =$

$(-(\square^{\frac{1}{2}}) \cdot f_1)(\sup A) - (-(\square^{\frac{1}{2}}) \cdot f_1)(\inf A)$.

(40) Suppose that $A \subseteq Z$ and $g = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f(x) = x \cdot (a^{\mathbf{2}} - x^2)^{-\frac{1}{2}}$ and $f_1(x) = a^{\mathbf{2}}$ and $g(x) > 0$ and $Z \subseteq \operatorname{dom}((\square^{\frac{1}{2}}) \cdot g)$ and $Z = \operatorname{dom} f$ and f is continuous on A. Then $\int\limits_{A} f(x)dx = (-(\square^{\frac{1}{2}}) \cdot g)(\sup A) - (-(\square^{\frac{1}{2}}) \cdot g)(\inf A)$.

(41) Suppose that

(i) $A \subseteq Z$,

(ii) $n > 0$,

(iii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function cos})(x)}{(\text{the function sin})(x)^{n+1}}$ and (the function sin)$(x) \neq 0$,

(iv) $Z \subseteq \operatorname{dom}((\square^n) \cdot \frac{1}{\text{the function sin}})$,

(v) $Z = \operatorname{dom} f$, and

(vi) f is continuous on A.

Then $\int\limits_{A} f(x)dx = ((-\frac{1}{n})\,((\square^n) \cdot \frac{1}{\text{the function sin}}))(\sup A) - ((-\frac{1}{n})\,((\square^n) \cdot \frac{1}{\text{the function sin}}))(\inf A)$.

(42) Suppose that

(i) $A \subseteq Z$,

(ii) $n > 0$,

(iii) for every x such that $x \in Z$ holds $f(x) = \frac{(\text{the function sin})(x)}{(\text{the function cos})(x)^{n+1}}$ and (the function cos)$(x) \neq 0$,

(iv) $Z \subseteq \operatorname{dom}((\square^n) \cdot \frac{1}{\text{the function cos}})$,

(v) $Z = \operatorname{dom} f$, and

(vi) f is continuous on A.

Then $\int\limits_{A} f(x)dx = (\frac{1}{n}\,((\square^n) \cdot \frac{1}{\text{the function cos}}))(\sup A) - (\frac{1}{n}\,((\square^n) \cdot \frac{1}{\text{the function cos}}))(\inf A)$.

(43) Suppose that $A \subseteq Z$ and $f = \frac{\frac{1}{g_1+g_2}}{f_2}$ and $f_2 =$ the function arccot and $Z \subseteq]-1,1[$ and $g_2 = \square^2$ and for every x such that $x \in Z$ holds $f(x) = \frac{1}{(1+x^{\mathbf{2}}) \cdot (\text{the function arccot})(x)}$ and $g_1(x) = 1$ and $f_2(x) > 0$ and $Z = \operatorname{dom} f$. Then $\int\limits_{A} f(x)dx = (-(\text{the function ln}) \cdot (\text{the function arccot}))(\sup A) - (-(\text{the function ln}) \cdot (\text{the function arccot}))(\inf A)$.

(44) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1,1[$,

(iii) for every x such that $x \in Z$ holds (the function arcsin)$(x) > 0$ and $f_1(x) = 1$,

(iv) $Z \subseteq \mathrm{dom}((\text{the function ln}) \cdot (\text{the function arcsin}))$,

(v) $Z = \mathrm{dom}\, f$, and

(vi) $f = \frac{1}{((\square^{\frac{1}{2}}) \cdot (f_1 - \square^2))\ (\text{the function arcsin})}$.

Then $\int\limits_A f(x)dx = ((\text{the function ln}) \cdot (\text{the function arcsin}))(\sup A) - ((\text{the function ln}) \cdot (\text{the function arcsin}))(\inf A)$.

(45) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f_1(x) = 1$ and $(\text{the function arccos})(x) > 0$,

(iv) $Z \subseteq \mathrm{dom}((\text{the function ln}) \cdot (\text{the function arccos}))$,

(v) $Z = \mathrm{dom}\, f$, and

(vi) $f = \frac{1}{((\square^{\frac{1}{2}}) \cdot (f_1 - \square^2))\ (\text{the function arccos})}$.

Then $\int\limits_A f(x)dx = (-(\text{the function ln}) \cdot (\text{the function arccos}))(\sup A) - (-(\text{the function ln}) \cdot (\text{the function arccos}))(\inf A)$.

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Partial Differentiation of Real Ternary Functions

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Summary. In this article, we shall extend the result of [19] to discuss partial differentiation of real ternary functions (refer to [8] and [16] for partial differentiation).

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The notation and terminology used here have been introduced in the following papers: [7], [12], [13], [14], [1], [2], [3], [4], [5], [8], [19], [15], [9], [18], [6], [11], [10], and [17].

1. Preliminaries

For simplicity, we use the following convention: D denotes a set, x, x_0, y, y_0, z, z_0, r, s, t denote real numbers, p, a, u, u_0 denote elements of $\mathcal{R}^3$, f, f_1, f_2, f_3, g denote partial functions from $\mathcal{R}^3$ to $\mathbb{R}$, R denotes a rest, and L denotes a linear function.

One can prove the following three propositions:

(1) $\operatorname{dom}\operatorname{proj}(1,3) = \mathcal{R}^3$ and $\operatorname{rng}\operatorname{proj}(1,3) = \mathbb{R}$ and for all elements x, y, z of $\mathbb{R}$ holds $(\operatorname{proj}(1,3))(\langle x, y, z\rangle) = x$.

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(2) dom proj(2, 3) = $\mathcal{R}^3$ and rng proj(2, 3) = $\mathbb{R}$ and for all elements x, y, z of $\mathbb{R}$ holds $(\text{proj}(2,3))(\langle x, y, z\rangle) = y$.

(3) dom proj(3, 3) = $\mathcal{R}^3$ and rng proj(3, 3) = $\mathbb{R}$ and for all elements x, y, z of $\mathbb{R}$ holds $(\text{proj}(3,3))(\langle x, y, z\rangle) = z$.

2. Partial Differentiation of Real Ternary Functions

One can prove the following propositions:

(4) If $u = \langle x, y, z\rangle$ and f is partially differentiable in u w.r.t. coordinate number 1, then SVF1$(1, f, u)$ is differentiable in x.

(5) If $u = \langle x, y, z\rangle$ and f is partially differentiable in u w.r.t. coordinate number 2, then SVF1$(2, f, u)$ is differentiable in y.

(6) If $u = \langle x, y, z\rangle$ and f is partially differentiable in u w.r.t. coordinate number 3, then SVF1$(3, f, u)$ is differentiable in z.

(7) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent

(i) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0\rangle$ and SVF1$(1, f, u)$ is differentiable in x_0,

(ii) f is partially differentiable in u w.r.t. coordinate number 1.

(8) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent

(i) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0\rangle$ and SVF1$(2, f, u)$ is differentiable in y_0,

(ii) f is partially differentiable in u w.r.t. coordinate number 2.

(9) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent

(i) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0\rangle$ and SVF1$(3, f, u)$ is differentiable in z_0,

(ii) f is partially differentiable in u w.r.t. coordinate number 3.

(10) Suppose $u = \langle x_0, y_0, z_0\rangle$ and f is partially differentiable in u w.r.t. coordinate number 1. Then there exists a neighbourhood N of x_0 such that $N \subseteq$ dom SVF1$(1, f, u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\text{SVF1}(1, f, u))(x) - (\text{SVF1}(1, f, u))(x_0) = L(x - x_0) + R(x - x_0)$.

(11) Suppose $u = \langle x_0, y_0, z_0\rangle$ and f is partially differentiable in u w.r.t. coordinate number 2. Then there exists a neighbourhood N of y_0 such that $N \subseteq$ dom SVF1$(2, f, u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\text{SVF1}(2, f, u))(y) - (\text{SVF1}(2, f, u))(y_0) = L(y - y_0) + R(y - y_0)$.

(12) Suppose $u = \langle x_0, y_0, z_0 \rangle$ and f is partially differentiable in u w.r.t. coordinate number 3. Then there exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, f, u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, f, u))(z) - (\mathrm{SVF1}(3, f, u))(z_0) = L(z - z_0) + R(z - z_0)$.

(13) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent
  (i) f is partially differentiable in u w.r.t. coordinate number 1,
  (ii) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(1, f, u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, f, u))(x) - (\mathrm{SVF1}(1, f, u))(x_0) = L(x - x_0) + R(x - x_0)$.

(14) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent
  (i) f is partially differentiable in u w.r.t. coordinate number 2,
  (ii) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(2, f, u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, f, u))(y) - (\mathrm{SVF1}(2, f, u))(y_0) = L(y - y_0) + R(y - y_0)$.

(15) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent
  (i) f is partially differentiable in u w.r.t. coordinate number 3,
  (ii) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, f, u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, f, u))(z) - (\mathrm{SVF1}(3, f, u))(z_0) = L(z - z_0) + R(z - z_0)$.

(16) Suppose $u = \langle x_0, y_0, z_0 \rangle$ and f is partially differentiable in u w.r.t. coordinate number 1. Then $r = \operatorname{partdiff}(f, u, 1)$ if and only if there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(1, f, u)$ and there exist L, R such that $r = L(1)$ and for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, f, u))(x) - (\mathrm{SVF1}(1, f, u))(x_0) = L(x - x_0) + R(x - x_0)$.

(17) Suppose $u = \langle x_0, y_0, z_0 \rangle$ and f is partially differentiable in u w.r.t. coordinate number 2. Then $r = \operatorname{partdiff}(f, u, 2)$ if and only if there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(2, f, u)$ and there exist L, R such that $r = L(1)$ and for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, f, u))(y) - (\mathrm{SVF1}(2, f, u))(y_0) = L(y - y_0) + R(y - y_0)$.

(18) Suppose $u = \langle x_0, y_0, z_0 \rangle$ and f is partially differentiable in u w.r.t. coordinate number 3. Then $r = \operatorname{partdiff}(f, u, 3)$ if and only if there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, f, u)$ and there

exist L, R such that $r = L(1)$ and for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, f, u))(z) - (\mathrm{SVF1}(3, f, u))(z_0) = L(z - z_0) + R(z - z_0)$.

(19) If $u = \langle x_0, y_0, z_0 \rangle$, then $\mathrm{partdiff}(f, u, 1) = (\mathrm{SVF1}(1, f, u))'(x_0)$.

(20) If $u = \langle x_0, y_0, z_0 \rangle$, then $\mathrm{partdiff}(f, u, 2) = (\mathrm{SVF1}(2, f, u))'(y_0)$.

(21) If $u = \langle x_0, y_0, z_0 \rangle$, then $\mathrm{partdiff}(f, u, 3) = (\mathrm{SVF1}(3, f, u))'(z_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. We say that f is partially differentiable w.r.t. 1st coordinate on D if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $D \subseteq \mathrm{dom}\, f$, and

(ii) for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partially differentiable in u w.r.t. coordinate number 1.

We say that f is partially differentiable w.r.t. 2nd coordinate on D if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) $D \subseteq \mathrm{dom}\, f$, and

(ii) for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partially differentiable in u w.r.t. coordinate number 2.

We say that f is partially differentiable w.r.t. 3rd coordinate on D if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $D \subseteq \mathrm{dom}\, f$, and

(ii) for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f{\restriction}D$ is partially differentiable in u w.r.t. coordinate number 3.

The following three propositions are true:

(22) Suppose f is partially differentiable w.r.t. 1st coordinate on D. Then $D \subseteq \mathrm{dom}\, f$ and for every u such that $u \in D$ holds f is partially differentiable in u w.r.t. coordinate number 1.

(23) Suppose f is partially differentiable w.r.t. 2nd coordinate on D. Then $D \subseteq \mathrm{dom}\, f$ and for every u such that $u \in D$ holds f is partially differentiable in u w.r.t. coordinate number 2.

(24) Suppose f is partially differentiable w.r.t. 3rd coordinate on D. Then $D \subseteq \mathrm{dom}\, f$ and for every u such that $u \in D$ holds f is partially differentiable in u w.r.t. coordinate number 3.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partially differentiable w.r.t. 1st coordinate on D. The functor $f^{\mathrm{1st}}_{\restriction D}$ yielding a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ is defined as follows:

(Def. 4) $\mathrm{dom}(f^{\mathrm{1st}}_{\restriction D}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f^{\mathrm{1st}}_{\restriction D}(u) = \mathrm{partdiff}(f, u, 1)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partially differentiable w.r.t. 2nd coordinate on D. The functor $f^{\mathrm{2nd}}_{\restriction D}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined as follows:

(Def. 5) $\mathrm{dom}(f^{\mathrm{2nd}}_{\restriction D}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f^{\mathrm{2nd}}_{\restriction D}(u) = \mathrm{partdiff}(f, u, 2)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partially differentiable w.r.t. 3rd coordinate on D. The functor $f^{\mathrm{3rd}}_{\restriction D}$ yielding a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ is defined as follows:

(Def. 6) $\mathrm{dom}(f^{\mathrm{3rd}}_{\restriction D}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f^{\mathrm{3rd}}_{\restriction D}(u) = \mathrm{partdiff}(f, u, 3)$.

3. Main Properties of Partial Differentiation of Real Ternary Functions

We now state a number of propositions:

(25) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\mathrm{proj}(1,3))(u_0)$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 1 and $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(1, f, u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\mathrm{rng}\, c = \{(\mathrm{proj}(1,3))(u_0)\}$ and $\mathrm{rng}(h+c) \subseteq N$. Then $h^{-1}\,(\mathrm{SVF1}(1, f, u_0) \cdot (h + c) - \mathrm{SVF1}(1, f, u_0) \cdot c)$ is convergent and $\mathrm{partdiff}(f, u_0, 1) = \lim(h^{-1}\,(\mathrm{SVF1}(1, f, u_0) \cdot (h+c) - \mathrm{SVF1}(1, f, u_0) \cdot c))$.

(26) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\mathrm{proj}(2,3))(u_0)$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 2 and $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(2, f, u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\mathrm{rng}\, c = \{(\mathrm{proj}(2,3))(u_0)\}$ and $\mathrm{rng}(h+c) \subseteq N$. Then $h^{-1}\,(\mathrm{SVF1}(2, f, u_0) \cdot (h + c) - \mathrm{SVF1}(2, f, u_0) \cdot c)$ is convergent and $\mathrm{partdiff}(f, u_0, 2) = \lim(h^{-1}\,(\mathrm{SVF1}(2, f, u_0) \cdot (h+c) - \mathrm{SVF1}(2, f, u_0) \cdot c))$.

(27) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\mathrm{proj}(3,3))(u_0)$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 3 and $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(3, f, u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\mathrm{rng}\, c = \{(\mathrm{proj}(3,3))(u_0)\}$ and $\mathrm{rng}(h+c) \subseteq N$. Then $h^{-1}\,(\mathrm{SVF1}(3, f, u_0) \cdot (h + c) - \mathrm{SVF1}(3, f, u_0) \cdot c)$ is convergent and $\mathrm{partdiff}(f, u_0, 3) = \lim(h^{-1}\,(\mathrm{SVF1}(3, f, u_0) \cdot (h+c) - \mathrm{SVF1}(3, f, u_0) \cdot c))$.

(28) Suppose that

(i) f_1 is partially differentiable in u_0 w.r.t. coordinate number 1, and

(ii) f_2 is partially differentiable in u_0 w.r.t. coordinate number 1.

Then $f_1\, f_2$ is partially differentiable in u_0 w.r.t. coordinate number 1.

(29) Suppose that

(i) f_1 is partially differentiable in u_0 w.r.t. coordinate number 2, and

(ii) f_2 is partially differentiable in u_0 w.r.t. coordinate number 2.

Then $f_1 \, f_2$ is partially differentiable in u_0 w.r.t. coordinate number 2.

(30) Suppose that
 (i) f_1 is partially differentiable in u_0 w.r.t. coordinate number 3, and
 (ii) f_2 is partially differentiable in u_0 w.r.t. coordinate number 3.

 Then $f_1 \, f_2$ is partially differentiable in u_0 w.r.t. coordinate number 3.

(31) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 1. Then $\mathrm{SVF1}(1, f, u_0)$ is continuous in $(\mathrm{proj}(1,3))(u_0)$.

(32) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 2. Then $\mathrm{SVF1}(2, f, u_0)$ is continuous in $(\mathrm{proj}(2,3))(u_0)$.

(33) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 3. Then $\mathrm{SVF1}(3, f, u_0)$ is continuous in $(\mathrm{proj}(3,3))(u_0)$.

(34) Suppose f is partially differentiable in u_0 w.r.t. coordinate number 1. Then there exists R such that $R(0) = 0$ and R is continuous in 0.

(35) Suppose f is partially differentiable in u_0 w.r.t. coordinate number 2. Then there exists R such that $R(0) = 0$ and R is continuous in 0.

(36) Suppose f is partially differentiable in u_0 w.r.t. coordinate number 3. Then there exists R such that $R(0) = 0$ and R is continuous in 0.

4. Grads and Curl

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let p be an element of $\mathcal{R}^3$. The functor $\mathrm{grad}(f,p)$ yields an element of $\mathcal{R}^3$ and is defined as follows:

(Def. 7) $\mathrm{grad}(f,p) = \mathrm{partdiff}(f,p,1) \cdot e_1 + \mathrm{partdiff}(f,p,2) \cdot e_2 + \mathrm{partdiff}(f,p,3) \cdot e_3$.

We now state several propositions:

(37) $\mathrm{grad}(f,p) = [\mathrm{partdiff}(f,p,1), \mathrm{partdiff}(f,p,2), \mathrm{partdiff}(f,p,3)]$.

(38) Suppose that
 (i) f is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3, and
 (ii) g is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3.

 Then $\mathrm{grad}(f+g,p) = \mathrm{grad}(f,p) + \mathrm{grad}(g,p)$.

(39) Suppose that
 (i) f is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3, and

(ii) g is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3.

Then $\mathrm{grad}(f - g, p) = \mathrm{grad}(f, p) - \mathrm{grad}(g, p)$.

(40) Suppose that

(i) f is partially differentiable in p w.r.t. coordinate number 1,

(ii) f is partially differentiable in p w.r.t. coordinate number 2, and

(iii) f is partially differentiable in p w.r.t. coordinate number 3.

Then $\mathrm{grad}(r\, f, p) = r \cdot \mathrm{grad}(f, p)$.

(41) Suppose that

(i) f is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3, and

(ii) g is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3.

Then $\mathrm{grad}(s\, f + t\, g, p) = s \cdot \mathrm{grad}(f, p) + t \cdot \mathrm{grad}(g, p)$.

(42) Suppose that

(i) f is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3, and

(ii) g is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3.

Then $\mathrm{grad}(s\, f - t\, g, p) = s \cdot \mathrm{grad}(f, p) - t \cdot \mathrm{grad}(g, p)$.

(43) If f is total and constant, then $\mathrm{grad}(f, p) = 0_{\mathcal{E}^3_{\mathrm{T}}}$.

Let a be an element of $\mathcal{R}^3$. The functor unitvector a yields an element of $\mathcal{R}^3$ and is defined as follows:

(Def. 8) $\mathrm{unitvector}\, a = [\frac{a(1)}{\sqrt{a(1)^2 + a(2)^2 + a(3)^2}}, \frac{a(2)}{\sqrt{a(1)^2 + a(2)^2 + a(3)^2}}, \frac{a(3)}{\sqrt{a(1)^2 + a(2)^2 + a(3)^2}}]$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let p, a be elements of $\mathcal{R}^3$. The functor $\mathrm{Directiondiff}(f, p, a)$ yielding a real number is defined by:

(Def. 9) $\mathrm{Directiondiff}(f, p, a) = \mathrm{partdiff}(f, p, 1) \cdot (\mathrm{unitvector}\, a)(1) + \mathrm{partdiff}(f, p, 2) \cdot (\mathrm{unitvector}\, a)(2) + \mathrm{partdiff}(f, p, 3) \cdot (\mathrm{unitvector}\, a)(3)$.

