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Sperner's Lemma

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Summary. In this article we introduce and prove properties of simplicial complexes in real linear spaces which are necessary to formulate Sperner's lemma. The lemma states that for a function f, which for an arbitrary vertex v of the barycentric subdivision $\mathcal{B}$ of simplex $\mathcal{K}$ assigns some vertex from a face of $\mathcal{K}$ which contains v, we can find a simplex S of $\mathcal{B}$ which satisfies $f(S) = \mathcal{K}$ (see [10]).

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The notation and terminology used in this paper have been introduced in the following papers: [2], [11], [19], [9], [6], [7], [1], [5], [3], [4], [13], [15], [12], [22], [23], [16], [18], [20], [14], [17], [21], and [8].

1. Preliminaries

We follow the rules: x, y, X denote sets and n, k denote natural numbers.

The following two propositions are true:

(1) Let R be a binary relation and C be a cardinal number. If for every x such that $x \in X$ holds $\mathrm{Card}(R^{\circ}x) = C$, then $\mathrm{Card}\, R = \mathrm{Card}(R{\restriction}(\mathrm{dom}\, R \setminus X)) + C \cdot \mathrm{Card}\, X$.

(2) Let Y be a non empty finite set. Suppose $\mathrm{Card}\, X = \overline{\overline{Y}} + 1$. Let f be a function from X into Y. Suppose f is onto. Then there exists y such that $y \in Y$ and $\mathrm{Card}(f^{-1}(\{y\})) = 2$ and for every x such that $x \in Y$ and $x \neq y$ holds $\mathrm{Card}(f^{-1}(\{x\})) = 1$.

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Let X be a 1-sorted structure. A simplicial complex structure of X is a simplicial complex structure of the carrier of X. A simplicial complex of X is a simplicial complex of the carrier of X.

Let X be a 1-sorted structure, let K be a simplicial complex structure of X, and let A be a subset of K. The functor ${}^{@}A$ yielding a subset of X is defined by:

(Def. 1) ${}^{@}A = A$.

Let X be a 1-sorted structure, let K be a simplicial complex structure of X, and let A be a family of subsets of K. The functor ${}^{@}A$ yielding a family of subsets of X is defined by:

(Def. 2) ${}^{@}A = A$.

We now state the proposition

(3) Let X be a 1-sorted structure and K be a subset-closed simplicial complex structure of X. Suppose K is total. Let S be a finite subset of K. Suppose S is simplex-like. Then the complex of $\{{}^{@}S\}$ is a subsimplicial complex of K.

2. The Area of an Abstract Simplicial Complex

For simplicity, we adopt the following rules: R_1 denotes a non empty RLS structure, K_1, K_2, K_3 denote simplicial complex structures of R_1, V denotes a real linear space, and K_4 denotes a non void simplicial complex of V.

Let us consider R_1, K_1. The functor $|K_1|$ yields a subset of R_1 and is defined by:

(Def. 3) $x \in |K_1|$ iff there exists a subset A of K_1 such that A is simplex-like and $x \in \mathrm{conv}{}^{@}A$.

One can prove the following propositions:

(4) If the topology of $K_2 \subseteq$ the topology of K_3, then $|K_2| \subseteq |K_3|$.

(5) For every subset A of K_1 such that A is simplex-like holds $\mathrm{conv}{}^{@}A \subseteq |K_1|$.

(6) Let K be a subset-closed simplicial complex structure of V. Then $x \in |K|$ if and only if there exists a subset A of K such that A is simplex-like and $x \in \mathrm{Int}({}^{@}A)$.

(7) $|K_1|$ is empty iff K_1 is empty-membered.

(8) For every subset A of R_1 holds $|\text{the complex of } \{A\}| = \mathrm{conv}\, A$.

(9) For all families A, B of subsets of R_1 holds $|\text{the complex of } A \cup B| = |\text{the complex of } A| \cup |\text{the complex of } B|$.

3. The Subdivision of a Simplicial Complex

Let us consider R_1, K_1. A simplicial complex structure of R_1 is said to be a subdivision structure of K_1 if it satisfies the conditions (Def. 4).

(Def. 4)(i) $|K_1| \subseteq |\mathrm{it}|$, and

(ii) for every subset A of it such that A is simplex-like there exists a subset B of K_1 such that B is simplex-like and $\mathrm{conv}^{@}A \subseteq \mathrm{conv}^{@}B$.

The following proposition is true

(10) For every subdivision structure P of K_1 holds $|K_1| = |P|$.