The following propositions are true:

(44) If $a = \langle x_0, y_0, z_0 \rangle$, then $\mathrm{Directiondiff}(f, p, a) = \frac{\mathrm{partdiff}(f,p,1) \cdot x_0}{\sqrt{{x_0}^2 + {y_0}^2 + {z_0}^2}} + \frac{\mathrm{partdiff}(f,p,2) \cdot y_0}{\sqrt{{x_0}^2 + {y_0}^2 + {z_0}^2}} + \frac{\mathrm{partdiff}(f,p,3) \cdot z_0}{\sqrt{{x_0}^2 + {y_0}^2 + {z_0}^2}}$.

(45) $\mathrm{Directiondiff}(f, p, a) = |(\mathrm{grad}(f, p), \mathrm{unitvector}\, a)|$.

Let f_1, f_2, f_3 be partial functions from $\mathcal{R}^3$ to $\mathbb{R}$ and let p be an element of $\mathcal{R}^3$. The functor $\mathrm{curl}(f_1, f_2, f_3, p)$ yields an element of $\mathcal{R}^3$ and is defined by:

(Def. 10) $\mathrm{curl}(f_1, f_2, f_3, p) = (\mathrm{partdiff}(f_3, p, 2) - \mathrm{partdiff}(f_2, p, 3)) \cdot e_1 + (\mathrm{partdiff}(f_1, p, 3) - \mathrm{partdiff}(f_3, p, 1)) \cdot e_2 + (\mathrm{partdiff}(f_2, p, 1) - \mathrm{partdiff}(f_1, p, 2)) \cdot e_3$.

Next we state the proposition

(46) $\mathrm{curl}(f_1, f_2, f_3, p) = [\mathrm{partdiff}(f_3, p, 2) - \mathrm{partdiff}(f_2, p, 3), \mathrm{partdiff}(f_1, p, 3) - \mathrm{partdiff}(f_3, p, 1), \mathrm{partdiff}(f_2, p, 1) - \mathrm{partdiff}(f_1, p, 2)]$.

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Fixpoint Theorem for Continuous Functions on Chain-Complete Posets

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Summary. This text includes the definition of chain-complete poset, fixpoint theorem on it, and the definition of the function space of continuous functions on chain-complete posets [10].

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The papers [8], [4], [5], [3], [1], [9], [7], [11], [13], [12], [2], [14], and [6] provide the notation and terminology for this paper.

1. Monotone Functions, Chain and Chain-complete Posets

Let P be a non empty poset. Observe that there exists a chain of P which is non empty.

Let I_1 be a relational structure. We say that I_1 is chain-complete if and only if:

(Def. 1) I_1 is lower-bounded and for every chain L of I_1 such that L is non empty holds sup L exists in I_1.

One can prove the following proposition

(1) Let P_1, P_2 be non empty posets, K be a non empty chain of P_1, and h be a monotone function from P_1 into P_2. Then $h^\circ K$ is a non empty chain of P_2.

Let us note that there exists a poset which is strict, chain-complete, and non empty.

Let us mention that every relational structure which is chain-complete is also lower-bounded.

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For simplicity, we adopt the following rules: x, y denote sets, P, Q denote strict chain-complete non empty posets, L denotes a non empty chain of P, M denotes a non empty chain of Q, p denotes an element of P, f denotes a monotone function from P into Q, and g, g_1, g_2 denote monotone functions from P into P.

We now state the proposition

(2) $\sup(f^\circ L) \leq f(\sup L)$.

2. Fixpoint Theorem for Continuous Functions on Chain-complete Posets

Let P be a non empty poset, let g be a monotone function from P into P, and let p be an element of P. The functor $\mathrm{iterSet}(g, p)$ yields a non empty set and is defined by:

(Def. 2) $\mathrm{iterSet}(g, p) = \{x \in P\colon \bigvee_{n\,:\,\text{natural number}} x = g^n(p)\}$.

Next we state the proposition

(3) $\mathrm{iterSet}(g, \bot_P)$ is a non empty chain of P.

Let us consider P and let g be a monotone function from P into P. The functor iter-min g yields a non empty chain of P and is defined by:

(Def. 3) $\text{iter-min}\, g = \mathrm{iterSet}(g, \bot_P)$.

The following propositions are true:

(4) $\sup \text{iter-min}\, g = \sup(g^\circ\, \text{iter-min}\, g)$.

(5) If $g_1 \leq g_2$, then $\sup \text{iter-min}\, g_1 \leq \sup \text{iter-min}\, g_2$.

Let P, Q be non empty posets and let f be a function from P into Q. We say that f is continuous if and only if:

(Def. 4) f is monotone and for every non empty chain L of P holds f preserves sup of L.

We now state two propositions:

(6) For every function f from P into Q holds f is continuous iff f is monotone and for every L holds $f(\sup L) = \sup(f^\circ L)$.

(7) For every element z of Q holds $P \longmapsto z$ is continuous.

Let us consider P, Q. Observe that there exists a function from P into Q which is continuous.

Let us consider P, Q. One can verify that every function from P into Q which is continuous is also monotone.

The following proposition is true

(8) For every monotone function f from P into Q such that for every L holds $f(\sup L) \leq \sup(f^\circ L)$ holds f is continuous.

Let us consider P and let g be a monotone function from P into P. Let us assume that g is continuous. The least fixpoint of g yields an element of P and is defined by the conditions (Def. 5).

(Def. 5)(i) The least fixpoint of g is a fixpoint of g, and

(ii) for every p such that p is a fixpoint of g holds the least fixpoint of $g \leq p$.

One can prove the following propositions:

(9) For every continuous function g from P into P holds the least fixpoint of $g = \sup \text{iter-min}\, g$.

(10) Let g_1, g_2 be continuous functions from P into P. If $g_1 \leq g_2$, then the least fixpoint of $g_1 \leq$ the least fixpoint of g_2.

3. FUNCTION SPACE OF CONTINUOUS FUNCTIONS ON CHAIN-COMPLETE POSETS

Let us consider P, Q. The functor $\text{ConFuncs}(P, Q)$ yields a non empty set and is defined by the condition (Def. 6).

(Def. 6) $\text{ConFuncs}(P, Q) = \{x; x$ ranges over elements of (the carrier of $Q)^{\text{the carrier of } P}$: $\bigvee_{f\colon \text{continuous function from } P \text{ into } Q}\ f = x\}$.

Let us consider P, Q. The functor $\text{ConRelat}(P, Q)$ yielding a binary relation on $\text{ConFuncs}(P, Q)$ is defined by the condition (Def. 7).

(Def. 7) Let given x, y. Then $\langle x, y\rangle \in \text{ConRelat}(P, Q)$ if and only if the following conditions are satisfied:

(i) $x \in \text{ConFuncs}(P, Q)$,

(ii) $y \in \text{ConFuncs}(P, Q)$, and

(iii) there exist functions f, g from P into Q such that $x = f$ and $y = g$ and $f \leq g$.

Let us consider P, Q. One can verify the following observations:

* $\text{ConRelat}(P, Q)$ is reflexive,
* $\text{ConRelat}(P, Q)$ is transitive, and
* $\text{ConRelat}(P, Q)$ is antisymmetric.

Let us consider P, Q. The functor $\text{ConPoset}(P, Q)$ yielding a strict non empty poset is defined as follows:

(Def. 8) $\text{ConPoset}(P, Q) = \langle \text{ConFuncs}(P, Q), \text{ConRelat}(P, Q)\rangle$.

In the sequel F is a non empty chain of $\text{ConPoset}(P, Q)$.

Let us consider P, Q, F, p. The functor F-image(p) yielding a non empty chain of Q is defined as follows:

(Def. 9) $F\text{-image}(p) = \{x \in Q\colon \bigvee_{f\colon \text{continuous function from } P \text{ into } Q} (f \in F \wedge x = f(p))\}$.

Let us consider P, Q, F. The functor sup-func F yields a function from P into Q and is defined as follows:

(Def. 10) For all p, M such that $M = F$-image(p) holds (sup-func $F)(p) = \sup M$.

Let us consider P, Q, F. One can check that sup-func F is continuous.

The following proposition is true

(11) Sup F exists in ConPoset(P, Q) and sup-func $F = \bigsqcup_{\mathrm{ConPoset}(P,Q)} F$.

Let us consider P, Q. The functor min-func(P, Q) yielding a function from P into Q is defined as follows:

(Def. 11) min-func$(P, Q) = P \longmapsto \perp_Q$.

Let us consider P, Q. One can check that min-func(P, Q) is continuous.

The following proposition is true

(12) For every element f of ConPoset(P, Q) such that $f =$ min-func(P, Q) holds $f \leq$ the carrier of ConPoset(P, Q).

Let us consider P, Q. Note that ConPoset(P, Q) is chain-complete.

4. Continuity of Fixpoint Function from ConPoset(P, P) into P

Let us consider P. The functor fix-func P yielding a function from ConPoset(P, P) into P is defined by the condition (Def. 12).

(Def. 12) Let g be an element of ConPoset(P, P) and h be a continuous function from P into P. If $g = h$, then (fix-func $P)(g) =$ the least fixpoint of h.

Let us consider P. One can check that fix-func P is continuous.

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Nilpotent Groups

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Summary. This article describes the concept of the nilpotent group and some properties of the nilpotent groups.

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The papers [2], [3], [4], [6], [7], [5], [8], [9], [10], and [1] provide the terminology and notation for this paper.

For simplicity, we adopt the following convention: x denotes a set, G denotes a group, A, B, H, H_1, H_2 denote subgroups of G, a, b, c denote elements of G, F denotes a finite sequence of elements of the carrier of G, and i, j denote elements of $\mathbb{N}$.

One can prove the following propositions:

(1) $a^b = a \cdot [a, b]$.

(2) $[a, b]^{-1} = [a, b^{-1}]^b$.

(3) $[a, b]^{-1} = [a^{-1}, b]^a$.

(4) $([a, b^{-1}]^b)^{-1} = [b^{-1}, a]^b$.

(5) $[a,\ b^{-1},\ c]^b = [[a, b^{-1}]^b, c^b]$.

(6) $[a, b^{-1}]^b = [b, a]$.

(7) $[a,\ b^{-1},\ c]^b = [b,\ a,\ c^b]$.

(8) $[a,\ b,\ c^a] \cdot [c,\ a,\ b^c] \cdot [b,\ c,\ a^b] = \mathbf{1}_G$.

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(9) $[A, B]$ is a subgroup of $[B, A]$.

(10) $[A, B] = [B, A]$.

Let us consider G, A, B. Let us note that the functor $[A, B]$ is commutative.

One can prove the following propositions:

(11) If B is a subgroup of A, then the commutators of A & $B \subseteq \overline{A}$.

(12) If B is a subgroup of A, then $[A, B]$ is a subgroup of A.

(13) If B is a subgroup of A, then $[B, A]$ is a subgroup of A.

(14) If $[H_1, \Omega_G]$ is a subgroup of H_2, then $[H_1 \cap H, H]$ is a subgroup of $H_2 \cap H$.

(15) $[H_1, H_2]$ is a subgroup of $[H_1, \Omega_G]$.

(16) A is a normal subgroup of G iff $[A, \Omega_G]$ is a subgroup of A.

Let us consider G. The normal subgroups of G yields a set and is defined by:

(Def. 1) $x \in$ the normal subgroups of G iff x is a strict normal subgroup of G.

Let us consider G. One can verify that the normal subgroups of G is non empty.

Next we state three propositions:

(17) Let F be a finite sequence of elements of the normal subgroups of G and given j. If $j \in \operatorname{dom} F$, then $F(j)$ is a strict normal subgroup of G.

(18) The normal subgroups of $G \subseteq \operatorname{SubGr} G$.

(19) Every finite sequence of elements of the normal subgroups of G is a finite sequence of elements of $\operatorname{SubGr} G$.

Let I_1 be a group. We say that I_1 is nilpotent if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exists a finite sequence F of elements of the normal subgroups of I_1 such that

(i) $\operatorname{len} F > 0$,

(ii) $F(1) = \Omega_{(I_1)}$,

(iii) $F(\operatorname{len} F) = \{\mathbf{1}\}_{(I_1)}$, and

(iv) for every i such that i, $i+1 \in \operatorname{dom} F$ and for all strict normal subgroups G_1, G_2 of I_1 such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a subgroup of G_1 and ${}^{G_1}/_{(G_2)_{(G_1)}}$ is a subgroup of $\mathrm{Z}({}^{I_1}/_{G_2})$.

Let us note that there exists a group which is nilpotent and strict.

We now state four propositions:

(20) Let G_1 be a subgroup of G and N be a strict normal subgroup of G. Suppose N is a subgroup of G_1 and ${}^{G_1}/_{(N)_{(G_1)}}$ is a subgroup of $\mathrm{Z}({}^{G}/_{N})$. Then $[G_1, \Omega_G]$ is a subgroup of N.

(21) Let G_1 be a subgroup of G and N be a normal subgroup of G. Suppose N is a strict subgroup of G_1 and $[G_1, \Omega_G]$ is a strict subgroup of N. Then ${}^{G_1}/_{(N)_{(G_1)}}$ is a subgroup of $\mathrm{Z}({}^{G}/_{N})$.

(22) Let G be a group. Then G is nilpotent if and only if there exists a finite sequence F of elements of the normal subgroups of G such that $\operatorname{len} F > 0$ and $F(1) = \Omega_G$ and $F(\operatorname{len} F) = \{\mathbf{1}\}_G$ and for every i such that $i, i+1 \in \operatorname{dom} F$ and for all strict normal subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a subgroup of G_1 and $[G_1, \Omega_G]$ is a subgroup of G_2.

(23) Let G be a group, H, G_1 be subgroups of G, G_2 be a strict normal subgroup of G, H_1 be a subgroup of H, and H_2 be a normal subgroup of H. Suppose G_2 is a subgroup of G_1 and ${}^{G_1}/_{(G_2)_{(G_1)}}$ is a subgroup of $\mathrm{Z}({}^{G}/_{G_2})$ and $H_1 = G_1 \cap H$ and $H_2 = G_2 \cap H$. Then ${}^{H_1}/_{(H_2)_{(H_1)}}$ is a subgroup of $\mathrm{Z}({}^{H}/_{H_2})$.

Let G be a nilpotent group. Note that every subgroup of G is nilpotent.

Let us mention that every group which is commutative is also nilpotent and every group which is cyclic is also nilpotent.

We now state four propositions:

(24) Let G, H be strict groups, h be a homomorphism from G to H, A be a strict subgroup of G, and a, b be elements of G. Then $h(a) \cdot h(b) \cdot h^{\circ}A = h^{\circ}(a \cdot b \cdot A)$ and $h^{\circ}A \cdot h(a) \cdot h(b) = h^{\circ}(A \cdot a \cdot b)$.

(25) Let G, H be strict groups, h be a homomorphism from G to H, A be a strict subgroup of G, a, b be elements of G, H_1 be a subgroup of $\operatorname{Im} h$, and a_1, b_1 be elements of $\operatorname{Im} h$. If $a_1 = h(a)$ and $b_1 = h(b)$ and $H_1 = h^{\circ}A$, then $a_1 \cdot b_1 \cdot H_1 = h(a) \cdot h(b) \cdot h^{\circ}A$.

(26) Let G, H be strict groups, h be a homomorphism from G to H, G_1 be a strict subgroup of G, G_2 be a strict normal subgroup of G, H_1 be a strict subgroup of $\operatorname{Im} h$, and H_2 be a strict normal subgroup of $\operatorname{Im} h$. Suppose G_2 is a strict subgroup of G_1 and ${}^{G_1}/_{(G_2)_{(G_1)}}$ is a subgroup of $\mathrm{Z}({}^{G}/_{G_2})$ and $H_1 = h^{\circ}G_1$ and $H_2 = h^{\circ}G_2$. Then ${}^{H_1}/_{(H_2)_{(H_1)}}$ is a subgroup of $\mathrm{Z}({}^{\operatorname{Im} h}/_{H_2})$.

(27) Let G, H be strict groups, h be a homomorphism from G to H, and A be a strict normal subgroup of G. Then $h^{\circ}A$ is a strict normal subgroup of $\operatorname{Im} h$.

Let G be a strict nilpotent group, let H be a strict group, and let h be a homomorphism from G to H. One can check that $\operatorname{Im} h$ is nilpotent.

Let G be a strict nilpotent group and let N be a strict normal subgroup of G. Note that ${}^{G}/_{N}$ is nilpotent.

One can prove the following three propositions:

(28) Let G be a group. Given a finite sequence F of elements of the normal subgroups of G such that

(i) $\operatorname{len} F > 0$,

(ii) $F(1) = \Omega_G$,

(iii) $F(\operatorname{len} F) = \{\mathbf{1}\}_G$, and

(iv) for every i such that i, $i+1 \in \operatorname{dom} F$ and for every strict normal subgroup G_1 of G such that $G_1 = F(i)$ holds $[G_1, \Omega_G] = F(i+1)$.
Then G is nilpotent.

(29) Let G be a group. Given a finite sequence F of elements of the normal subgroups of G such that
(i) $\operatorname{len} F > 0$,
(ii) $F(1) = \Omega_G$,
(iii) $F(\operatorname{len} F) = \{\mathbf{1}\}_G$, and
(iv) for every i such that i, $i+1 \in \operatorname{dom} F$ and for all strict normal subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a subgroup of G_1 and ${}^{G}/_{G_2}$ is a commutative group.
Then G is nilpotent.

(30) Let G be a group. Given a finite sequence F of elements of the normal subgroups of G such that
(i) $\operatorname{len} F > 0$,
(ii) $F(1) = \Omega_G$,
(iii) $F(\operatorname{len} F) = \{\mathbf{1}\}_G$, and
(iv) for every i such that i, $i+1 \in \operatorname{dom} F$ and for all strict normal subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a subgroup of G_1 and ${}^{G}/_{G_2}$ is a cyclic group.
Then G is nilpotent.

Let us mention that every group which is nilpotent is also solvable.

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Difference and Difference Quotient. Part III

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Summary. In this article, we give some important theorems of forward difference, backward difference, central difference and difference quotient and forward difference, backward difference, central difference and difference quotient formulas of some special functions.

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The terminology and notation used in this paper have been introduced in the following papers: [6], [2], [1], [4], [11], [7], [5], [8], [12], [9], [10], and [3].

We follow the rules: n, m are elements of $\mathbb{N}$, h, k, r, r_1, r_2, x, x_0, x_1, x_2, x_3 are real numbers, and f, f_1, f_2 are functions from $\mathbb{R}$ into $\mathbb{R}$.

Next we state a number of propositions:

(1) $(\delta_h[f])(x) = (\Delta_{\frac{h}{2}}[f])(x) - (\Delta_{-\frac{h}{2}}[f])(x)$.

(2) $(\Delta_{-\frac{h}{2}}[f])(x) = -(\nabla_{\frac{h}{2}}[f])(x)$.

(3) $(\delta_h[f])(x) = (\nabla_{\frac{h}{2}}[f])(x) - (\nabla_{-\frac{h}{2}}[f])(x)$.

(4) $(\vec{\Delta}_h[r\, f_1 + f_2])(n+1)(x) = r \cdot (\vec{\Delta}_h[f_1])(n+1)(x) + (\vec{\Delta}_h[f_2])(n+1)(x)$.

(5) $(\vec{\Delta}_h[f_1 + r\, f_2])(n+1)(x) = (\vec{\Delta}_h[f_1])(n+1)(x) + r \cdot (\vec{\Delta}_h[f_2])(n+1)(x)$.

(6) $(\vec{\Delta}_h[r_1\, f_1 - r_2\, f_2])(n+1)(x) = r_1 \cdot (\vec{\Delta}_h[f_1])(n+1)(x) - r_2 \cdot (\vec{\Delta}_h[f_2])(n+1)(x)$.

(7) $(\vec{\Delta}_h[f])(1) = \Delta_h[f]$.

(8) $(\vec{\nabla}_h[r\, f_1 + f_2])(n+1)(x) = r \cdot (\vec{\nabla}_h[f_1])(n+1)(x) + (\vec{\nabla}_h[f_2])(n+1)(x)$.

(9) $(\vec{\nabla}_h[f_1 + r\, f_2])(n+1)(x) = (\vec{\nabla}_h[f_1])(n+1)(x) + r \cdot (\vec{\nabla}_h[f_2])(n+1)(x)$.

(10) $(\vec{\nabla}_h[r_1\, f_1 - r_2\, f_2])(n+1)(x) = r_1 \cdot (\vec{\nabla}_h[f_1])(n+1)(x) - r_2 \cdot (\vec{\nabla}_h[f_2])(n+1)(x)$.

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(11) $(\vec{\nabla}_h[f])(1) = \nabla_h[f]$.

(12) $(\vec{\nabla}_h[(\vec{\nabla}_h[f])(m)])(n)(x) = (\vec{\nabla}_h[f])(m+n)(x)$.

(13) $(\vec{\delta}_h[r\,f_1 + f_2])(n+1)(x) = r \cdot (\vec{\delta}_h[f_1])(n+1)(x) + (\vec{\delta}_h[f_2])(n+1)(x)$.

(14) $(\vec{\delta}_h[f_1 + r\,f_2])(n+1)(x) = (\vec{\delta}_h[f_1])(n+1)(x) + r \cdot (\vec{\delta}_h[f_2])(n+1)(x)$.

(15) $(\vec{\delta}_h[r_1\,f_1 - r_2\,f_2])(n+1)(x) = r_1 \cdot (\vec{\delta}_h[f_1])(n+1)(x) - r_2 \cdot (\vec{\delta}_h[f_2])(n+1)(x)$.

(16) $(\vec{\delta}_h[f])(1) = \delta_h[f]$.

(17) $(\vec{\delta}_h[(\vec{\delta}_h[f])(m)])(n)(x) = (\vec{\delta}_h[f])(m+n)(x)$.

(18) If $(\vec{\Delta}_h[f])(n)(x) = (\vec{\delta}_h[f])(n)(x + \frac{n}{2} \cdot h)$, then $(\vec{\nabla}_h[f])(n)(x) = (\vec{\delta}_h[f])(n)(x - \frac{n}{2} \cdot h)$.

(19) If $(\vec{\Delta}_h[f])(n)(x) = (\vec{\delta}_h[f])(n)(x + \frac{n-1}{2} \cdot h + \frac{h}{2})$, then $(\vec{\nabla}_h[f])(n)(x) = (\vec{\delta}_h[f])(n)(x - \frac{n-1}{2} \cdot h - \frac{h}{2})$.

(20) $\Delta[f](x, x+h) = \frac{(\Delta_h[f])(x)}{h}$.

(21) $\Delta[f](x-h, x) = \frac{(\nabla_h[f])(x)}{h}$.

(22) $\Delta[f](x - \frac{h}{2}, x + \frac{h}{2}) = \frac{(\delta_h[f])(x)}{h}$.

(23) $\Delta[f](x - \frac{h}{2}, x + \frac{h}{2}) = \frac{(\vec{\delta}_h[f])(1)(x)}{h}$.

(24) If $h \neq 0$, then $\Delta[f](x-h, x, x+h) = \frac{(\vec{\delta}_h[f])(2)(x)}{2 \cdot h \cdot h}$.

(25) $\Delta[f_1 - f_2](x_0, x_1) = \Delta[f_1](x_0, x_1) - \Delta[f_2](x_0, x_1)$.

(26) $\Delta[r\,f_1 + f_2](x_0, x_1) = r \cdot \Delta[f_1](x_0, x_1) + \Delta[f_2](x_0, x_1)$.

(27) $\Delta[r\,f_1 - f_2](x_0, x_1) = r \cdot \Delta[f_1](x_0, x_1) - \Delta[f_2](x_0, x_1)$.

(28) $\Delta[f_1 + r\,f_2](x_0, x_1) = \Delta[f_1](x_0, x_1) + r \cdot \Delta[f_2](x_0, x_1)$.

(29) $\Delta[f_1 - r\,f_2](x_0, x_1) = \Delta[f_1](x_0, x_1) - r \cdot \Delta[f_2](x_0, x_1)$.

(30) $\Delta[r_1\,f_1 - r_2\,f_2](x_0, x_1) = r_1 \cdot \Delta[f_1](x_0, x_1) - r_2 \cdot \Delta[f_2](x_0, x_1)$.

(31) $(\vec{\nabla}_h[f_1\,f_2])(1)(x) = f_1(x) \cdot (\vec{\nabla}_h[f_2])(1)(x) + f_2(x-h) \cdot (\vec{\nabla}_h[f_1])(1)(x)$.

(32) If x_0, x_1, x_2 are mutually different, then $\Delta[f](x_0, x_1, x_2) = \Delta[f](x_0, x_2, x_1)$.

In the sequel S is a sequence of real sequences.

We now state a number of propositions:

(33) Suppose that for all natural numbers n, i such that $i \leq n$ holds $S(n)(i) = \binom{n}{i} \cdot (\vec{\nabla}_h[f_1])(i)(x) \cdot (\vec{\nabla}_h[f_2])(n -' i)(x - i \cdot h)$. Then $(\vec{\nabla}_h[f_1\,f_2])(1)(x) = \sum_{\kappa=0}^{1} S(1)(\kappa)$ and $(\vec{\nabla}_h[f_1\,f_2])(2)(x) = \sum_{\kappa=0}^{2} S(2)(\kappa)$.

(34) $(\vec{\delta}_h[f_1\,f_2])(1)(x) = f_1(x + \frac{h}{2}) \cdot (\vec{\delta}_h[f_2])(1)(x) + f_2(x - \frac{h}{2}) \cdot (\vec{\delta}_h[f_1])(1)(x)$.

(35) Suppose that for all natural numbers n, i such that $i \leq n$ holds $S(n)(i) = \binom{n}{i} \cdot (\vec{\delta}_h[f_1])(i)(x + (n -' i) \cdot \frac{h}{2}) \cdot (\vec{\delta}_h[f_2])(n -' i)(x - i \cdot \frac{h}{2})$. Then $(\vec{\delta}_h[f_1\,f_2])(1)(x) = \sum_{\kappa=0}^{1} S(1)(\kappa)$ and $(\vec{\delta}_h[f_1\,f_2])(2)(x) = \sum_{\kappa=0}^{2} S(2)(\kappa)$.

(36) If for every x holds $f(x) = \sqrt{x}$ and $x_0 \neq x_1$ and $x_0 > 0$ and $x_1 > 0$, then $\Delta[f](x_0, x_1) = \frac{1}{\sqrt{x_0} + \sqrt{x_1}}$.

(37) Suppose for every x holds $f(x) = \sqrt{x}$ and x_0, x_1, x_2 are mutually different and $x_0 > 0$ and $x_1 > 0$ and $x_2 > 0$. Then $\Delta[f](x_0, x_1, x_2) = -\frac{1}{(\sqrt{x_0}+\sqrt{x_1})\cdot(\sqrt{x_0}+\sqrt{x_2})\cdot(\sqrt{x_1}+\sqrt{x_2})}$.

(38) Suppose for every x holds $f(x) = \sqrt{x}$ and x_0, x_1, x_2, x_3 are mutually different and $x_0 > 0$ and $x_1 > 0$ and $x_2 > 0$ and $x_3 > 0$. Then $\Delta[f](x_0, x_1, x_2, x_3) = \frac{\sqrt{x_0}+\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}}{(\sqrt{x_0}+\sqrt{x_1})\cdot(\sqrt{x_0}+\sqrt{x_2})\cdot(\sqrt{x_0}+\sqrt{x_3})\cdot(\sqrt{x_1}+\sqrt{x_2})\cdot(\sqrt{x_1}+\sqrt{x_3})\cdot(\sqrt{x_2}+\sqrt{x_3})}$.

(39) If for every x holds $f(x) = \sqrt{x}$ and $x > 0$ and $x + h > 0$, then $(\Delta_h[f])(x) = \sqrt{x+h} - \sqrt{x}$.

(40) If for every x holds $f(x) = \sqrt{x}$ and $x > 0$ and $x - h > 0$, then $(\nabla_h[f])(x) = \sqrt{x} - \sqrt{x-h}$.

(41) If for every x holds $f(x) = \sqrt{x}$ and $x + \frac{h}{2} > 0$ and $x - \frac{h}{2} > 0$, then $(\delta_h[f])(x) = \sqrt{x+\frac{h}{2}} - \sqrt{x-\frac{h}{2}}$.

(42) If for every x holds $f(x) = x^{\mathbf{2}}$ and $x_0 \neq x_1$, then $\Delta[f](x_0, x_1) = x_0 + x_1$.

(43) If for every x holds $f(x) = x^{\mathbf{2}}$ and x_0, x_1, x_2 are mutually different, then $\Delta[f](x_0, x_1, x_2) = 1$.

(44) If for every x holds $f(x) = x^{\mathbf{2}}$ and x_0, x_1, x_2, x_3 are mutually different, then $\Delta[f](x_0, x_1, x_2, x_3) = 0$.

(45) If for every x holds $f(x) = x^{\mathbf{2}}$, then $(\Delta_h[f])(x) = 2 \cdot x \cdot h + h^{\mathbf{2}}$.

(46) If for every x holds $f(x) = x^{\mathbf{2}}$, then $(\nabla_h[f])(x) = h \cdot (2 \cdot x - h)$.

(47) If for every x holds $f(x) = x^{\mathbf{2}}$, then $(\delta_h[f])(x) = 2 \cdot h \cdot x$.

(48) If for every x holds $f(x) = \frac{k}{x^{\mathbf{2}}}$ and $x_0 \neq x_1$ and $x_0 \neq 0$ and $x_1 \neq 0$, then $\Delta[f](x_0, x_1) = -\frac{k}{x_0 \cdot x_1} \cdot (\frac{1}{x_0} + \frac{1}{x_1})$.

(49) Suppose for every x holds $f(x) = \frac{k}{x^{\mathbf{2}}}$ and $x_0 \neq 0$ and $x_1 \neq 0$ and $x_2 \neq 0$ and x_0, x_1, x_2 are mutually different. Then $\Delta[f](x_0, x_1, x_2) = \frac{k}{x_0 \cdot x_1 \cdot x_2} \cdot (\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2})$.

(50) If for every x holds $f(x) = \frac{k}{x^{\mathbf{2}}}$ and $x \neq 0$ and $x+h \neq 0$, then $(\Delta_h[f])(x) = \frac{(-k)\cdot h\cdot(2\cdot x+h)}{(x^{\mathbf{2}}+h\cdot x)^{\mathbf{2}}}$.

(51) If for every x holds $f(x) = \frac{k}{x^{\mathbf{2}}}$ and $x \neq 0$ and $x-h \neq 0$, then $(\nabla_h[f])(x) = \frac{(-k)\cdot h\cdot(2\cdot x-h)}{(x^{\mathbf{2}}-x\cdot h)^{\mathbf{2}}}$.

(52) If for every x holds $f(x) = \frac{k}{x^{\mathbf{2}}}$ and $x + \frac{h}{2} \neq 0$ and $x - \frac{h}{2} \neq 0$, then $(\delta_h[f])(x) = \frac{-2\cdot h\cdot k\cdot x}{(x^{\mathbf{2}}-(\frac{h}{2})^{\mathbf{2}})^{\mathbf{2}}}$.

(53) $\Delta[(\text{the function sin})\ (\text{the function sin})\ (\text{the function sin})](x_0, x_1) = \frac{\frac{1}{2}\cdot(3\cdot\cos(\frac{x_0+x_1}{2})\cdot\sin(\frac{x_0-x_1}{2})-\cos(\frac{3\cdot(x_0+x_1)}{2})\cdot\sin(\frac{3\cdot(x_0-x_1)}{2}))}{x_0-x_1}$.

(54) $(\Delta_h[(\text{the function sin})\ (\text{the function sin})\ (\text{the function sin})])(x) = \frac{1}{2} \cdot (3 \cdot \cos(\frac{2\cdot x+h}{2}) \cdot \sin(\frac{h}{2}) - \cos(\frac{3\cdot(2\cdot x+h)}{2}) \cdot \sin(\frac{3\cdot h}{2}))$.

(55) $(\nabla_h[(\text{the function sin})\ (\text{the function sin})\ (\text{the function sin})])(x) = \frac{1}{2} \cdot (3 \cdot \cos(\frac{2\cdot x-h}{2}) \cdot \sin(\frac{h}{2}) - \cos(\frac{3\cdot(2\cdot x-h)}{2}) \cdot \sin(\frac{3\cdot h}{2}))$.

(56) $(\delta_h[(\text{the function sin})\ (\text{the function sin})\ (\text{the function sin})])(x) = \frac{1}{2} \cdot (3 \cdot \cos x \cdot \sin(\frac{h}{2}) - \cos(3 \cdot x) \cdot \sin(\frac{3\cdot h}{2}))$.

(57) $\Delta[(\text{the function cos})\ (\text{the function cos})\ (\text{the function cos})](x_0, x_1) = -\frac{\frac{1}{2}\cdot(3\cdot\sin(\frac{x_0+x_1}{2})\cdot\sin(\frac{x_0-x_1}{2})+\sin(\frac{3\cdot x_0+3\cdot x_1}{2})\cdot\sin(\frac{3\cdot x_0-3\cdot x_1}{2}))}{x_0-x_1}$.

(58) $(\Delta_h[(\text{the function cos})\ (\text{the function cos})\ (\text{the function cos})])(x) = -\frac{1}{2} \cdot (3 \cdot \sin(\frac{2\cdot x+h}{2}) \cdot \sin(\frac{h}{2}) + \sin(\frac{3\cdot(2\cdot x+h)}{2}) \cdot \sin(\frac{3\cdot h}{2}))$.

(59) $(\nabla_h[(\text{the function cos})\ (\text{the function cos})\ (\text{the function cos})])(x) = -\frac{1}{2} \cdot (3 \cdot \sin(\frac{2\cdot x-h}{2}) \cdot \sin(\frac{h}{2}) + \sin(\frac{3\cdot(2\cdot x-h)}{2}) \cdot \sin(\frac{3\cdot h}{2}))$.

(60) $(\delta_h[(\text{the function cos})\ (\text{the function cos})\ (\text{the function cos})])(x) = -\frac{1}{2} \cdot (3 \cdot \sin x \cdot \sin(\frac{h}{2}) + \sin(3 \cdot x) \cdot \sin(\frac{3\cdot h}{2}))$.

(61) If for every x holds $f(x) = \frac{1}{\sin x}$ and $\sin x_0 \neq 0$ and $\sin x_1 \neq 0$, then $\Delta[f](x_0, x_1) = -\frac{\frac{2\cdot(\sin x_1-\sin x_0)}{\cos(x_0+x_1)-\cos(x_0-x_1)}}{x_0-x_1}$.

(62) If for every x holds $f(x) = \frac{1}{\sin x}$ and $\sin x \neq 0$ and $\sin(x + h) \neq 0$, then $(\Delta_h[f])(x) = -\frac{2\cdot(\sin x-\sin(x+h))}{\cos(2\cdot x+h)-\cos h}$.

(63) If for every x holds $f(x) = \frac{1}{\sin x}$ and $\sin x \neq 0$ and $\sin(x - h) \neq 0$, then $(\nabla_h[f])(x) = \frac{(-2)\cdot(\sin(x-h)-\sin x)}{\cos(2\cdot x-h)-\cos h}$.

(64) If for every x holds $f(x) = \frac{1}{\sin x}$ and $\sin(x + \frac{h}{2}) \neq 0$ and $\sin(x - \frac{h}{2}) \neq 0$, then $(\delta_h[f])(x) = -\frac{2\cdot(\sin(x-\frac{h}{2})-\sin(x+\frac{h}{2}))}{\cos(2\cdot x)-\cos h}$.

(65) If for every x holds $f(x) = \frac{1}{\cos x}$ and $x_0 \neq x_1$ and $\cos x_0 \neq 0$ and $\cos x_1 \neq 0$, then $\Delta[f](x_0, x_1) = \frac{\frac{2\cdot(\cos x_1-\cos x_0)}{\cos(x_0+x_1)+\cos(x_0-x_1)}}{x_0-x_1}$.

(66) If for every x holds $f(x) = \frac{1}{\cos x}$ and $\cos x \neq 0$ and $\cos(x + h) \neq 0$, then $(\Delta_h[f])(x) = \frac{2\cdot(\cos x-\cos(x+h))}{\cos(2\cdot x+h)+\cos h}$.

(67) If for every x holds $f(x) = \frac{1}{\cos x}$ and $\cos x \neq 0$ and $\cos(x - h) \neq 0$, then $(\nabla_h[f])(x) = \frac{2\cdot(\cos(x-h)-\cos x)}{\cos(2\cdot x-h)+\cos h}$.

(68) If for every x holds $f(x) = \frac{1}{\cos x}$ and $\cos(x + \frac{h}{2}) \neq 0$ and $\cos(x - \frac{h}{2}) \neq 0$, then $(\delta_h[f])(x) = \frac{2\cdot(\cos(x-\frac{h}{2})-\cos(x+\frac{h}{2}))}{\cos(2\cdot x)+\cos h}$.

(69) Suppose for every x holds $f(x) = \frac{1}{(\sin x)^{\mathbf{2}}}$ and $x_0 \neq x_1$ and $\sin x_0 \neq 0$ and $\sin x_1 \neq 0$. Then $\Delta[f](x_0, x_1) = \frac{16\cdot\cos(\frac{x_1+x_0}{2})\cdot\sin(\frac{x_1-x_0}{2})\cdot\cos(\frac{x_1-x_0}{2})\cdot\sin(\frac{x_1+x_0}{2})}{(\cos(x_0+x_1)-\cos(x_0-x_1))^{\mathbf{2}}\cdot(x_0-x_1)}$.

(70) If for every x holds $f(x) = \frac{1}{(\sin x)^{\mathbf{2}}}$ and $\sin x \neq 0$ and $\sin(x+h) \neq 0$, then $(\Delta_h[f])(x) = \frac{16\cdot\cos(\frac{2\cdot x+h}{2})\cdot\sin(\frac{-h}{2})\cdot\cos(\frac{-h}{2})\cdot\sin(\frac{2\cdot x+h}{2})}{(\cos(2\cdot x+h)-\cos h)^{\mathbf{2}}}$.

(71) If for every x holds $f(x) = \frac{1}{(\sin x)^{\mathbf{2}}}$ and $\sin x \neq 0$ and $\sin(x-h) \neq 0$, then

$(\nabla_h[f])(x) = \frac{16\cdot\cos(\frac{2\cdot x-h}{2})\cdot\sin(\frac{-h}{2})\cdot\cos(\frac{-h}{2})\cdot\sin(\frac{2\cdot x-h}{2})}{(\cos(2\cdot x-h)-\cos h)^{\mathbf{2}}}$.

(72) If for every x holds $f(x) = \frac{1}{(\sin x)^{\mathbf{2}}}$ and $\sin(x+\frac{h}{2}) \neq 0$ and $\sin(x-\frac{h}{2}) \neq 0$, then $(\delta_h[f])(x) = \frac{16\cdot\cos x\cdot\sin(\frac{-h}{2})\cdot\cos(\frac{-h}{2})\cdot\sin x}{(\cos(2\cdot x)-\cos h)^{\mathbf{2}}}$.