Let us consider R_1 and let K_1 be a simplicial complex structure of R_1 with a non-empty element. Observe that every subdivision structure of K_1 has a non-empty element.

We now state four propositions:

(11) K_1 is a subdivision structure of K_1.

(12) The complex of the topology of K_1 is a subdivision structure of K_1.

(13) Let K be a subset-closed simplicial complex structure of V and S_1 be a family of subsets of K. Suppose $S_1 = \mathrm{SubFin}(\text{the topology of } K)$. Then the complex of S_1 is a subdivision structure of K.

(14) For every subdivision structure P_1 of K_1 holds every subdivision structure of P_1 is a subdivision structure of K_1.

Let us consider V and let K be a simplicial complex structure of V. Note that there exists a subdivision structure of K which is finite-membered and subset-closed.

Let us consider V and let K be a simplicial complex structure of V. A subdivision of K is a finite-membered subset-closed subdivision structure of K.

We now state the proposition

(15) Let K be a simplicial complex of V with empty element. Suppose $|K| \subseteq \Omega_K$. Let B be a function from $2_{+}^{\text{the carrier of } V}$ into the carrier of V. Suppose that for every simplex S of K such that S is non empty holds $B(S) \in \mathrm{conv}^{@}S$. Then $\mathrm{subdivision}(B, K)$ is a subdivision structure of K.

Let us consider V, K_4. One can verify that there exists a subdivision of K_4 which is non void.

4. The Barycentric Subdivision

Let us consider V, K_4. Let us assume that $|K_4| \subseteq \Omega_{(K_4)}$. The functor $\mathrm{BCS}\,K_4$ yields a non void subdivision of K_4 and is defined by:

(Def. 5) $\mathrm{BCS}\,K_4 = \mathrm{subdivision}(\text{the center of mass of } V, K_4)$.

Let us consider n and let us consider V, K_4. Let us assume that $|K_4| \subseteq \Omega_{(K_4)}$. The functor $\mathrm{BCS}(n, K_4)$ yields a non void subdivision of K_4 and is defined by:

(Def. 6) $\mathrm{BCS}(n, K_4) = \mathrm{subdivision}(n, \text{the center of mass of } V, K_4)$.

Next we state several propositions:

(16) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\mathrm{BCS}(0, K_4) = K_4$.

(17) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\mathrm{BCS}(1, K_4) = \mathrm{BCS}\, K_4$.

(18) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\Omega_{\mathrm{BCS}(n,K_4)} = \Omega_{(K_4)}$.

(19) If $|K_4| \subseteq \Omega_{(K_4)}$, then $|\mathrm{BCS}(n, K_4)| = |K_4|$.

(20) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\mathrm{BCS}(n+1, K_4) = \mathrm{BCS}\,\mathrm{BCS}(n, K_4)$.

(21) If $|K_4| \subseteq \Omega_{(K_4)}$ and $\mathrm{degree}(K_4) \leq 0$, then the topological structure of $K_4 = \mathrm{BCS}\, K_4$.

(22) If $n > 0$ and $|K_4| \subseteq \Omega_{(K_4)}$ and $\mathrm{degree}(K_4) \leq 0$, then the topological structure of $K_4 = \mathrm{BCS}(n, K_4)$.

(23) Let S_2 be a non void subsimplicial complex of K_4. If $|K_4| \subseteq \Omega_{(K_4)}$ and $|S_2| \subseteq \Omega_{(S_2)}$, then $\mathrm{BCS}(n, S_2)$ is a subsimplicial complex of $\mathrm{BCS}(n, K_4)$.

(24) If $|K_4| \subseteq \Omega_{(K_4)}$, then $\mathrm{Vertices}\, K_4 \subseteq \mathrm{Vertices}\,\mathrm{BCS}(n, K_4)$.

Let us consider n, V and let K be a non void total simplicial complex of V. Note that $\mathrm{BCS}(n, K)$ is total.

Let us consider n, V and let K be a non void finite-vertices total simplicial complex of V. Note that $\mathrm{BCS}(n, K)$ is finite-vertices.

5. Selected Properties of Simplicial Complexes

Let us consider V and let K be a simplicial complex structure of V. We say that K is affinely-independent if and only if:

(Def. 7) For every subset A of K such that A is simplex-like holds ${}^{@}A$ is affinely-independent.

Let us consider R_1, K_1. We say that K_1 is simplex-join-closed if and only if:

(Def. 8) For all subsets A, B of K_1 such that A is simplex-like and B is simplex-like holds $\mathrm{conv}^{@}A \cap \mathrm{conv}^{@}B = \mathrm{conv}^{@}A \cap B$.