(73) Suppose for every x holds $f(x) = \frac{1}{(\cos x)^{\mathbf{2}}}$ and $x_0 \neq x_1$ and $\cos x_0 \neq 0$ and $\cos x_1 \neq 0$. Then $\Delta[f](x_0, x_1) = \frac{\frac{(-16)\cdot\sin(\frac{x_1+x_0}{2})\cdot\sin(\frac{x_1-x_0}{2})\cdot\cos(\frac{x_1+x_0}{2})\cdot\cos(\frac{x_1-x_0}{2})}{(\cos(x_0+x_1)+\cos(x_0-x_1))^{\mathbf{2}}}}{x_0-x_1}$.

(74) If for every x holds $f(x) = \frac{1}{(\cos x)^{\mathbf{2}}}$ and $\cos x \neq 0$ and $\cos(x+h) \neq 0$, then $(\Delta_h[f])(x) = \frac{(-16)\cdot\sin(\frac{2\cdot x+h}{2})\cdot\sin(\frac{-h}{2})\cdot\cos(\frac{2\cdot x+h}{2})\cdot\cos(\frac{-h}{2})}{(\cos(2\cdot x+h)+\cos h)^{\mathbf{2}}}$.

(75) If for every x holds $f(x) = \frac{1}{(\cos x)^{\mathbf{2}}}$ and $\cos x \neq 0$ and $\cos(x-h) \neq 0$, then $(\nabla_h[f])(x) = \frac{(-16)\cdot\sin(\frac{2\cdot x-h}{2})\cdot\sin(\frac{-h}{2})\cdot\cos(\frac{2\cdot x-h}{2})\cdot\cos(\frac{-h}{2})}{(\cos(2\cdot x-h)+\cos h)^{\mathbf{2}}}$.

(76) If for every x holds $f(x) = \frac{1}{(\cos x)^{\mathbf{2}}}$ and $\cos(x+\frac{h}{2}) \neq 0$ and $\cos(x-\frac{h}{2}) \neq 0$, then $(\delta_h[f])(x) = \frac{(-16)\cdot\sin x\cdot\sin(\frac{-h}{2})\cdot\cos x\cdot\cos(\frac{-h}{2})}{(\cos(2\cdot x)+\cos h)^{\mathbf{2}}}$.

(77) Suppose $x_0 \in \operatorname{dom}$(the function tan) and $x_1 \in \operatorname{dom}$(the function tan). Then Δ[(the function tan) (the function sin)]$(x_0, x_1) = \frac{(\frac{1}{\cos x_0}-\cos x_0-\frac{1}{\cos x_1})+\cos x_1}{x_0-x_1}$.

(78) Suppose that

(i) for every x holds $f(x) =$ ((the function tan) (the function sin))(x),

(ii) $x \in \operatorname{dom}$(the function tan), and

(iii) $x + h \in \operatorname{dom}$(the function tan).

Then $(\Delta_h[f])(x) = (\frac{1}{\cos(x+h)} - \cos(x+h) - \frac{1}{\cos x}) + \cos x$.

(79) Suppose that

(i) for every x holds $f(x) =$ ((the function tan) (the function sin))(x),

(ii) $x \in \operatorname{dom}$(the function tan), and

(iii) $x - h \in \operatorname{dom}$(the function tan).

Then $(\nabla_h[f])(x) = (\frac{1}{\cos x} - \cos x - \frac{1}{\cos(x-h)}) + \cos(x-h)$.

(80) Suppose that

(i) for every x holds $f(x) =$ ((the function tan) (the function sin))(x),

(ii) $x + \frac{h}{2} \in \operatorname{dom}$(the function tan), and

(iii) $x - \frac{h}{2} \in \operatorname{dom}$(the function tan).

Then $(\delta_h[f])(x) = (\frac{1}{\cos(x+\frac{h}{2})} - \cos(x+\frac{h}{2}) - \frac{1}{\cos(x-\frac{h}{2})}) + \cos(x-\frac{h}{2})$.

(81) Suppose for every x holds $f(x) =$ ((the function tan) (the function cos))(x) and $x_0 \in \operatorname{dom}$(the function tan) and $x_1 \in \operatorname{dom}$(the function tan). Then $\Delta[f](x_0, x_1) = \frac{\sin x_0-\sin x_1}{x_0-x_1}$.

(82) Suppose that

(i) for every x holds $f(x) =$ ((the function tan) (the function cos))(x),

(ii) $x \in \operatorname{dom}$(the function tan), and

(iii) $x+h \in \mathrm{dom}$ (the function tan).
Then $(\Delta_h[f])(x) = \sin(x+h) - \sin x$.

(83) Suppose that
(i) for every x holds $f(x) =$ ((the function tan) (the function cos))(x),
(ii) $x \in \mathrm{dom}$ (the function tan), and
(iii) $x-h \in \mathrm{dom}$ (the function tan).
Then $(\nabla_h[f])(x) = \sin x - \sin(x-h)$.

(84) Suppose that
(i) for every x holds $f(x) =$ ((the function tan) (the function cos))(x),
(ii) $x+\frac{h}{2} \in \mathrm{dom}$ (the function tan), and
(iii) $x-\frac{h}{2} \in \mathrm{dom}$ (the function tan).
Then $(\delta_h[f])(x) = \sin(x+\frac{h}{2}) - \sin(x-\frac{h}{2})$.

(85) Suppose for every x holds $f(x) =$ ((the function cot) (the function cos))(x) and $x_0 \in \mathrm{dom}$ (the function cot) and $x_1 \in \mathrm{dom}$ (the function cot). Then $\Delta[f](x_0, x_1) = \frac{(\frac{1}{\sin x_0} - \sin x_0 - \frac{1}{\sin x_1}) + \sin x_1}{x_0 - x_1}$.

(86) Suppose that
(i) for every x holds $f(x) =$ ((the function cot) (the function cos))(x),
(ii) $x \in \mathrm{dom}$ (the function cot), and
(iii) $x+h \in \mathrm{dom}$ (the function cot).
Then $(\Delta_h[f])(x) = (\frac{1}{\sin(x+h)} - \sin(x+h) - \frac{1}{\sin x}) + \sin x$.

(87) Suppose that
(i) for every x holds $f(x) =$ ((the function cot) (the function cos))(x),
(ii) $x \in \mathrm{dom}$ (the function cot), and
(iii) $x-h \in \mathrm{dom}$ (the function cot).
Then $(\nabla_h[f])(x) = (\frac{1}{\sin x} - \sin x - \frac{1}{\sin(x-h)}) + \sin(x-h)$.

(88) Suppose that
(i) for every x holds $f(x) =$ ((the function cot) (the function cos))(x),
(ii) $x+\frac{h}{2} \in \mathrm{dom}$ (the function cot), and
(iii) $x-\frac{h}{2} \in \mathrm{dom}$ (the function cot).
Then $(\delta_h[f])(x) = (\frac{1}{\sin(x+\frac{h}{2})} - \sin(x+\frac{h}{2}) - \frac{1}{\sin(x-\frac{h}{2})}) + \sin(x-\frac{h}{2})$.

(89) Suppose for every x holds $f(x) =$ ((the function cot) (the function sin))(x) and $x_0 \in \mathrm{dom}$ (the function cot) and $x_1 \in \mathrm{dom}$ (the function cot). Then $\Delta[f](x_0, x_1) = \frac{\cos x_0 - \cos x_1}{x_0 - x_1}$.

(90) Suppose that
(i) for every x holds $f(x) =$ ((the function cot) (the function sin))(x),
(ii) $x \in \mathrm{dom}$ (the function cot), and
(iii) $x+h \in \mathrm{dom}$ (the function cot).
Then $(\Delta_h[f])(x) = \cos(x+h) - \cos x$.

(91) Suppose that
(i) for every x holds $f(x) =$ ((the function cot) (the function sin))(x),

(ii) $x \in \mathrm{dom}\,(\text{the function cot})$, and
(iii) $x - h \in \mathrm{dom}\,(\text{the function cot})$.
Then $(\nabla_h[f])(x) = \cos x - \cos(x - h)$.

(92) Suppose that
(i) for every x holds $f(x) = ((\text{the function cot})\ (\text{the function sin}))(x)$,
(ii) $x + \frac{h}{2} \in \mathrm{dom}\,(\text{the function cot})$, and
(iii) $x - \frac{h}{2} \in \mathrm{dom}\,(\text{the function cot})$.
Then $(\delta_h[f])(x) = \cos(x + \frac{h}{2}) - \cos(x - \frac{h}{2})$.

(93) Suppose for every x holds $f(x) = ((\text{the function tan})\ (\text{the function tan}))(x)$ and $x_0 \in \mathrm{dom}\,(\text{the function tan})$ and $x_1 \in \mathrm{dom}\,(\text{the function tan})$. Then $\Delta[f](x_0, x_1) = \frac{(\cos x_1)^{\mathbf{2}} - (\cos x_0)^{\mathbf{2}}}{(\cos x_0 \cdot \cos x_1)^{\mathbf{2}} \cdot (x_0 - x_1)}$.

(94) Suppose that
(i) for every x holds $f(x) = ((\text{the function tan})\ (\text{the function tan}))(x)$,
(ii) $x \in \mathrm{dom}\,(\text{the function tan})$, and
(iii) $x + h \in \mathrm{dom}\,(\text{the function tan})$.
Then $(\Delta_h[f])(x) = -\frac{\frac{1}{2} \cdot (\cos(2 \cdot (x+h)) - \cos(2 \cdot x))}{(\cos(x+h) \cdot \cos x)^{\mathbf{2}}}$.

(95) Suppose that
(i) for every x holds $f(x) = ((\text{the function tan})\ (\text{the function tan}))(x)$,
(ii) $x \in \mathrm{dom}\,(\text{the function tan})$, and
(iii) $x - h \in \mathrm{dom}\,(\text{the function tan})$.
Then $(\nabla_h[f])(x) = -\frac{\frac{1}{2} \cdot (\cos(2 \cdot x) - \cos(2 \cdot (h - x)))}{(\cos x \cdot \cos(x-h))^{\mathbf{2}}}$.

(96) Suppose that
(i) for every x holds $f(x) = ((\text{the function tan})\ (\text{the function tan}))(x)$,
(ii) $x + \frac{h}{2} \in \mathrm{dom}\,(\text{the function tan})$, and
(iii) $x - \frac{h}{2} \in \mathrm{dom}\,(\text{the function tan})$.
Then $(\delta_h[f])(x) = -\frac{\frac{1}{2} \cdot (\cos(h + 2 \cdot x) - \cos(h - 2 \cdot x))}{(\cos(x + \frac{h}{2}) \cdot \cos(x - \frac{h}{2}))^{\mathbf{2}}}$.

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A Model of Mizar Concepts – Unification

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Summary. The aim of this paper is to develop a formal theory of Mizar linguistic concepts following the ideas from [6] and [7]. The theory presented is an abstraction from the existing implementation of the Mizar system and is devoted to the formalization of Mizar expressions. The concepts formalized here are: standarized constructor signature, arity-rich signatures, and the unification of Mizar expressions.

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The notation and terminology used in this paper are introduced in the following articles: [20], [21], [12], [22], [10], [14], [13], [17], [18], [15], [1], [8], [11], [2], [3], [4], [19], [16], [5], [9], and [7]. For simplicity the abbreviation $\mathfrak{M}$ = MaxConstrSign is introduced.

1. Preliminary

In this paper i, j denote natural numbers.

Next we state two propositions:

(1) For every pair set x holds $x = \langle x_{\mathbf{1}}, x_{\mathbf{2}} \rangle$.

(2) For every infinite set X there exist sets x_1, x_2 such that $x_1, x_2 \in X$ and $x_1 \neq x_2$.

In this article we present several logical schemes. The scheme *MinimalElement* deals with a finite non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

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There exists a set x such that $x \in \mathcal{A}$ and for every set y such that $y \in \mathcal{A}$ holds not $\mathcal{P}[y, x]$

provided the parameters have the following properties:

- For all sets x, y such that x, $y \in \mathcal{A}$ and $\mathcal{P}[x, y]$ holds not $\mathcal{P}[y, x]$, and
- For all sets x, y, z such that x, y, $z \in \mathcal{A}$ and $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

The scheme *FiniteC* deals with a finite set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{A}]$

provided the following condition is satisfied:

- For every subset A of $\mathcal{A}$ such that for every set B such that $B \subset A$ holds $\mathcal{P}[B]$ holds $\mathcal{P}[A]$.

The scheme *Numeration* deals with a finite set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists an one-to-one finite sequence s such that $\operatorname{rng} s = \mathcal{A}$ and for all i, j such that i, $j \in \operatorname{dom} s$ and $\mathcal{P}[s(i), s(j)]$ holds $i < j$

provided the parameters satisfy the following conditions:

- For all sets x, y such that x, $y \in \mathcal{A}$ and $\mathcal{P}[x, y]$ holds not $\mathcal{P}[y, x]$, and
- For all sets x, y, z such that x, y, $z \in \mathcal{A}$ and $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

One can prove the following two propositions:

(3) For every variable x holds $\operatorname{varcl} \operatorname{vars}(x) = \operatorname{vars}(x)$.

(4) Let $\mathfrak{C}$ be an initialized constructor signature and e be an expression of $\mathfrak{C}$. Then e is compound if and only if it is not true that there exists an element x of Vars such that $e = x_{\mathfrak{C}}$.

2. Standardized Constructor Signature

Let us note that there exists a quasi-locus sequence which is empty.

Let $\mathfrak{C}$ be a constructor signature. We say that $\mathfrak{C}$ is standardized if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let o be an operation symbol of $\mathfrak{C}$. Suppose o is constructor. Then $o \in$ Constructors and $o_{\mathbf{1}} =$ the result sort of o and $\operatorname{Card}((o_{\mathbf{2}})_{\mathbf{1}}) = \operatorname{len} \operatorname{Arity}(o)$.

The following proposition is true

(5) Let $\mathfrak{C}$ be a constructor signature. Suppose $\mathfrak{C}$ is standardized. Let o be an operation symbol of $\mathfrak{C}$. Then o is constructor if and only if $o \in$ Constructors .

Let us note that $\mathfrak{M}$ is standardized.

Let us observe that there exists a constructor signature which is initialized, standardized, and strict.

Let $\mathfrak{C}$ be an initialized standardized constructor signature and let c be a constructor operation symbol of $\mathfrak{C}$. The loci of c yielding a quasi-locus sequence is defined by:

(Def. 2) The loci of $c = (c_{\mathbf{2}})_{\mathbf{1}}$.

Let $\mathfrak{C}$ be a constructor signature. One can verify that there exists a subsignature of $\mathfrak{C}$ which is constructor.

Let $\mathfrak{C}$ be an initialized constructor signature. Note that there exists a constructor subsignature of $\mathfrak{C}$ which is initialized.

Let $\mathfrak{C}$ be a standardized constructor signature. One can verify that every constructor subsignature of $\mathfrak{C}$ is standardized.

One can prove the following two propositions:

(6) Let S_1, S_2 be standardized constructor signatures. Suppose the operation symbols of S_1 = the operation symbols of S_2. Then the many sorted signature of S_1 = the many sorted signature of S_2.

(7) For every constructor signature $\mathfrak{C}$ holds $\mathfrak{C}$ is standardized iff $\mathfrak{C}$ is a subsignature of $\mathfrak{M}$.

Let $\mathfrak{C}$ be an initialized constructor signature. Observe that there exists a quasi-term of $\mathfrak{C}$ which is non compound.

Let us mention that every element of Vars is pair.

The following propositions are true:

(8) For every element x of Vars such that $\mathrm{vars}(x)$ is natural holds $\mathrm{vars}(x) = 0$.

(9) Vars misses Constructors.

(10) For every element x of Vars holds $x \neq *$ and $x \neq$ **non**.

(11) For every standardized constructor signature $\mathfrak{C}$ holds Vars misses the operation symbols of $\mathfrak{C}$.

(12) Let $\mathfrak{C}$ be an initialized standardized constructor signature and e be an expression of $\mathfrak{C}$. Then

(i) there exists an element x of Vars such that $e = x_{\mathfrak{C}}$ and $e(\emptyset) = \langle x,$ **term**$\rangle$, or

(ii) there exists an operation symbol o of $\mathfrak{C}$ such that $e(\emptyset) = \langle o,$ the carrier of $\mathfrak{C}\rangle$ but $o \in$ Constructors or $o = *$ or $o =$ **non**.

Let $\mathfrak{C}$ be an initialized standardized constructor signature and let e be an expression of $\mathfrak{C}$. Note that $e(\emptyset)$ is pair.

The following propositions are true:

(13) Let $\mathfrak{C}$ be an initialized constructor signature, e be an expression of $\mathfrak{C}$, and o be an operation symbol of $\mathfrak{C}$. Suppose $e(\emptyset) = \langle o,$ the carrier of $\mathfrak{C}\rangle$. Then e is an expression of $\mathfrak{C}$ from the result sort of o.

(14) Let $\mathfrak{C}$ be an initialized standardized constructor signature and e be an expression of $\mathfrak{C}$. Then
  (i) if $e(\emptyset)_{\mathbf{1}} = *$, then e is an expression of $\mathfrak{C}$ from $\mathbf{type}_{\mathfrak{C}}$, and
  (ii) if $e(\emptyset)_{\mathbf{1}} = \mathbf{non}$, then e is an expression of $\mathfrak{C}$ from $\mathbf{adj}_{\mathfrak{C}}$.

(15) Let $\mathfrak{C}$ be an initialized standardized constructor signature and e be an expression of $\mathfrak{C}$. Then
  (i) $e(\emptyset)_{\mathbf{1}} \in$ Vars and $e(\emptyset)_{\mathbf{2}} = \mathbf{term}$ and e is a quasi-term of $\mathfrak{C}$, or
  (ii) $e(\emptyset)_{\mathbf{2}} =$ the carrier of $\mathfrak{C}$ but $e(\emptyset)_{\mathbf{1}} \in$ Constructors and $e(\emptyset)_{\mathbf{1}} \in$ the operation symbols of $\mathfrak{C}$ or $e(\emptyset)_{\mathbf{1}} = *$ or $e(\emptyset)_{\mathbf{1}} = \mathbf{non}$.

(16) Let $\mathfrak{C}$ be an initialized standardized constructor signature and e be an expression of $\mathfrak{C}$. If $e(\emptyset)_{\mathbf{1}} \in$ Constructors, then $e \in$ (the sorts of $\mathrm{Free}_{\mathfrak{C}}(\mathrm{Vars}\,\mathfrak{C}))((e(\emptyset)_{\mathbf{1}})_{\mathbf{1}})$.

(17) Let $\mathfrak{C}$ be an initialized standardized constructor signature and e be an expression of $\mathfrak{C}$. Then $e(\emptyset)_{\mathbf{1}} \notin$ Vars if and only if $e(\emptyset)_{\mathbf{1}}$ is an operation symbol of $\mathfrak{C}$.

(18) Let $\mathfrak{C}$ be an initialized standardized constructor signature and e be an expression of $\mathfrak{C}$. If $e(\emptyset)_{\mathbf{1}} \in$ Vars, then there exists an element x of Vars such that $x = e(\emptyset)_{\mathbf{1}}$ and $e = x_{\mathfrak{C}}$.

(19) Let $\mathfrak{C}$ be an initialized standardized constructor signature and e be an expression of $\mathfrak{C}$. Suppose $e(\emptyset)_{\mathbf{1}} = *$. Then there exists an expression α of $\mathfrak{C}$ from $\mathbf{adj}_{\mathfrak{C}}$ and there exists an expression q of $\mathfrak{C}$ from $\mathbf{type}_{\mathfrak{C}}$ such that $e = \langle *, 3\rangle$-tree(α, q).

(20) Let $\mathfrak{C}$ be an initialized standardized constructor signature and e be an expression of $\mathfrak{C}$. If $e(\emptyset)_{\mathbf{1}} = \mathbf{non}$, then there exists an expression α of $\mathfrak{C}$ from $\mathbf{adj}_{\mathfrak{C}}$ such that $e = \langle \mathbf{non}, 3\rangle$-tree$(\alpha)$.

(21) Let $\mathfrak{C}$ be an initialized standardized constructor signature and e be an expression of $\mathfrak{C}$. Suppose $e(\emptyset)_{\mathbf{1}} \in$ Constructors. Then there exists an operation symbol o of $\mathfrak{C}$ such that $o = e(\emptyset)_{\mathbf{1}}$ and the result sort of $o = o_{\mathbf{1}}$ and e is an expression of $\mathfrak{C}$ from the result sort of o.

(22) Let $\mathfrak{C}$ be an initialized standardized constructor signature and τ be a quasi-term of $\mathfrak{C}$. Then τ is compound if and only if $\tau(\emptyset)_{\mathbf{1}} \in$ Constructors and $(\tau(\emptyset)_{\mathbf{1}})_{\mathbf{1}} = \mathbf{term}$.

(23) Let $\mathfrak{C}$ be an initialized standardized constructor signature and τ be an expression of $\mathfrak{C}$. Then τ is a non compound quasi-term of $\mathfrak{C}$ if and only if $\tau(\emptyset)_{\mathbf{1}} \in$ Vars.

(24) Let $\mathfrak{C}$ be an initialized standardized constructor signature and τ be an expression of $\mathfrak{C}$. Then τ is a quasi-term of $\mathfrak{C}$ if and only if $\tau(\emptyset)_{\mathbf{1}} \in$ Constructors and $(\tau(\emptyset)_{\mathbf{1}})_{\mathbf{1}} = \mathbf{term}$ or $\tau(\emptyset)_{\mathbf{1}} \in$ Vars.

(25) Let $\mathfrak{C}$ be an initialized standardized constructor signature and α be an expression of $\mathfrak{C}$. Then α is a positive quasi-adjective of $\mathfrak{C}$ if and only if

$\alpha(\emptyset)_{\mathbf{1}} \in$ Constructors and $(\alpha(\emptyset)_{\mathbf{1}})_{\mathbf{1}} = \mathbf{adj}$.

(26) Let $\mathfrak{C}$ be an initialized standardized constructor signature and α be a quasi-adjective of $\mathfrak{C}$. Then α is negative if and only if $\alpha(\emptyset)_{\mathbf{1}} = \mathbf{non}$.

(27) Let $\mathfrak{C}$ be an initialized standardized constructor signature and τ be an expression of $\mathfrak{C}$. Then τ is a pure expression of $\mathfrak{C}$ from $\mathbf{type}_{\mathfrak{C}}$ if and only if $\tau(\emptyset)_{\mathbf{1}} \in$ Constructors and $(\tau(\emptyset)_{\mathbf{1}})_{\mathbf{1}} = \mathbf{type}$.

3. Expressions

In the sequel i is a natural number, x is a variable, and ℓ is a quasi-locus sequence.

An expression is an expression of $\mathfrak{M}$. A valuation is a valuation of $\mathfrak{M}$. A quasi-adjective is a quasi-adjective of $\mathfrak{M}$. The subset QuasiAdjs of $\mathrm{Free}_{\mathfrak{M}}(\mathrm{Vars}\,\mathfrak{M})$ is defined as follows:

(Def. 3) QuasiAdjs = QuasiAdjs $\mathfrak{M}$.

A quasi-term is a quasi-term of $\mathfrak{M}$. The subset QuasiTerms of $\mathrm{Free}_{\mathfrak{M}}(\mathrm{Vars}\,\mathfrak{M})$ is defined as follows:

(Def. 4) QuasiTerms = QuasiTerms $\mathfrak{M}$.

A quasi-type is a quasi-type of $\mathfrak{M}$. The functor QuasiTypes is defined as follows:

(Def. 5) QuasiTypes = QuasiTypes $\mathfrak{M}$.

One can verify the following observations:

* QuasiAdjs is non empty,
* QuasiTerms is non empty, and
* QuasiTypes is non empty.

Modes is a non empty subset of Constructors. Then Attrs is a non empty subset of Constructors. Then Funcs is a non empty subset of Constructors.

In the sequel $\mathfrak{C}$ denotes an initialized constructor signature.

The element set-constr of Modes is defined by:

(Def. 6) set-constr $= \langle \mathbf{type}, \langle \emptyset, 0 \rangle \rangle$.

One can prove the following propositions:

(28) The kind of set-constr $= \mathbf{type}$ and the loci of set-constr $= \emptyset$ and the index of set-constr $= 0$.

(29) Constructors $= \{\mathbf{type}, \mathbf{adj}, \mathbf{term}\} \times (\mathrm{QuasiLoci} \times \mathbb{N})$.

(30) $\langle \mathrm{rng}\,\ell, i \rangle \in$ Vars and $\ell \frown \langle \langle \mathrm{rng}\,\ell, i \rangle \rangle$ is a quasi-locus sequence.

(31) There exists ℓ such that $\mathrm{len}\,\ell = i$.

(32) For every finite subset X of Vars there exists ℓ such that $\mathrm{rng}\,\ell = \mathrm{varcl}\,X$.

(33) Let X, o be sets and p be a decorated tree yielding finite sequence. Given $\mathfrak{C}$ such that $X = \bigcup$(the sorts of $\mathrm{Free}_{\mathfrak{C}}(\mathrm{Vars}\,\mathfrak{C})$). If o-tree$(p) \in X$, then p is a finite sequence of elements of X.

Let us consider $\mathfrak{C}$ and let e be an expression of $\mathfrak{C}$. An expression of $\mathfrak{C}$ is called a subexpression of e if:

(Def. 7) It $\in$ Subtrees(e).

The functor constrs e is defined by:

(Def. 8) $\text{constrs}\, e = \pi_1(\text{rng}\, e) \cap \{o : o$ ranges over constructor operation symbols of $\mathfrak{C}\}$.

The functor main-constr e is defined by:

(Def. 9) $\text{main-constr}\, e = \begin{cases} e(\emptyset)_{\mathbf{1}}, & \text{if } e \text{ is compound,} \\ \emptyset, & \text{otherwise.} \end{cases}$

The functor args e yields a finite sequence of elements of $\text{Free}_{\mathfrak{C}}(\text{Vars}\,\mathfrak{C})$ and is defined by:

(Def. 10) $e = e(\emptyset)\text{-tree}(\text{args}\, e)$.

Next we state three propositions:

(34) For every $\mathfrak{C}$ holds every expression e of $\mathfrak{C}$ is a subexpression of e.

(35) $\text{main-constr}(x_{\mathfrak{C}}) = \emptyset$.

(36) Let c be a constructor operation symbol of $\mathfrak{C}$ and p be a finite sequence of elements of QuasiTerms $\mathfrak{C}$. If $\text{len}\, p = \text{len}\,\text{Arity}(c)$, then $\text{main-constr}(c^{\rightarrow}(p)) = c$.

Let us consider $\mathfrak{C}$ and let e be an expression of $\mathfrak{C}$. We say that e is constructor if and only if:

(Def. 11) e is compound and main-constr e is a constructor operation symbol of $\mathfrak{C}$.

Let us consider $\mathfrak{C}$. Observe that every expression of $\mathfrak{C}$ which is constructor is also compound.

Let us consider $\mathfrak{C}$. Observe that there exists an expression of $\mathfrak{C}$ which is constructor.

Let us consider $\mathfrak{C}$ and let e be a constructor expression of $\mathfrak{C}$. One can verify that there exists a subexpression of e which is constructor.

Let S be a non void signature, let X be a non empty yielding many sorted set indexed by S, and let τ be an element of $\text{Free}_S(X)$. Observe that rng τ is relation-like.

One can prove the following proposition

(37) For every constructor expression e of $\mathfrak{C}$ holds main-constr $e \in$ constrs e.

4. Arity

For simplicity, we follow the rules: α is a quasi-adjective, τ, τ_1, τ_2 are quasi-terms, ϑ is a quasi-type, and c is an element of Constructors.

Let $\mathfrak{C}$ be a non void signature. We say that $\mathfrak{C}$ is arity-rich if and only if the condition (Def. 12) is satisfied.

(Def. 12) Let n be a natural number and s be a sort symbol of $\mathfrak{C}$. Then $\{o; o$ ranges over operation symbols of $\mathfrak{C}$: the result sort of $o = s \ \wedge\ \operatorname{len}\operatorname{Arity}(o) = n\}$ is infinite.

Let o be an operation symbol of $\mathfrak{C}$. We say that o is nullary if and only if:

(Def. 13) $\operatorname{Arity}(o) = \emptyset$.

We say that o is unary if and only if:

(Def. 14) $\operatorname{len}\operatorname{Arity}(o) = 1$.

We say that o is binary if and only if:

(Def. 15) $\operatorname{len}\operatorname{Arity}(o) = 2$.

The following proposition is true

(38) Let $\mathfrak{C}$ be a non void signature and o be an operation symbol of $\mathfrak{C}$. Then
- (i) if o is nullary, then o is not unary,
- (ii) if o is nullary, then o is not binary, and
- (iii) if o is unary, then o is not binary.

Let $\mathfrak{C}$ be a constructor signature. Observe that $\mathbf{non}_{\mathfrak{C}}$ is unary and $*_{\mathfrak{C}}$ is binary.

Let $\mathfrak{C}$ be a constructor signature. Note that every operation symbol of $\mathfrak{C}$ which is nullary is also constructor.

The following proposition is true

(39) Let $\mathfrak{C}$ be a constructor signature. Then $\mathfrak{C}$ is initialized if and only if there exists an operation symbol m of $\mathbf{type}_{\mathfrak{C}}$ and there exists an operation symbol α of $\mathbf{adj}_{\mathfrak{C}}$ such that m is nullary and α is nullary.

Let $\mathfrak{C}$ be an initialized constructor signature. One can verify that there exists an operation symbol of $\mathbf{type}_{\mathfrak{C}}$ which is nullary and constructor and there exists an operation symbol of $\mathbf{adj}_{\mathfrak{C}}$ which is nullary and constructor.

Let $\mathfrak{C}$ be an initialized constructor signature. Observe that there exists an operation symbol of $\mathfrak{C}$ which is nullary and constructor.

One can check that every non void signature which is arity-rich has also an operation for each sort and every constructor signature which is arity-rich is also initialized.

One can check that $\mathfrak{M}$ is arity-rich.

Let us mention that there exists a constructor signature which is arity-rich and initialized.

Let $\mathfrak{C}$ be an arity-rich constructor signature and let s be a sort symbol of $\mathfrak{C}$. One can verify the following observations:

* there exists an operation symbol of s which is nullary and constructor,
* there exists an operation symbol of s which is unary and constructor, and
* there exists an operation symbol of s which is binary and constructor.

Let $\mathfrak{C}$ be an arity-rich constructor signature. One can check that there exists an operation symbol of $\mathfrak{C}$ which is unary and constructor and there exists an operation symbol of $\mathfrak{C}$ which is binary and constructor.

The following proposition is true

(40) Let o be a nullary operation symbol of $\mathfrak{C}$. Then $\langle o$, the carrier of $\mathfrak{C}\rangle$-tree$(\emptyset)$ is an expression of $\mathfrak{C}$ from the result sort of o.

Let $\mathfrak{C}$ be an initialized constructor signature and let m be a nullary constructor operation symbol of $\mathbf{type}_{\mathfrak{C}}$. Then $m_{\rm t}$ is a pure expression of $\mathfrak{C}$ from $\mathbf{type}_{\mathfrak{C}}$.

Let c be an element of Constructors. The functor ${}^{@}c$ yielding a constructor operation symbol of $\mathfrak{M}$ is defined by:

(Def. 16) ${}^{@}c = c$.

Let m be an element of Modes. Then ${}^{@}m$ is a constructor operation symbol of $\mathbf{type}_{\mathfrak{M}}$.

Let us note that ${}^{@}$set-constr is nullary.

We now state the proposition

(41) Arity$({}^{@}$set-constr$) = \emptyset$.

The quasi-type set-type is defined by:

(Def. 17) set-type $= \emptyset_{\mathrm{QuasiAdjs}\,\mathfrak{M}} * ({}^{@}\text{set-constr})_{\rm t}$.

The following proposition is true

(42) adjs set-type $= \emptyset$ and the base of set-type $= ({}^{@}\text{set-constr})_{\rm t}$.

Let ℓ be a finite sequence of elements of Vars. The functor args ℓ yields a finite sequence of elements of QuasiTerms $\mathfrak{M}$ and is defined as follows:

(Def. 18) len args ℓ = len ℓ and for every i such that $i \in \operatorname{dom} \ell$ holds $(\operatorname{args} \ell)(i) = (\ell_i)_{\mathfrak{M}}$.

Let us consider c. The base expression of c yields an expression and is defined as follows:

(Def. 19) The base expression of $c = ({}^{@}c)^{\neg}$(args (the loci of c)).

Next we state several propositions:

(43) For every operation symbol o of $\mathfrak{M}$ holds o is constructor iff $o \in$ Constructors.

(44) For every nullary operation symbol m of $\mathfrak{M}$ holds main-constr$(m_{\rm t}) = m$.

(45) For every unary constructor operation symbol m of $\mathfrak{M}$ and for every τ holds main-constr$(m(\tau)) = m$.

(46) For every α holds main-constr$(\mathbf{non}_{\mathfrak{M}}(\alpha)) = \mathbf{non}$.

(47) For every binary constructor operation symbol m of $\mathfrak{M}$ and for all τ_1, τ_2 holds main-constr$(m(\tau_1, \tau_2)) = m$.

(48) For every expression q of $\mathfrak{M}$ from $\mathbf{type}_{\mathfrak{M}}$ and for every α holds main-constr$(*_{\mathfrak{M}}(\alpha, q)) = *$.

Let ϑ be a quasi-type. The functor $\operatorname{constrs}\vartheta$ is defined by:

(Def. 20) $\operatorname{constrs}\vartheta = \operatorname{constrs}(\text{the base of } \vartheta) \cup \bigcup\{\operatorname{constrs}\alpha : \alpha \in \operatorname{adjs}\vartheta\}$.

The following two propositions are true:

(49) For every pure expression q of $\mathfrak{M}$ from $\mathbf{type}_{\mathfrak{M}}$ and for every finite subset A of QuasiAdjs $\mathfrak{M}$ holds $\operatorname{constrs}(A * q) = \operatorname{constrs} q \cup \bigcup\{\operatorname{constrs}\alpha : \alpha \in A\}$.

(50) $\operatorname{constrs}(\alpha * \vartheta) = \operatorname{constrs}\alpha \cup \operatorname{constrs}\vartheta$.

5. Unification

Let $\mathfrak{C}$ be an initialized constructor signature and let τ, p be expressions of $\mathfrak{C}$. We say that τ matches p if and only if:

(Def. 21) There exists a valuation f of $\mathfrak{C}$ such that $\tau = p[f]$.

Let us note that the predicate τ matches p is reflexive.

The following proposition is true

(51) For all expressions τ_1, τ_2, τ_3 of $\mathfrak{C}$ such that τ_1 matches τ_2 and τ_2 matches τ_3 holds τ_1 matches τ_3.

Let $\mathfrak{C}$ be an initialized constructor signature and let A, B be subsets of QuasiAdjs $\mathfrak{C}$. We say that A matches B if and only if:

(Def. 22) There exists a valuation f of $\mathfrak{C}$ such that $B[f] \subseteq A$.

Let us note that the predicate A matches B is reflexive.

The following proposition is true

(52) For all subsets A_1, A_2, A_3 of QuasiAdjs $\mathfrak{C}$ such that A_1 matches A_2 and A_2 matches A_3 holds A_1 matches A_3.

Let $\mathfrak{C}$ be an initialized constructor signature and let ϑ, P be quasi-types of $\mathfrak{C}$. We say that ϑ matches P if and only if:

(Def. 23) There exists a valuation f of $\mathfrak{C}$ such that $(\operatorname{adjs} P)[f] \subseteq \operatorname{adjs}\vartheta$ and (the base of $P)[f] =$ the base of ϑ.

Let us note that the predicate ϑ matches P is reflexive.

One can prove the following proposition

(53) For all quasi-types ϑ_1, ϑ_2, ϑ_3 of $\mathfrak{C}$ such that ϑ_1 matches ϑ_2 and ϑ_2 matches ϑ_3 holds ϑ_1 matches ϑ_3.

Let $\mathfrak{C}$ be an initialized constructor signature, let τ_1, τ_2 be expressions of $\mathfrak{C}$, and let f be a valuation of $\mathfrak{C}$. We say that f unifies τ_1 with τ_2 if and only if:

(Def. 24) $\tau_1[f] = \tau_2[f]$.

The following proposition is true

(54) Let τ_1, τ_2 be expressions of $\mathfrak{C}$ and f be a valuation of $\mathfrak{C}$. If f unifies τ_1 with τ_2, then f unifies τ_2 with τ_1.

Let $\mathfrak{C}$ be an initialized constructor signature and let τ_1, τ_2 be expressions of $\mathfrak{C}$. We say that τ_1 and τ_2 are unifiable if and only if:

(Def. 25) There exists a valuation f of $\mathfrak{C}$ such that f unifies τ_1 with τ_2.

Let us notice that the predicate τ_1 and τ_2 are unifiable is reflexive and symmetric.

Let $\mathfrak{C}$ be an initialized constructor signature and let τ_1, τ_2 be expressions of $\mathfrak{C}$. We say that τ_1 and τ_2 are weakly-unifiable if and only if:

(Def. 26) There exists an irrelevant one-to-one valuation g of $\mathfrak{C}$ such that $\mathrm{Var}\,\tau_2 \subseteq \mathrm{dom}\,g$ and τ_1 and $\tau_2[g]$ are unifiable.

Let us note that the predicate τ_1 and τ_2 are weakly-unifiable is reflexive.

We now state the proposition

(55) For all expressions τ_1, τ_2 of $\mathfrak{C}$ such that τ_1 and τ_2 are unifiable holds τ_1 and τ_2 are weakly-unifiable.

Let $\mathfrak{C}$ be an initialized constructor signature and let τ, τ_1, τ_2 be expressions of $\mathfrak{C}$. We say that τ is a unification of τ_1 and τ_2 if and only if:

(Def. 27) There exists a valuation f of $\mathfrak{C}$ such that f unifies τ_1 with τ_2 and $\tau = \tau_1[f]$.

We now state two propositions:

(56) For all expressions τ_1, τ_2, τ of $\mathfrak{C}$ such that τ is a unification of τ_1 and τ_2 holds τ is a unification of τ_2 and τ_1.

(57) For all expressions τ_1, τ_2, τ of $\mathfrak{C}$ such that τ is a unification of τ_1 and τ_2 holds τ matches τ_1 and τ matches τ_2.

Let $\mathfrak{C}$ be an initialized constructor signature and let τ, τ_1, τ_2 be expressions of $\mathfrak{C}$. We say that τ is a general-unification of τ_1 and τ_2 if and only if the conditions (Def. 28) are satisfied.

(Def. 28)(i) τ is a unification of τ_1 and τ_2, and

(ii) for every expression u of $\mathfrak{C}$ such that u is a unification of τ_1 and τ_2 holds u matches τ.

6. Type Distribution

The following three propositions are true:

(58) Let n be a natural number and s be a sort symbol of $\mathfrak{M}$. Then there exists a constructor operation symbol m of s such that $\mathrm{len}\,\mathrm{Arity}(m) = n$.

(59) Let given ℓ, s be a sort symbol of $\mathfrak{M}$, and m be a constructor operation symbol of s. If $\mathrm{len}\,\mathrm{Arity}(m) = \mathrm{len}\,\ell$, then $\mathrm{Var}(m^{\rightarrow}(\mathrm{args}\,\ell)) = \mathrm{rng}\,\ell$.