Let us consider V. Note that every simplicial complex structure of V which is empty-membered is also affinely-independent. Let F be an affinely-independent family of subsets of V. Observe that the complex of F is affinely-independent.

Let us consider R_1. One can verify that every simplicial complex structure of R_1 which is empty-membered is also simplex-join-closed.

Let us consider V and let I be an affinely-independent subset of V. One can check that the complex of $\{I\}$ is simplex-join-closed.

Let us consider V. One can check that there exists a subset of V which is non empty, trivial, and affinely-independent.

Let us consider V. One can check that there exists a simplicial complex of V which is finite-vertices, affinely-independent, simplex-join-closed, and total and has a non-empty element.

Let us consider V and let K be an affinely-independent simplicial complex structure of V. One can verify that every subsimplicial complex of K is affinely-independent.

Let us consider V and let K be a simplex-join-closed simplicial complex structure of V. One can check that every subsimplicial complex of K is simplex-join-closed.

Next we state the proposition

(25) Let K be a subset-closed simplicial complex structure of V. Then K is simplex-join-closed if and only if for all subsets A, B of K such that A is simplex-like and B is simplex-like and $\mathrm{Int}({}^{@}A)$ meets $\mathrm{Int}({}^{@}B)$ holds $A = B$.

For simplicity, we follow the rules: K_5 is a simplex-join-closed simplicial complex of V, A_1, B_1 are subsets of K_5, K_6 is a non void affinely-independent simplicial complex of V, K_7 is a non void affinely-independent simplex-join-closed simplicial complex of V, and K is a non void affinely-independent simplex-join-closed total simplicial complex of V.

Let us consider V, K_6 and let S be a simplex of K_6. Note that ${}^{@}S$ is affinely-independent.

One can prove the following propositions:

(26) If A_1 is simplex-like and B_1 is simplex-like and $\mathrm{Int}({}^{@}A_1)$ meets $\mathrm{conv}^{@}B_1$, then $A_1 \subseteq B_1$.

(27) If A_1 is simplex-like and ${}^{@}A_1$ is affinely-independent and B_1 is simplex-like, then $\mathrm{Int}({}^{@}A_1) \subseteq \mathrm{conv}^{@}B_1$ iff $A_1 \subseteq B_1$.

(28) If $|K_6| \subseteq \Omega_{(K_6)}$, then $\mathrm{BCS}\, K_6$ is affinely-independent.

Let us consider V and let K_6 be a non void affinely-independent total simplicial complex of V. Observe that $\mathrm{BCS}\, K_6$ is affinely-independent. Let us consider n. Observe that $\mathrm{BCS}(n, K_6)$ is affinely-independent.

Let us consider V, K_7. One can verify that (the center of mass of V)$\restriction$the topology of K_7 is one-to-one.

We now state the proposition

(29) If $|K_7| \subseteq \Omega_{(K_7)}$, then $\mathrm{BCS}\, K_7$ is simplex-join-closed.

Let us consider V, K. Note that $\mathrm{BCS}\, K$ is simplex-join-closed. Let us consider n. Observe that $\mathrm{BCS}(n, K)$ is simplex-join-closed.

The following four propositions are true:

(30) Suppose $|K_4| \subseteq \Omega_{(K_4)}$ and for every n such that $n \leq \mathrm{degree}(K_4)$ there exists a simplex S of K_4 such that $\overline{\overline{S}} = n + 1$ and ${}^{@}S$ is affinely-independent. Then $\mathrm{degree}(K_4) = \mathrm{degree}(\mathrm{BCS}\, K_4)$.

(31) If $|K_6| \subseteq \Omega_{(K_6)}$, then $\mathrm{degree}(K_6) = \mathrm{degree}(\mathrm{BCS}\, K_6)$.

(32) If $|K_6| \subseteq \Omega_{(K_6)}$, then $\mathrm{degree}(K_6) = \mathrm{degree}(\mathrm{BCS}(n, K_6))$.

(33) Let S be a simplex-like family of subsets of K_7. If S has non empty elements, then $\mathrm{Card}\,S = \mathrm{Card}((\text{the center of mass of } V)^{\circ}S)$.

For simplicity, we adopt the following convention: A_2 denotes a finite affinely-independent subset of V, A_3, B_2 denote finite subsets of V, B denotes a subset of V, S, T denote finite families of subsets of V, S_3 denotes a $\subseteq$-linear finite finite-membered family of subsets of V, S_4, T_1 denote finite simplex-like families of subsets of K, and A_4 denotes a simplex of K.