(60) Let X be a finite subset of Vars. Suppose $\mathrm{varcl}\,X = X$. Let s be a sort symbol of $\mathfrak{M}$. Then there exists a constructor operation symbol m of s and there exists a finite sequence p of elements of $\mathrm{QuasiTerms}\,\mathfrak{M}$ such that $\mathrm{len}\,p = \mathrm{len}\,\mathrm{Arity}(m)$ and $\mathrm{vars}(m^{\rightarrow}(p)) = X$.

Let d be a partial function from Vars to QuasiTypes. We say that d is even if and only if:

(Def. 29) For all x, ϑ such that $x \in \operatorname{dom} d$ and $\vartheta = d(x)$ holds $\operatorname{vars}(\vartheta) = \operatorname{vars}(x)$.

Let ℓ be a quasi-locus sequence. A partial function from Vars to QuasiTypes is said to be a type-distribution for ℓ if:

(Def. 30) $\operatorname{dom} \mathrm{it} = \operatorname{rng} \ell$ and it is even.

We now state the proposition

(61) For every empty quasi-locus sequence ℓ holds $\emptyset$ is a type-distribution for ℓ.

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Representation of the Fibonacci and Lucas Numbers in Terms of Floor and Ceiling

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Summary. In the paper we show how to express the Fibonacci numbers and Lucas numbers using the floor and ceiling operations.

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The notation and terminology used here have been introduced in the following papers: [7], [3], [8], [11], [10], [1], [4], [6], [2], [5], and [9].

1. Preliminaries

One can prove the following propositions:

(1) For all real numbers a, b and for every natural number c holds $(\frac{a}{b})^c = \frac{a^c}{b^c}$.

(2) For every real number a and for all integer numbers b, c such that $a \neq 0$ holds $a^{b+c} = a^b \cdot a^c$.

(3) For every natural number n and for every real number a such that n is even and $a \neq 0$ holds $(-a)^n = a^n$.

(4) For every natural number n and for every real number a such that n is odd and $a \neq 0$ holds $(-a)^n = -a^n$.

(5) $|\overline{\tau}| < 1$.

(6) For every natural number n and for every non empty real number r such that n is even holds $r^n > 0$.

(7) For every natural number n and for every real number r such that n is odd and $r < 0$ holds $r^n < 0$.

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(8) For every natural number n such that $n \neq 0$ holds $\overline{\tau}^n < \frac{1}{2}$.

(9) For all natural numbers n, m and for every real number r such that m is odd and $n \geq m$ and $r < 0$ and $r > -1$ holds $r^n \geq r^m$.

(10) For all natural numbers n, m such that m is odd and $n \geq m$ holds $\overline{\tau}^n \geq \overline{\tau}^m$.

(11) For all natural numbers n, m such that n is even and m is even and $n \geq m$ holds $\overline{\tau}^n \leq \overline{\tau}^m$.

(12) For all non empty natural numbers m, n such that $m \geq n$ holds $\mathrm{Luc}(m) \geq \mathrm{Luc}(n)$.

(13) For every non empty natural number n holds $\tau^n > \overline{\tau}^n$.

(14) For every natural number n such that $n > 1$ holds $-\frac{1}{2} < \overline{\tau}^n$.

(15) For every natural number n such that $n > 2$ holds $\overline{\tau}^n \geq -\frac{1}{\sqrt{5}}$.

(16) For every natural number n such that $n \geq 2$ holds $\overline{\tau}^n \leq \frac{1}{\sqrt{5}}$.

(17) For every natural number n holds $\frac{\overline{\tau}^n}{\sqrt{5}} + \frac{1}{2} > 0$ and $\frac{\overline{\tau}^n}{\sqrt{5}} + \frac{1}{2} < 1$.

2. Formulas for the Fibonacci Numbers

Next we state two propositions:

(18) For every natural number n holds $\lfloor \frac{\tau^n}{\sqrt{5}} + \frac{1}{2} \rfloor = \mathrm{Fib}(n)$.

(19) For every natural number n such that $n \neq 0$ holds $\lceil \frac{\tau^n}{\sqrt{5}} - \frac{1}{2} \rceil = \mathrm{Fib}(n)$.

We now state a number of propositions:

(20) For every natural number n such that $n \neq 0$ holds $\lfloor \frac{\tau^{2 \cdot n}}{\sqrt{5}} \rfloor = \mathrm{Fib}(2 \cdot n)$.

(21) For every natural number n holds $\lceil \frac{\tau^{2 \cdot n + 1}}{\sqrt{5}} \rceil = \mathrm{Fib}(2 \cdot n + 1)$.

(22) For every natural number n such that $n \geq 2$ and n is even holds $\mathrm{Fib}(n + 1) = \lfloor \tau \cdot \mathrm{Fib}(n) + 1 \rfloor$.

(23) For every natural number n such that $n \geq 2$ and n is odd holds $\mathrm{Fib}(n + 1) = \lceil \tau \cdot \mathrm{Fib}(n) - 1 \rceil$.

(24) For every natural number n such that $n \geq 2$ holds $\mathrm{Fib}(n + 1) = \lfloor \frac{\mathrm{Fib}(n) + \sqrt{5} \cdot \mathrm{Fib}(n) + 1}{2} \rfloor$.

(25) For every natural number n such that $n \geq 2$ holds $\mathrm{Fib}(n + 1) = \lceil \frac{(\mathrm{Fib}(n) + \sqrt{5} \cdot \mathrm{Fib}(n)) - 1}{2} \rceil$.

(26) For every natural number n holds $\mathrm{Fib}(n+1) = \frac{\mathrm{Fib}(n) + \sqrt{5 \cdot \mathrm{Fib}(n)^{\mathbf{2}} + 4 \cdot (-1)^n}}{2}$.

(27) For every natural number n such that $n \geq 2$ holds $\mathrm{Fib}(n + 1) = \lfloor \frac{\mathrm{Fib}(n) + 1 + \sqrt{(5 \cdot \mathrm{Fib}(n)^{\mathbf{2}} - 2 \cdot \mathrm{Fib}(n)) + 1}}{2} \rfloor$.

(28) For every natural number n such that $n \geq 2$ holds $\mathrm{Fib}(n) = \lfloor \frac{1}{\tau} \cdot (\mathrm{Fib}(n + 1) + \frac{1}{2}) \rfloor$.

(29) For all natural numbers n, k such that $n \geq k > 1$ or $k = 1$ and $n > k$ holds $\lfloor \tau^k \cdot \mathrm{Fib}(n) + \frac{1}{2} \rfloor = \mathrm{Fib}(n+k)$.

3. Formulas for the Lucas Numbers

Next we state a number of propositions:

(30) For every natural number n such that $n \geq 2$ holds $\mathrm{Luc}(n) = \lfloor \tau^n + \frac{1}{2} \rfloor$.

(31) For every natural number n such that $n \geq 2$ holds $\mathrm{Luc}(n) = \lceil \tau^n - \frac{1}{2} \rceil$.

(32) For every natural number n such that $n \geq 2$ holds $\mathrm{Luc}(2 \cdot n) = \lceil \tau^{2 \cdot n} \rceil$.

(33) For every natural number n such that $n \geq 2$ holds $\mathrm{Luc}(2 \cdot n + 1) = \lfloor \tau^{2 \cdot n + 1} \rfloor$.

(34) For every natural number n such that $n \geq 2$ and n is odd holds $\mathrm{Luc}(n+1) = \lfloor \tau \cdot \mathrm{Luc}(n) + 1 \rfloor$.

(35) For every natural number n such that $n \geq 2$ and n is even holds $\mathrm{Luc}(n+1) = \lceil \tau \cdot \mathrm{Luc}(n) - 1 \rceil$.

(36) For every natural number n such that $n \neq 1$ holds $\mathrm{Luc}(n+1) = \frac{\mathrm{Luc}(n) + \sqrt{5 \cdot (\mathrm{Luc}(n)^{\mathbf{2}} - 4 \cdot (-1)^n)}}{2}$.

(37) For every natural number n such that $n \geq 4$ holds $\mathrm{Luc}(n+1) = \lfloor \frac{\mathrm{Luc}(n) + 1 + \sqrt{(5 \cdot \mathrm{Luc}(n)^{\mathbf{2}} - 2 \cdot \mathrm{Luc}(n)) + 1}}{2} \rfloor$.

(38) For every natural number n such that $n > 2$ holds $\mathrm{Luc}(n) = \lfloor \frac{1}{\tau} \cdot (\mathrm{Luc}(n+1) + \frac{1}{2}) \rfloor$.

(39) For all natural numbers n, k such that $n \geq 4$ and $k \geq 1$ and $n > k$ and n is odd holds $\mathrm{Luc}(n+k) = \lfloor \tau^k \cdot \mathrm{Luc}(n) + 1 \rfloor$.

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The Correspondence Between n-dimensional Euclidean Space and the Product of n Real Lines

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Summary. In the article we prove that a family of open n-hypercubes is a basis of n-dimensional Euclidean space. The equality of the space and the product of n real lines has been proven.

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The terminology and notation used in this paper have been introduced in the following papers: [2], [6], [10], [4], [7], [18], [8], [13], [1], [3], [5], [15], [16], [17], [21], [22], [9], [19], [20], [11], [14], and [12].

For simplicity, we use the following convention: x, y are sets, i, n are natural numbers, r, s are real numbers, and f_1, f_2 are n-long real-valued finite sequences.

Let s be a real number and let r be a non positive real number. One can check the following observations:

* $]s - r, s + r[$ is empty,
* $[s - r, s + r[$ is empty, and
* $]s - r, s + r]$ is empty.

Let s be a real number and let r be a negative real number. Observe that $[s - r, s + r]$ is empty.

Let f be an empty yielding function and let us consider x. Observe that $f(x)$ is empty.

Let us consider i. Observe that $i \mapsto 0$ is empty yielding.

Let f be an n-long complex-valued finite sequence. One can check the following observations:

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* $-f$ is n-long,
* f^{-1} is n-long,
* $f^{\mathbf{2}}$ is n-long, and
* $|f|$ is n-long.

Let g be an n-long complex-valued finite sequence. One can verify the following observations:

* $f+g$ is n-long,
* $f-g$ is n-long,
* $f\,g$ is n-long, and
* f/g is n-long.

Let c be a complex number and let f be an n-long complex-valued finite sequence. One can check the following observations:

* $c+f$ is n-long,
* $f-c$ is n-long, and
* $c\,f$ is n-long.

Let f be a real-valued function. Note that $\{f\}$ is real-functions-membered. Let g be a real-valued function. One can verify that $\{f, g\}$ is real-functions-membered.

Let D be a set and let us consider n. Note that D^n is finite sequence-membered.

Let us consider n. Note that $\mathcal{R}^n$ is finite sequence-membered.

Let us consider n. Observe that $\mathcal{R}^n$ is real-functions-membered.

Let us consider x, y and let f be an n-long finite sequence. Observe that $f +\cdot (x, y)$ is n-long.

One can prove the following three propositions:

(1) For every n-long finite sequence f such that f is empty holds $n = 0$.

(2) For every n-long real-valued finite sequence f holds $f \in \mathcal{R}^n$.

(3) For all complex-valued functions f, g holds $|f - g| = |g - f|$.

Let us consider f_1, f_2. The functor max-diff-index(f_1, f_2) yields a natural number and is defined as follows:

(Def. 1) max-diff-index(f_1, f_2) is the element of $|f_1 - f_2|^{-1}(\{\sup \operatorname{rng}|f_1 - f_2|\})$.

Let us note that the functor max-diff-index(f_1, f_2) is commutative.

One can prove the following propositions:

(4) If $n \neq 0$, then max-diff-index$(f_1, f_2) \in \operatorname{dom} f_1$.

(5) $|f_1 - f_2|(x) \leq |f_1 - f_2|$(max-diff-index$(f_1, f_2)$).

One can verify that the metric space of real numbers is real-membered.

Let us observe that $(\mathcal{E}^0)_{\mathrm{top}}$ is trivial.

Let us consider n. Observe that $\mathcal{E}^n$ is constituted finite sequences.

Let us consider n. One can verify that every point of $\mathcal{E}^n$ is real-valued.

Let us consider n. One can check that every point of $\mathcal{E}^n$ is n-long.

The following two propositions are true:

(6) The open set family of $\mathcal{E}^0 = \{\emptyset, \{\emptyset\}\}$.

(7) For every subset B of $\mathcal{E}^0$ holds $B = \emptyset$ or $B = \{\emptyset\}$.

In the sequel e, e_1 are points of $\mathcal{E}^n$.

Let us consider n, e. The functor ${}^{@}e$ yields a point of $(\mathcal{E}^n)_{\rm top}$ and is defined by:

(Def. 2) ${}^{@}e = e$.

Let us consider n, e and let r be a non positive real number. Observe that $\mathrm{Ball}(e, r)$ is empty.

Let us consider n, e and let r be a positive real number. Note that $\mathrm{Ball}(e, r)$ is non empty.

We now state three propositions:

(8) For all points p_1, p_2 of $\mathcal{E}^n_{\rm T}$ such that $i \in \mathrm{dom}\, p_1$ holds $(p_1(i) - p_2(i))^{\mathbf{2}} \leq \sum^2(p_1 - p_2)$.

(9) Let n be an element of $\mathbb{N}$ and a, o, p be elements of $\mathcal{E}^n_{\rm T}$. If $a \in \mathrm{Ball}(o, r)$, then for every set x holds $|(a - o)(x)| < r$ and $|a(x) - o(x)| < r$.

(10) For all points a, o of $\mathcal{E}^n$ such that $a \in \mathrm{Ball}(o, r)$ and for every set x holds $|(a - o)(x)| < r$ and $|a(x) - o(x)| < r$.

Let f be a real-valued function and let r be a real number. The functor $\mathrm{Intervals}(f, r)$ yields a function and is defined as follows:

(Def. 3) $\mathrm{dom}\,\mathrm{Intervals}(f, r) = \mathrm{dom}\, f$ and for every set x such that $x \in \mathrm{dom}\, f$ holds $(\mathrm{Intervals}(f, r))(x) =]f(x) - r, f(x) + r[$.

Let us consider r. Note that $\mathrm{Intervals}(\emptyset, r)$ is empty.

Let f be a real-valued finite sequence and let us consider r. One can check that $\mathrm{Intervals}(f, r)$ is finite sequence-like.

Let us consider n, e, r. The functor $\mathrm{OpenHypercube}(e, r)$ yielding a subset of $(\mathcal{E}^n)_{\rm top}$ is defined by:

(Def. 4) $\mathrm{OpenHypercube}(e, r) = \prod \mathrm{Intervals}(e, r)$.

Next we state the proposition

(11) If $0 < r$, then $e \in \mathrm{OpenHypercube}(e, r)$.

Let n be a non zero natural number, let e be a point of $\mathcal{E}^n$, and let r be a non positive real number. Observe that $\mathrm{OpenHypercube}(e, r)$ is empty.

One can prove the following proposition

(12) For every point e of $\mathcal{E}^0$ holds $\mathrm{OpenHypercube}(e, r) = \{\emptyset\}$.

Let e be a point of $\mathcal{E}^0$ and let us consider r. Note that $\mathrm{OpenHypercube}(e, r)$ is non empty.

Let us consider n, e and let r be a positive real number. One can check that $\mathrm{OpenHypercube}(e, r)$ is non empty.

One can prove the following propositions:

(13) If $r \leq s$, then $\mathrm{OpenHypercube}(e, r) \subseteq \mathrm{OpenHypercube}(e, s)$.

(14) If $n \neq 0$ or $0 < r$ and if $e_1 \in \mathrm{OpenHypercube}(e, r)$, then for every set x holds $|(e_1 - e)(x)| < r$ and $|e_1(x) - e(x)| < r$.

(15) If $n \neq 0$ and $e_1 \in \mathrm{OpenHypercube}(e, r)$, then $\sum^2(e_1 - e) < n \cdot r^{\mathbf{2}}$.

(16) If $n \neq 0$ and $e_1 \in \mathrm{OpenHypercube}(e, r)$, then $\rho(e_1, e) < r \cdot \sqrt{n}$.

(17) If $n \neq 0$, then $\mathrm{OpenHypercube}(e, \frac{r}{\sqrt{n}}) \subseteq \mathrm{Ball}(e, r)$.

(18) If $n \neq 0$, then $\mathrm{OpenHypercube}(e, r) \subseteq \mathrm{Ball}(e, r \cdot \sqrt{n})$.

(19) If $e_1 \in \mathrm{Ball}(e, r)$, then there exists a non zero element m of $\mathbb{N}$ such that $\mathrm{OpenHypercube}(e_1, \frac{1}{m}) \subseteq \mathrm{Ball}(e, r)$.

(20) If $n \neq 0$ and $e_1 \in \mathrm{OpenHypercube}(e, r)$, then $r > |e_1 - e|(\text{max-diff-index}(e_1, e))$.

(21) $\mathrm{OpenHypercube}(e_1, r - |e_1 - e|(\text{max-diff-index}(e_1, e))) \subseteq \mathrm{OpenHypercube}(e, r)$.

(22) $\mathrm{Ball}(e, r) \subseteq \mathrm{OpenHypercube}(e, r)$.

Let us consider n, e, r. Observe that $\mathrm{OpenHypercube}(e, r)$ is open.

We now state two propositions:

(23) Let V be a subset of $(\mathcal{E}^n)_{\mathrm{top}}$. Suppose V is open. Let e be a point of $\mathcal{E}^n$. If $e \in V$, then there exists a non zero element m of $\mathbb{N}$ such that $\mathrm{OpenHypercube}(e, \frac{1}{m}) \subseteq V$.

(24) Let V be a subset of $(\mathcal{E}^n)_{\mathrm{top}}$. Suppose that for every point e of $\mathcal{E}^n$ such that $e \in V$ there exists a real number r such that $r > 0$ and $\mathrm{OpenHypercube}(e, r) \subseteq V$. Then V is open.

Let us consider n, e. The functor $\mathrm{OpenHypercubes}\, e$ yields a family of subsets of $(\mathcal{E}^n)_{\mathrm{top}}$ and is defined by:

(Def. 5) $\mathrm{OpenHypercubes}\, e = \{\mathrm{OpenHypercube}(e, \frac{1}{m}) : m$ ranges over non zero elements of $\mathbb{N}\}$.

Let us consider n, e. Observe that $\mathrm{OpenHypercubes}\, e$ is non empty, open, and e-quasi-basis.

Next we state four propositions:

(25) For every 1-sorted yielding many sorted set J indexed by $\mathrm{Seg}\, n$ such that $J = \mathrm{Seg}\, n \longmapsto \mathbb{R}^{\mathbf{1}}$ holds $\mathbb{R}^{\mathrm{Seg}\, n} = \prod(\text{the support of } J)$.

(26) Let J be a topological space yielding many sorted set indexed by $\mathrm{Seg}\, n$. Suppose $n \neq 0$ and $J = \mathrm{Seg}\, n \longmapsto \mathbb{R}^{\mathbf{1}}$. Let P_1 be a family of subsets of $(\mathcal{E}^n)_{\mathrm{top}}$. If $P_1 =$ the product prebasis for J, then P_1 is quasi-prebasis.

(27) Let J be a topological space yielding many sorted set indexed by $\mathrm{Seg}\, n$. Suppose $J = \mathrm{Seg}\, n \longmapsto \mathbb{R}^{\mathbf{1}}$. Let P_1 be a family of subsets of $(\mathcal{E}^n)_{\mathrm{top}}$. If $P_1 =$ the product prebasis for J, then P_1 is open.

(28) $(\mathcal{E}^n)_{\mathrm{top}} = \prod(\mathrm{Seg}\, n \longmapsto \mathbb{R}^{\mathbf{1}})$.

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Affine Independence in Vector Spaces

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Summary. In this article we describe the notion of affinely independent subset of a real linear space. First we prove selected theorems concerning operations on linear combinations. Then we introduce affine independence and prove the equivalence of various definitions of this notion. We also introduce the notion of the affine hull, i.e. a subset generated by a set of vectors which is an intersection of all affine sets including the given set. Finally, we introduce and prove selected properties of the barycentric coordinates.

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The terminology and notation used here are introduced in the following papers: [1], [6], [10], [2], [3], [8], [15], [13], [12], [11], [7], [5], [9], [14], and [4].

1. Preliminaries

For simplicity, we adopt the following convention: x, y are sets, r, s are real numbers, S is a non empty additive loop structure, L_1, L_2, L_3 are linear combinations of S, G is an Abelian add-associative right zeroed right complementable non empty additive loop structure, L_4, L_5, L_6 are linear combinations of G, g, h are elements of G, R_1 is a non empty RLS structure, R is a real linear space-like non empty RLS structure, A_1 is a subset of R, L_7, L_8, L_9 are linear combinations of R, V is a real linear space, v, v_1, v_2, w, p are vectors of V, A, B are subsets of V, F_1, F_2 are families of subsets of V, and L, L_{10}, L_{11} are linear combinations of V.

Let us consider R_1 and let A be an empty subset of R_1. Note that $\operatorname{conv} A$ is empty.

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Let us consider R_1 and let A be a non empty subset of R_1. One can check that $\operatorname{conv} A$ is non empty.

One can prove the following propositions:

(1) For every element v of R holds $\operatorname{conv}\{v\} = \{v\}$.

(2) For every subset A of R_1 holds $A \subseteq \operatorname{conv} A$.

(3) For all subsets A, B of R_1 such that $A \subseteq B$ holds $\operatorname{conv} A \subseteq \operatorname{conv} B$.

(4) For all subsets S, A of R_1 such that $A \subseteq \operatorname{conv} S$ holds $\operatorname{conv} S = \operatorname{conv} S \cup A$.

(5) Let V be an add-associative non empty additive loop structure, A be a subset of V, and v, w be elements of V. Then $(v + w) + A = v + (w + A)$.

(6) For every Abelian right zeroed non empty additive loop structure V and for every subset A of V holds $0_V + A = A$.

(7) For every subset A of G holds $\operatorname{Card} A = \operatorname{Card}(g + A)$.

(8) For every element v of S holds $v + \emptyset_S = \emptyset_S$.

(9) For all subsets A, B of R_1 such that $A \subseteq B$ holds $r \cdot A \subseteq r \cdot B$.

(10) $(r \cdot s) \cdot A_1 = r \cdot (s \cdot A_1)$.

(11) $1 \cdot A_1 = A_1$.

(12) $0 \cdot A \subseteq \{0_V\}$.

(13) For every finite sequence F of elements of S holds $(L_2 + L_3) \cdot F = L_2 \cdot F + L_3 \cdot F$.

(14) For every finite sequence F of elements of V holds $(r \cdot L) \cdot F = r \cdot (L \cdot F)$.

(15) Suppose A is linearly independent and $A \subseteq B$ and $\operatorname{Lin}(B) = V$. Then there exists a linearly independent subset I of V such that $A \subseteq I \subseteq B$ and $\operatorname{Lin}(I) = V$.

2. Two Transformations of Linear Combinations

Let us consider G, L_4, g. The functor $g + L_4$ yielding a linear combination of G is defined as follows:

(Def. 1) $(g + L_4)(h) = L_4(h - g)$.

Next we state several propositions:

(16) The support of $g + L_4 = g +$ the support of L_4.

(17) $g + (L_5 + L_6) = (g + L_5) + (g + L_6)$.

(18) $v + r \cdot L = r \cdot (v + L)$.

(19) $g + (h + L_4) = (g + h) + L_4$.

(20) $g + \mathbf{0}_{\mathrm{LC}_G} = \mathbf{0}_{\mathrm{LC}_G}$.

(21) $0_G + L_4 = L_4$.

Let us consider R, L_7, r. The functor $r \circ L_7$ yields a linear combination of R and is defined as follows:

(Def. 2)(i) For every element v of R holds $(r \circ L_7)(v) = L_7(r^{-1} \cdot v)$ if $r \neq 0$,

(ii) $r \circ L_7 = \mathbf{0}_{\mathrm{LC}_R}$, otherwise.

The following propositions are true:

(22) The support of $r \circ L_7 \subseteq r \cdot$ (the support of L_7).

(23) If $r \neq 0$, then the support of $r \circ L_7 = r \cdot$ (the support of L_7).

(24) $r \circ (L_8 + L_9) = r \circ L_8 + r \circ L_9$.

(25) $r \cdot (s \circ L) = s \circ (r \cdot L)$.

(26) $r \circ \mathbf{0}_{\mathrm{LC}_R} = \mathbf{0}_{\mathrm{LC}_R}$.

(27) $r \circ (s \circ L_7) = (r \cdot s) \circ L_7$.

(28) $1 \circ L_7 = L_7$.

3. The Sum of Coefficients of a Linear Combination

Let us consider S, L_1. The functor $\mathrm{sum}\, L_1$ yields a real number and is defined as follows:

(Def. 3) There exists a finite sequence F of elements of S such that F is one-to-one and $\mathrm{rng}\, F =$ the support of L_1 and $\mathrm{sum}\, L_1 = \sum(L_1 \cdot F)$.

One can prove the following propositions:

(29) For every finite sequence F of elements of S such that the support of L_1 misses $\mathrm{rng}\, F$ holds $\sum(L_1 \cdot F) = 0$.

(30) Let F be a finite sequence of elements of S. If F is one-to-one and the support of $L_1 \subseteq \mathrm{rng}\, F$, then $\mathrm{sum}\, L_1 = \sum(L_1 \cdot F)$.

(31) $\mathrm{sum}\, \mathbf{0}_{\mathrm{LC}_S} = 0$.

(32) For every element v of S such that the support of $L_1 \subseteq \{v\}$ holds $\mathrm{sum}\, L_1 = L_1(v)$.

(33) For all elements v_1, v_2 of S such that the support of $L_1 \subseteq \{v_1, v_2\}$ and $v_1 \neq v_2$ holds $\mathrm{sum}\, L_1 = L_1(v_1) + L_1(v_2)$.

(34) $\mathrm{sum}\, L_2 + L_3 = \mathrm{sum}\, L_2 + \mathrm{sum}\, L_3$.

(35) $\mathrm{sum}\, r \cdot L = r \cdot \mathrm{sum}\, L$.

(36) $\mathrm{sum}\, L_{10} - L_{11} = \mathrm{sum}\, L_{10} - \mathrm{sum}\, L_{11}$.

(37) $\mathrm{sum}\, L_4 = \mathrm{sum}\, g + L_4$.

(38) If $r \neq 0$, then $\mathrm{sum}\, L_7 = \mathrm{sum}\, r \circ L_7$.

(39) $\sum(v + L) = \mathrm{sum}\, L \cdot v + \sum L$.

(40) $\sum(r \circ L) = r \cdot \sum L$.

4. Affine Independence of Vectors

Let us consider V, A. We say that A is affinely independent if and only if:

(Def. 4) A is empty or there exists v such that $v \in A$ and $(-v + A) \setminus \{0_V\}$ is linearly independent.

Let us consider V. Observe that every subset of V which is empty is also affinely independent. Let us consider v. One can check that $\{v\}$ is affinely independent. Let us consider w. Observe that $\{v, w\}$ is affinely independent.

Let us consider V. Note that there exists a subset of V which is non empty, trivial, and affinely independent.

We now state three propositions:

(41) A is affinely independent iff for every v such that $v \in A$ holds $(-v+A) \setminus \{0_V\}$ is linearly independent.

(42) A is affinely independent if and only if for every linear combination L of A such that $\sum L = 0_V$ and $\mathrm{sum}\, L = 0$ holds the support of $L = \emptyset$.

(43) If A is affinely independent and $B \subseteq A$, then B is affinely independent.

Let us consider V. Note that every subset of V which is linearly independent is also affinely independent.

In the sequel I denotes an affinely independent subset of V.

Let us consider V, I, v. Observe that $v + I$ is affinely independent.

One can prove the following proposition

(44) If $v + A$ is affinely independent, then A is affinely independent.

Let us consider V, I, r. One can check that $r \cdot I$ is affinely independent.

The following propositions are true:

(45) If $r \cdot A$ is affinely independent and $r \neq 0$, then A is affinely independent.

(46) If $0_V \in A$, then A is affinely independent iff $A \setminus \{0_V\}$ is linearly independent.

Let us consider V and let F be a family of subsets of V. We say that F is affinely independent if and only if:

(Def. 5) If $A \in F$, then A is affinely independent.

Let us consider V. Observe that every family of subsets of V which is empty is also affinely independent. Let us consider I. One can check that $\{I\}$ is affinely independent.

Let us consider V. Note that there exists a family of subsets of V which is empty and affinely independent and there exists a family of subsets of V which is non empty and affinely independent.

Next we state two propositions:

(47) If F_1 is affinely independent and F_2 is affinely independent, then $F_1 \cup F_2$ is affinely independent.

(48) If $F_1 \subseteq F_2$ and F_2 is affinely independent, then F_1 is affinely independent.

5. Affine Hull

Let us consider R_1 and let A be a subset of R_1. The functor $\mathrm{Affin}\, A$ yields a subset of R_1 and is defined as follows:

(Def. 6) $\mathrm{Affin}\, A = \bigcap\{B; B$ ranges over affine subsets of R_1: $A \subseteq B\}$.

Let us consider R_1 and let A be a subset of R_1. Observe that $\mathrm{Affin}\, A$ is affine.

Let us consider R_1 and let A be an empty subset of R_1. Note that $\mathrm{Affin}\, A$ is empty.

Let us consider R_1 and let A be a non empty subset of R_1. Note that $\mathrm{Affin}\, A$ is non empty.

One can prove the following propositions:

(49) For every subset A of R_1 holds $A \subseteq \mathrm{Affin}\, A$.

(50) For every affine subset A of R_1 holds $A = \mathrm{Affin}\, A$.

(51) For all subsets A, B of R_1 such that $A \subseteq B$ and B is affine holds $\mathrm{Affin}\, A \subseteq B$.

(52) For all subsets A, B of R_1 such that $A \subseteq B$ holds $\mathrm{Affin}\, A \subseteq \mathrm{Affin}\, B$.

(53) $\mathrm{Affin}(v + A) = v + \mathrm{Affin}\, A$.

(54) If A_1 is affine, then $r \cdot A_1$ is affine.

(55) If $r \neq 0$, then $\mathrm{Affin}(r \cdot A_1) = r \cdot \mathrm{Affin}\, A_1$.

(56) $\mathrm{Affin}(r \cdot A) = r \cdot \mathrm{Affin}\, A$.

(57) If $v \in \mathrm{Affin}\, A$, then $\mathrm{Affin}\, A = v + \mathrm{Up}(\mathrm{Lin}(-v + A))$.

(58) A is affinely independent iff for every B such that $B \subseteq A$ and $\mathrm{Affin}\, A = \mathrm{Affin}\, B$ holds $A = B$.

(59) $\mathrm{Affin}\, A = \{\sum L; L$ ranges over linear combinations of A: $\mathrm{sum}\, L = 1\}$.

(60) If $I \subseteq A$, then there exists an affinely independent subset I_1 of V such that $I \subseteq I_1 \subseteq A$ and $\mathrm{Affin}\, I_1 = \mathrm{Affin}\, A$.

(61) Let A, B be finite subsets of V. Suppose A is affinely independent and $\mathrm{Affin}\, A = \mathrm{Affin}\, B$ and $\overline{\overline{B}} \leq \overline{\overline{A}}$. Then B is affinely independent.

(62) L is convex iff $\mathrm{sum}\, L = 1$ and for every v holds $0 \leq L(v)$.

(63) If L is convex, then $L(x) \leq 1$.

(64) If L is convex and $L(x) = 1$, then the support of $L = \{x\}$.

(65) $\mathrm{conv}\, A \subseteq \mathrm{Affin}\, A$.

(66) If $x \in \mathrm{conv}\, A$ and $\mathrm{conv}\, A \setminus \{x\}$ is convex, then $x \in A$.

(67) $\mathrm{Affin}\,\mathrm{conv}\, A = \mathrm{Affin}\, A$.

(68) If $\mathrm{conv}\, A \subseteq \mathrm{conv}\, B$, then $\mathrm{Affin}\, A \subseteq \mathrm{Affin}\, B$.

(69) For all subsets A, B of R_1 such that $A \subseteq \mathrm{Affin}\, B$ holds $\mathrm{Affin}(A \cup B) = \mathrm{Affin}\, B$.

6. Barycentric Coordinates

Let us consider V and let us consider A. Let us assume that A is affinely independent. Let us consider x. Let us assume that $x \in \operatorname{Affin} A$. The functor $x \to A$ yielding a linear combination of A is defined by:

(Def. 7) $\sum(x \to A) = x$ and $\operatorname{sum} x \to A = 1$.

We now state a number of propositions:

(70) If $v_1, v_2 \in \operatorname{Affin} I$, then $(1-r) \cdot v_1 + r \cdot v_2 \to I = (1-r) \cdot (v_1 \to I) + r \cdot (v_2 \to I)$.

(71) If $x \in \operatorname{conv} I$, then $x \to I$ is convex and $0 \leq (x \to I)(v) \leq 1$.

(72) If $x \in \operatorname{conv} I$, then $(x \to I)(y) = 1$ iff $x = y$ and $x \in I$.

(73) For every I such that $x \in \operatorname{Affin} I$ and for every v such that $v \in I$ holds $0 \leq (x \to I)(v)$ holds $x \in \operatorname{conv} I$.

(74) If $x \in I$, then $\operatorname{conv} I \setminus \{x\}$ is convex.

(75) For every B such that $x \in \operatorname{Affin} I$ and for every y such that $y \in B$ holds $(x \to I)(y) = 0$ holds $x \in \operatorname{Affin}(I \setminus B)$ and $x \to I = x \to I \setminus B$.

(76) For every B such that $x \in \operatorname{conv} I$ and for every y such that $y \in B$ holds $(x \to I)(y) = 0$ holds $x \in \operatorname{conv} I \setminus B$.

(77) If $B \subseteq I$ and $x \in \operatorname{Affin} B$, then $x \to B = x \to I$.

(78) If $v_1, v_2 \in \operatorname{Affin} A$ and $r + s = 1$, then $r \cdot v_1 + s \cdot v_2 \in \operatorname{Affin} A$.

(79) For all finite subsets A, B of V such that A is affinely independent and $\operatorname{Affin} A \subseteq \operatorname{Affin} B$ holds $\overline{\overline{A}} \leq \overline{\overline{B}}$.

(80) Let A, B be finite subsets of V. Suppose A is affinely independent and $\operatorname{Affin} A \subseteq \operatorname{Affin} B$ and $\overline{\overline{A}} = \overline{\overline{B}}$. Then B is affinely independent.

(81) If $L_{10}(v) \neq L_{11}(v)$, then $(r \cdot L_{10} + (1-r) \cdot L_{11})(v) = s$ iff $r = \frac{L_{11}(v) - s}{L_{11}(v) - L_{10}(v)}$.

(82) $A \cup \{v\}$ is affinely independent iff A is affinely independent but $v \in A$ or $v \notin \operatorname{Affin} A$.

(83) If $w \notin \operatorname{Affin} A$ and $v_1, v_2 \in A$ and $r \neq 1$ and $r \cdot w + (1-r) \cdot v_1 = s \cdot w + (1-s) \cdot v_2$, then $r = s$ and $v_1 = v_2$.

(84) If $v \in I$ and $w \in \operatorname{Affin} I$ and $p \in \operatorname{Affin}(I \setminus \{v\})$ and $w = r \cdot v + (1-r) \cdot p$, then $r = (w \to I)(v)$.

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Abstract Simplicial Complexes

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Summary. In this article we define the notion of abstract simplicial complexes and operations on them. We introduce the following basic notions: simplex, face, vertex, degree, skeleton, subdivision and substructure, and prove some of their properties.

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The articles [2], [5], [6], [10], [8], [14], [1], [7], [3], [4], [11], [13], [16], [12], [15], and [9] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: x, y, X, Y, Z are sets, D is a non empty set, n, k are natural numbers, and i, i_1, i_2 are integers.

Let us consider X. We introduce X has empty element as an antonym of X has non empty elements.

Note that there exists a set which is empty and finite-membered and every set which is empty is also finite-membered. Let X be a finite set. Note that $\{X\}$ is finite-membered and 2^X is finite-membered. Let Y be a finite set. Observe that $\{X, Y\}$ is finite-membered.

Let X be a finite-membered set. Observe that every subset of X is finite-membered. Let Y be a finite-membered set. One can check that $X \cup Y$ is finite-membered.

Let X be a finite finite-membered set. Note that $\bigcup X$ is finite.

One can verify the following observations:

* every set which is empty is also subset-closed,

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* every set which has empty element is also non empty,
* every set which is non empty and subset-closed has also empty element, and
* there exists a set which has empty element.

Let us consider X. Observe that $\mathrm{SubFin}(X)$ is finite-membered and there exists a family of subsets of X which is subset-closed, finite, and finite-membered.

Let X be a subset-closed set. One can check that $\mathrm{SubFin}(X)$ is subset-closed.

Next we state the proposition

(1) Y is subset-closed iff for every X such that $X \in Y$ holds $2^X \subseteq Y$.

Let A, B be subset-closed sets. Note that $A \cup B$ is subset-closed and $A \cap B$ is subset-closed.

Let us consider X. The subset-closure of X yields a subset-closed set and is defined by the conditions (Def. 1).

(Def. 1)(i) $X \subseteq$ the subset-closure of X, and

(ii) for every Y such that $X \subseteq Y$ and Y is subset-closed holds the subset-closure of $X \subseteq Y$.

The following proposition is true

(2) $x \in$ the subset-closure of X iff there exists y such that $x \subseteq y$ and $y \in X$.

Let us consider X and let F be a family of subsets of X. Then the subset-closure of F is a subset-closed family of subsets of X.

Observe that the subset-closure of $\emptyset$ is empty. Let X be a non empty set. Note that the subset-closure of X is non empty.

Let X be a set with a non-empty element. One can check that the subset-closure of X has a non-empty element.

Let X be a finite-membered set. Note that the subset-closure of X is finite-membered.

The following propositions are true:

(3) If $X \subseteq Y$ and Y is subset-closed, then the subset-closure of $X \subseteq Y$.

(4) The subset-closure of $\{X\} = 2^X$.

(5) The subset-closure of $X \cup Y =$ (the subset-closure of X) $\cup$ (the subset-closure of Y).

(6) X is finer than Y iff the subset-closure of $X \subseteq$ the subset-closure of Y.

(7) If X is subset-closed, then the subset-closure of $X = X$.

(8) If the subset-closure of $X \subseteq X$, then X is subset-closed.

Let us consider Y, X and let n be a set. The subsets of X and Y with cardinality limited by n yields a family of subsets of Y and is defined by the condition (Def. 2).

(Def. 2) Let A be a subset of Y. Then $A \in$ the subsets of X and Y with cardinality limited by n if and only if $A \in X$ and $\mathrm{Card}\, A \subseteq \mathrm{Card}\, n$.

Let us consider D. One can verify that there exists a family of subsets of D which is finite, subset-closed, and finite-membered and has a non-empty element.

Let us consider Y, X and let n be a finite set. One can check that the subsets of X and Y with cardinality limited by n is finite-membered.

Let us consider Y, let X be a subset-closed set, and let n be a set. Note that the subsets of X and Y with cardinality limited by n is subset-closed.

Let us consider Y, let X be a set with empty element, and let n be a set. One can check that the subsets of X and Y with cardinality limited by n has empty element.

Let us consider D, let X be a subset-closed family of subsets of D with a non-empty element, and let n be a non empty set. Note that the subsets of X and D with cardinality limited by n has a non-empty element.

Let us consider X, let Y be a family of subsets of X, and let n be a set. We introduce the subsets of Y with cardinality limited by n as a synonym of the subsets of Y and X with cardinality limited by n.