The following propositions are true:

(34) Let S_6, S_5 be simplex-like families of subsets of K_7. Suppose that
 (i) $|K_7| \subseteq \Omega_{(K_7)}$,
 (ii) S_6 has non empty elements,
 (iii) $(\text{the center of mass of } V)^{\circ}S_5$ is a simplex of $\mathrm{BCS}\,K_7$, and
 (iv) $(\text{the center of mass of } V)^{\circ}S_6 \subseteq (\text{the center of mass of } V)^{\circ}S_5$.

Then $S_6 \subseteq S_5$ and S_5 is $\subseteq$-linear.

(35) Suppose S has non empty elements and $\bigcup S \subseteq A_2$ and $\overline{\overline{S}} + n + 1 \leq \overline{\overline{A_2}}$. Then the following statements are equivalent
 (i) B_2 is a simplex of $n + \overline{\overline{S}}$ and BCS (the complex of $\{A_2\}$) and (the center of mass of $V)^{\circ}S \subseteq B_2$,
 (ii) there exists T such that T misses S and $T \cup S$ is $\subseteq$-linear and has non empty elements and $\overline{\overline{T}} = n + 1$ and $\bigcup T \subseteq A_2$ and $B_2 = (\text{the center of mass of } V)^{\circ}S \cup (\text{the center of mass of } V)^{\circ}T$.

(36) Suppose S_3 has non empty elements and $\bigcup S_3 \subseteq A_2$. Then the following statements are equivalent
 (i) $(\text{the center of mass of } V)^{\circ}S_3$ is a simplex of $\overline{\overline{\bigcup S_3}} - 1$ and BCS (the complex of $\{A_2\}$),
 (ii) for every n such that $0 < n \leq \overline{\overline{\bigcup S_3}}$ there exists x such that $x \in S_3$ and $\mathrm{Card}\,x = n$.

(37) Let given S. Suppose S is $\subseteq$-linear and has non empty elements and $\overline{\overline{S}} = \mathrm{Card} \bigcup S$. Let given A_3, B_2. Suppose A_3 is non empty and A_3 misses $\bigcup S$ and $\bigcup S \cup A_3$ is affinely-independent and $\bigcup S \cup A_3 \subseteq B_2$. Then (the center of mass of $V)^{\circ}S \cup (\text{the center of mass of } V)^{\circ}\{\bigcup S \cup A_3\}$ is a simplex of $\overline{\overline{S}}$ and BCS (the complex of $\{B_2\}$).

(38) Let given S_3. Suppose S_3 has non empty elements and $\overline{\overline{S_3}} = \overline{\overline{\bigcup S_3}}$. Let v be an element of V. Suppose $v \notin \bigcup S_3$ and $\bigcup S_3 \cup \{v\}$ is affinely-independent. Then $\{S_6; S_6$ ranges over simplexes of $\overline{\overline{S_3}}$ and BCS (the complex of $\{\bigcup S_3 \cup \{v\}\}$): (the center of mass of $V)^{\circ}S_3 \subseteq S_6\} = \{(\text{the center of mass of } V)^{\circ}S_3 \cup (\text{the center of mass of } V)^{\circ}\{\bigcup S_3 \cup \{v\}\}\}$.

(39) Let given S_3. Suppose S_3 has non empty elements and $\overline{\overline{S_3}} + 1 = \overline{\overline{\bigcup S_3}}$ and $\bigcup S_3$ is affinely-independent. Then $\mathrm{Card}\{S_6; S_6$ ranges over simplexes of $\overline{\overline{S_3}}$ and BCS (the complex of $\{\bigcup S_3\}$): (the center of mass of $V)^{\circ}S_3 \subseteq S_6\} = 2$.

(40) Suppose A_2 is a simplex of K. Then B is a simplex of BCS (the complex of $\{A_2\}$) if and only if B is a simplex of BCS K and conv $B \subseteq$ conv A_2.

(41) Suppose S_4 has non empty elements and $\overline{\overline{S_4}} + n \leq \mathrm{degree}(K)$. Then the following statements are equivalent

(i) A_3 is a simplex of $n + \overline{\overline{S_4}}$ and BCS K and (the center of mass of $V)^\circ S_4 \subseteq A_3$,

(ii) there exists T_1 such that T_1 misses S_4 and $T_1 \cup S_4$ is $\subseteq$-linear and has non empty elements and $\overline{\overline{T_1}} = n + 1$ and $A_3 =$ (the center of mass of $V)^\circ S_4 \cup$ (the center of mass of $V)^\circ T_1$.