Let us observe that every set which is empty is also $\subseteq$-linear and there exists a set which is empty and $\subseteq$-linear.

Let X be a $\subseteq$-linear set. Note that every subset of X is $\subseteq$-linear.

The following propositions are true:

(9) If X is non empty, finite, and $\subseteq$-linear, then $\bigcup X \in X$.

(10) For every finite $\subseteq$-linear set X such that X has non empty elements holds $\operatorname{Card} X \subseteq \operatorname{Card} \bigcup X$.

(11) If X is $\subseteq$-linear and $\bigcup X$ misses x, then $X \cup \{\bigcup X \cup x\}$ is $\subseteq$-linear.

(12) Let X be a non empty set. Then there exists a family Y of subsets of X such that

(i) Y is $\subseteq$-linear and has non empty elements,

(ii) $X \in Y$,

(iii) $\operatorname{Card} X = \operatorname{Card} Y$, and

(iv) for every Z such that $Z \in Y$ and $\operatorname{Card} Z \neq 1$ there exists x such that $x \in Z$ and $Z \setminus \{x\} \in Y$.

(13) Let Y be a family of subsets of X. Suppose Y is finite and $\subseteq$-linear and has non empty elements and $X \in Y$. Then there exists a family Y' of subsets of X such that

(i) $Y \subseteq Y'$,

(ii) Y' is $\subseteq$-linear and has non empty elements,

(iii) $\operatorname{Card} X = \operatorname{Card} Y'$, and

(iv) for every Z such that $Z \in Y'$ and $\operatorname{Card} Z \neq 1$ there exists x such that $x \in Z$ and $Z \setminus \{x\} \in Y'$.

2. Simplicial Complexes

A simplicial complex structure is a topological structure.

In the sequel K denotes a simplicial complex structure.

Let us consider K and let A be a subset of K. We introduce A is simplex-like as a synonym of A is open.

Let us consider K and let S be a family of subsets of K. We introduce S is simplex-like as a synonym of S is open.

Let us consider K. One can check that there exists a family of subsets of K which is empty and simplex-like.

The following proposition is true

(14) For every family S of subsets of K holds S is simplex-like iff $S \subseteq$ the topology of K.

Let us consider K and let v be an element of K. We say that v is vertex-like if and only if:

(Def. 3) There exists a subset S of K such that S is simplex-like and $v \in S$.

Let us consider K. The functor Vertices K yielding a subset of K is defined by:

(Def. 4) For every element v of K holds $v \in \text{Vertices}\, K$ iff v is vertex-like.

Let K be a simplicial complex structure. A vertex of K is an element of Vertices K.

Let K be a simplicial complex structure. We say that K is finite-vertices if and only if:

(Def. 5) Vertices K is finite.

Let us consider K. We say that K is locally-finite if and only if:

(Def. 6) For every vertex v of K holds $\{S \subseteq K \colon S \text{ is simplex-like} \wedge v \in S\}$ is finite.

Let us consider K. We say that K is empty-membered if and only if:

(Def. 7) The topology of K is empty-membered.

We say that K has non empty elements if and only if:

(Def. 8) The topology of K has non empty elements.

Let us consider K. We introduce K has a non-empty element as an antonym of K is empty-membered. We introduce K has empty element as an antonym of K has non empty elements.

Let us consider X. A simplicial complex structure is said to be a simplicial complex structure of X if:

(Def. 9) $\Omega_{\rm it} \subseteq X$.

Let us consider X and let K_1 be a simplicial complex structure of X. We say that K_1 is total if and only if:

(Def. 10) $\Omega_{(K_1)} = X$.

One can check the following observations:

* every simplicial complex structure which has empty element is also non void,
* every simplicial complex structure which has a non-empty element is also non void,
* every simplicial complex structure which is non void and empty-membered has also empty element,
* every simplicial complex structure which is non void and subset-closed has also empty element,
* every simplicial complex structure which is empty-membered is also subset-closed and finite-vertices,
* every simplicial complex structure which is finite-vertices is also locally-finite and finite-degree, and
* every simplicial complex structure which is locally-finite and subset-closed is also finite-membered.

Let us consider X. Observe that there exists a simplicial complex structure of X which is empty, void, empty-membered, and strict.

Let us consider D. Note that there exists a simplicial complex structure of D which is non empty, non void, total, empty-membered, and strict and there exists a simplicial complex structure of D which is non empty, total, finite-vertices, subset-closed, and strict and has empty element and a non-empty element.

Let us observe that there exists a simplicial complex structure which is non empty, finite-vertices, subset-closed, and strict and has empty element and a non-empty element.

Let K be a simplicial complex structure with a non-empty element. Observe that Vertices K is non empty.

Let K be a finite-vertices simplicial complex structure. Note that every family of subsets of K which is simplex-like is also finite.

Let K be a finite-membered simplicial complex structure. Note that every family of subsets of K which is simplex-like is also finite-membered.

Next we state several propositions:

(15) Vertices K is empty iff K is empty-membered.

(16) Vertices $K = \bigcup$(the topology of K).

(17) For every subset S of K such that S is simplex-like holds $S \subseteq$ Vertices K.

(18) If K is finite-vertices, then the topology of K is finite.

(19) If the topology of K is finite and K is non finite-vertices, then K is non finite-membered.

(20) If K is subset-closed and the topology of K is finite, then K is finite-vertices.

3. The Simplicial Complex Generated on the Set

Let us consider X and let Y be a family of subsets of X. The complex of Y yielding a strict simplicial complex structure of X is defined as follows:

(Def. 11) The complex of $Y = \langle X, \text{the subset-closure of } Y \rangle$.

Let us consider X and let Y be a family of subsets of X. One can verify that the complex of Y is total and subset-closed.

Let us consider X and let Y be a non empty family of subsets of X. Note that the complex of Y has empty element.

Let us consider X and let Y be a finite-membered family of subsets of X. Note that the complex of Y is finite-membered.

Let us consider X and let Y be a finite finite-membered family of subsets of X. Observe that the complex of Y is finite-vertices.

One can prove the following proposition

(21) If K is subset-closed, then the topological structure of K = the complex of the topology of K.

Let us consider X. A simplicial complex of X is a finite-membered subset-closed simplicial complex structure of X.

Let K be a non void simplicial complex structure. A simplex of K is a simplex-like subset of K.

Let K be a simplicial complex structure with empty element. One can check that every subset of K which is empty is also simplex-like and there exists a simplex of K which is empty.

Let K be a non void finite-membered simplicial complex structure. Note that there exists a simplex of K which is finite.

4. The Degree of Simplicial Complexes

Let us consider K. The functor degree(K) yields an extended real number and is defined as follows:

(Def. 12)(i) For every finite subset S of K such that S is simplex-like holds $\overline{\overline{S}} \leq$ degree$(K)+1$ and there exists a subset S of K such that S is simplex-like and $\text{Card}\, S = \text{degree}(K) + 1$ if K is non void and finite-degree,

(ii) degree$(K) = -1$ if K is void,

(iii) degree$(K) = +\infty$, otherwise.

Let K be a finite-degree simplicial complex structure. Note that degree$(K)+1$ is natural and degree(K) is integer.

The following propositions are true:

(22) degree$(K) = -1$ iff K is empty-membered.

(23) $-1 \leq \text{degree}(K)$.

(24) For every finite subset S of K such that S is simplex-like holds $\overline{\overline{S}} \leq \mathrm{degree}(K) + 1$.

(25) Suppose K is non void or $i \geq -1$. Then $\mathrm{degree}(K) \leq i$ if and only if the following conditions are satisfied:

(i) K is finite-membered, and

(ii) for every finite subset S of K such that S is simplex-like holds $\overline{\overline{S}} \leq i+1$.

(26) For every finite subset A of X holds degree(the complex of $\{A\}$) $= \overline{\overline{A}} - 1$.

5. Subcomplexes

Let us consider X and let K_1 be a simplicial complex structure of X. A simplicial complex of X is said to be a subsimplicial complex of K_1 if:

(Def. 13) $\Omega_{\mathrm{it}} \subseteq \Omega_{(K_1)}$ and the topology of it $\subseteq$ the topology of K_1.

In the sequel K_1 denotes a simplicial complex structure of X and S_1 denotes a subsimplicial complex of K_1.

Let us consider X, K_1. One can check that there exists a subsimplicial complex of K_1 which is empty, void, and strict.

Let us consider X and let K_1 be a void simplicial complex structure of X. Observe that every subsimplicial complex of K_1 is void.

Let us consider D and let K_2 be a non void subset-closed simplicial complex structure of D. Note that there exists a subsimplicial complex of K_2 which is non void.

Let us consider X and let K_1 be a finite-vertices simplicial complex structure of X. One can check that every subsimplicial complex of K_1 is finite-vertices.

Let us consider X and let K_1 be a finite-degree simplicial complex structure of X. Note that every subsimplicial complex of K_1 is finite-degree.

Next we state several propositions:

(27) Every subsimplicial complex of S_1 is a subsimplicial complex of K_1.

(28) Let A be a subset of K_1 and S be a finite-membered family of subsets of A. Suppose the subset-closure of $S \subseteq$ the topology of K_1. Then the complex of S is a strict subsimplicial complex of K_1.

(29) Let K_1 be a subset-closed simplicial complex structure of X, A be a subset of K_1, and S be a finite-membered family of subsets of A. Suppose $S \subseteq$ the topology of K_1. Then the complex of S is a strict subsimplicial complex of K_1.

(30) Let Y_1, Y_2 be families of subsets of X. Suppose Y_1 is finite-membered and finer than Y_2. Then the complex of Y_1 is a subsimplicial complex of the complex of Y_2.

(31) $\mathrm{Vertices}\, S_1 \subseteq \mathrm{Vertices}\, K_1$.

(32) $\mathrm{degree}(S_1) \leq \mathrm{degree}(K_1)$.

Let us consider X, K_1, S_1. We say that S_1 is maximal if and only if:

(Def. 14) For every subset A of S_1 such that $A \in$ the topology of K_1 holds A is simplex-like.

We now state the proposition

(33) S_1 is maximal iff $2^{\Omega_{(S_1)}} \cap$ the topology of $K_1 \subseteq$ the topology of S_1.

Let us consider X, K_1. Note that there exists a subsimplicial complex of K_1 which is maximal and strict.

We now state three propositions:

(34) Let S_2 be a subsimplicial complex of S_1. Suppose S_1 is maximal and S_2 is maximal. Then S_2 is a maximal subsimplicial complex of K_1.

(35) Let S_2 be a subsimplicial complex of S_1. If S_2 is a maximal subsimplicial complex of K_1, then S_2 is maximal.

(36) Let K_3, K_4 be maximal subsimplicial complexes of K_1. Suppose $\Omega_{(K_3)} = \Omega_{(K_4)}$. Then the topological structure of $K_3 =$ the topological structure of K_4.

Let us consider X, let K_1 be a subset-closed simplicial complex structure of X, and let A be a subset of K_1. Let us assume that $2^A \cap$ the topology of K_1 is finite-membered. The functor $K_1 \restriction A$ yields a maximal strict subsimplicial complex of K_1 and is defined as follows:

(Def. 15) $\Omega_{K_1 \restriction A} = A$.

In the sequel S_3 denotes a simplicial complex of X.

Let us consider X, S_3 and let A be a subset of S_3. Then $S_3 \restriction A$ is a maximal strict subsimplicial complex of S_3 and it can be characterized by the condition:

(Def. 16) $\Omega_{S_3 \restriction A} = A$.

The following four propositions are true:

(37) For every subset A of S_3 holds the topology of $S_3 \restriction A = 2^A \cap$ the topology of S_3.

(38) For all subsets A, B of S_3 and for every subset B' of $S_3 \restriction A$ such that $B' = B$ holds $S_3 \restriction A \restriction B' = S_3 \restriction B$.

(39) $S_3 \restriction \Omega_{(S_3)} =$ the topological structure of S_3.

(40) For all subsets A, B of S_3 such that $A \subseteq B$ holds $S_3 \restriction A$ is a subsimplicial complex of $S_3 \restriction B$.

Let us observe that every integer is finite.

6. The Skeleton of a Simplicial Complex

Let us consider X, K_1 and let i be a real number. The skeleton of K_1 and i yielding a simplicial complex structure of X is defined by the condition (Def. 17).

(Def. 17) The skeleton of K_1 and i = the complex of the subsets of the topology of K_1 with cardinality limited by $i+1$.

Let us consider X, K_1. Observe that the skeleton of K_1 and -1 is empty-membered. Let us consider i. Note that the skeleton of K_1 and i is finite-degree.

Let us consider X, let K_1 be an empty-membered simplicial complex structure of X, and let us consider i. One can check that the skeleton of K_1 and i is empty-membered.

Let us consider D, let K_2 be a non void subset-closed simplicial complex structure of D, and let us consider i. One can check that the skeleton of K_2 and i is non void.

One can prove the following proposition

(41) If $-1 \leq i_1 \leq i_2$, then the skeleton of K_1 and i_1 is a subsimplicial complex of the skeleton of K_1 and i_2.

Let us consider X, let K_1 be a subset-closed simplicial complex structure of X, and let us consider i. Then the skeleton of K_1 and i is a subsimplicial complex of K_1.

We now state several propositions:

(42) If K_1 is subset-closed and the skeleton of K_1 and i is empty-membered, then K_1 is empty-membered or $i = -1$.

(43) degree(the skeleton of K_1 and i) $\leq$ degree(K_1).

(44) If $-1 \leq i$, then degree(the skeleton of K_1 and i) $\leq i$.

(45) If $-1 \leq i$ and the skeleton of K_1 and i = the topological structure of K_1, then degree(K_1) $\leq i$.

(46) If K_1 is subset-closed and degree(K_1) $\leq i$, then the skeleton of K_1 and i = the topological structure of K_1.

In the sequel K is a non void subset-closed simplicial complex structure.

Let us consider K and let i be a real number. Let us assume that i is integer. A finite simplex of K is said to be a simplex of i and K if:

(Def. 18)(i) $\overline{\overline{\mathrm{it}}} = i + 1$ if $-1 \leq i \leq$ degree(K),

(ii) it is empty, otherwise.

Let us consider K. Note that every simplex of -1 and K is empty.

The following three propositions are true:

(47) For every simplex S of i and K such that S is non empty holds i is natural.

(48) Every finite simplex S of K is a simplex of $\overline{\overline{S}} - 1$ and K.

(49) Let K be a non void subset-closed simplicial complex structure of D, S be a non void subsimplicial complex of K, i be an integer, and A be a simplex of i and S. If A is non empty or $i \leq$ degree(S) or degree(S) = degree(K), then A is a simplex of i and K.

Let us consider K and let i be a real number. Let us assume that i is integer and $i \leq \operatorname{degree}(K)$. Let S be a simplex of i and K. A simplex of $\max(i-1,-1)$ and K is said to be a face of S if:

(Def. 19) It $\subseteq S$.

One can prove the following proposition

(50) Let S be a simplex of n and K. Suppose $n \leq \operatorname{degree}(K)$. Then X is a face of S if and only if there exists x such that $x \in S$ and $S \setminus \{x\} = X$.

7. The Subdivision of a Simplicial Complex

In the sequel P is a function.

Let us consider X, K_1, P. The functor subdivision(P, K_1) yields a strict simplicial complex structure of X and is defined by the conditions (Def. 20).

(Def. 20)(i) $\Omega_{\operatorname{subdivision}(P,K_1)} = \Omega_{(K_1)}$, and

(ii) for every subset A of subdivision(P, K_1) holds A is simplex-like iff there exists a $\subseteq$-linear finite simplex-like family S of subsets of K_1 such that $A = P^{\circ}S$.

Let us consider X, K_1, P. One can verify that subdivision(P, K_1) is subset-closed and finite-membered and has empty element.

Let us consider X, let K_1 be a void simplicial complex structure of X, and let us consider P. Observe that subdivision(P, K_1) is empty-membered.

The following propositions are true:

(51) $\operatorname{degree}(\operatorname{subdivision}(P, K_1)) \leq \operatorname{degree}(K_1) + 1$.

(52) If $\operatorname{dom} P$ has non empty elements, then $\operatorname{degree}(\operatorname{subdivision}(P, K_1)) \leq \operatorname{degree}(K_1)$.

Let us consider X, let K_1 be a finite-degree simplicial complex structure of X, and let us consider P. Note that subdivision(P, K_1) is finite-degree.

Let us consider X, let K_1 be a finite-vertices simplicial complex structure of X, and let us consider P. One can check that subdivision(P, K_1) is finite-vertices.

One can prove the following propositions:

(53) Let K_1 be a subset-closed simplicial complex structure of X and given P. Suppose that

(i) $\operatorname{dom} P$ has non empty elements, and

(ii) for every n such that $n \leq \operatorname{degree}(K_1)$ there exists a subset S of K_1 such that S is simplex-like and $\operatorname{Card} S = n + 1$ and $2^S_+ \subseteq \operatorname{dom} P$ and $P^{\circ}2^S_+$ is a subset of K_1 and $P{\restriction}2^S_+$ is one-to-one.

Then $\operatorname{degree}(\operatorname{subdivision}(P, K_1)) = \operatorname{degree}(K_1)$.

(54) If $Y \subseteq Z$, then subdivision$(P{\restriction}Y, K_1)$ is a subsimplicial complex of subdivision$(P{\restriction}Z, K_1)$.

(55) If $\operatorname{dom} P \cap$ the topology of $K_1 \subseteq Y$, then $\operatorname{subdivision}(P \restriction Y, K_1) = \operatorname{subdivision}(P, K_1)$.

(56) If $Y \subseteq Z$, then $\operatorname{subdivision}(Y \restriction P, K_1)$ is a subsimplicial complex of $\operatorname{subdivision}(Z \restriction P, K_1)$.

(57) If P°(the topology of $K_1) \subseteq Y$, then $\operatorname{subdivision}(Y \restriction P, K_1) = \operatorname{subdivision}(P, K_1)$.

(58) $\operatorname{subdivision}(P, S_1)$ is a subsimplicial complex of $\operatorname{subdivision}(P, K_1)$.

(59) For every subset A of $\operatorname{subdivision}(P, K_1)$ such that $\operatorname{dom} P \subseteq$ the topology of S_1 and $A = \Omega_{(S_1)}$ holds $\operatorname{subdivision}(P, S_1) = \operatorname{subdivision}(P, K_1) \restriction A$.

(60) Let K_3, K_4 be simplicial complex structures of X. Suppose the topological structure of K_3 = the topological structure of K_4. Then $\operatorname{subdivision}(P, K_3) = \operatorname{subdivision}(P, K_4)$.

Let us consider X, K_1, P, n. The functor $\operatorname{subdivision}(n, P, K_1)$ yielding a simplicial complex structure of X is defined by the condition (Def. 21).

(Def. 21) There exists a function F such that

(i) $F(0) = K_1$,

(ii) $F(n) = \operatorname{subdivision}(n, P, K_1)$,

(iii) $\operatorname{dom} F = \mathbb{N}$, and

(iv) for every k and for every simplicial complex structure K_1' of X such that $K_1' = F(k)$ holds $F(k+1) = \operatorname{subdivision}(P, K_1')$.

Next we state several propositions:

(61) $\operatorname{subdivision}(0, P, K_1) = K_1$.

(62) $\operatorname{subdivision}(1, P, K_1) = \operatorname{subdivision}(P, K_1)$.

(63) For every natural number n_1 such that $n_1 = n + k$ holds $\operatorname{subdivision}(n_1, P, K_1) = \operatorname{subdivision}(n, P, \operatorname{subdivision}(k, P, K_1))$.

(64) $\Omega_{\operatorname{subdivision}(n,P,K_1)} = \Omega_{(K_1)}$.

(65) $\operatorname{subdivision}(n, P, S_1)$ is a subsimplicial complex of $\operatorname{subdivision}(n, P, K_1)$.

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