(42) Suppose S_4 is $\subseteq$-linear and has non empty elements and $\overline{\overline{S_4}} = \overline{\overline{\bigcup S_4}}$ and $\bigcup S_4 \subseteq A_4$ and $\overline{\overline{A_4}} = \overline{\overline{S_4}} + 1$. Then $\{S_6; S_6$ ranges over simplexes of $\overline{\overline{S_4}}$ and BCS K : (the center of mass of $V)^\circ S_4 \subseteq S_6 \wedge \mathrm{conv}^@ S_6 \subseteq \mathrm{conv}^@ A_4\} = \{$(the center of mass of $V)^\circ S_4 \cup$ (the center of mass of $V)^\circ \{A_4\}\}$.

(43) Suppose S_4 is $\subseteq$-linear and has non empty elements and $\overline{\overline{S_4}} + 1 = \overline{\overline{\bigcup S_4}}$. Then Card$\{S_6; S_6$ ranges over simplexes of $\overline{\overline{S_4}}$ and BCS K : (the center of mass of $V)^\circ S_4 \subseteq S_6 \ \wedge\ \mathrm{conv}^@ S_6 \subseteq \mathrm{conv}^@ \bigcup S_4\} = 2$.

(44) Let given A_3. Suppose that

(i) K is a subdivision of the complex of $\{A_3\}$,

(ii) $\overline{\overline{A_3}} = n + 1$,

(iii) $\mathrm{degree}(K) = n$, and

(iv) for every simplex S of $n - 1$ and K and for every X such that $X = \{S_6; S_6$ ranges over simplexes of n and K: $S \subseteq S_6\}$ holds if $\mathrm{conv}^@ S$ meets Int A_3, then Card $X = 2$ and if $\mathrm{conv}^@ S$ misses Int A_3, then Card $X = 1$.

Let S be a simplex of $n-1$ and BCS K and given X such that $X = \{S_6; S_6$ ranges over simplexes of n and BCS K : $S \subseteq S_6\}$. Then

(v) if $\mathrm{conv}^@ S$ meets Int A_3, then Card $X = 2$, and

(vi) if $\mathrm{conv}^@ S$ misses Int A_3, then Card $X = 1$.

(45) Let S be a simplex of $n - 1$ and BCS(k, the complex of $\{A_2\}$) such that $\overline{\overline{A_2}} = n + 1$ and $X = \{S_6; S_6$ ranges over simplexes of n and BCS(k, the complex of $\{A_2\}$): $S \subseteq S_6\}$. Then

(i) if $\mathrm{conv}^@ S$ meets Int A_2, then Card $X = 2$, and

(ii) if $\mathrm{conv}^@ S$ misses Int A_2, then Card $X = 1$.

6. The Main Theorem

In the sequel v is a vertex of BCS(k, the complex of $\{A_2\}$) and F is a function from Vertices BCS(k, the complex of $\{A_2\}$) into A_2.

The following two propositions are true:

(46) Let given F. Suppose that for all v, B such that $B \subseteq A_2$ and $v \in$ conv B holds $F(v) \in B$. Then there exists n such that Card$\{S; S$ ranges over

simplexes of $\overline{\overline{A_2}} - 1$ and $\mathrm{BCS}(k,$ the complex of $\{A_2\})$: $F^{\circ}S = A_2\} = 2 \cdot n + 1$.

(47) Let given F. Suppose that for all v, B such that $B \subseteq A_2$ and $v \in \mathrm{conv}\, B$ holds $F(v) \in B$. Then there exists a simplex S of $\overline{\overline{A_2}} - 1$ and $\mathrm{BCS}(k,$ the complex of $\{A_2\})$ such that $F^{\circ}S = A_2$.

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Counting Derangements, Non Bijective Functions and the Birthday Problem[1]

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Summary. The article provides counting derangements of finite sets and counting non bijective functions. We provide a recursive formula for the number of derangements of a finite set, together with an explicit formula involving the number e. We count the number of non-one-to-one functions between to finite sets and perform a computation to give explicitely a formalization of the birthday problem. The article is an extension of [10].

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The notation and terminology used here have been introduced in the following papers: [13], [16], [9], [1], [4], [7], [5], [6], [14], [2], [8], [3], [11], [12], [17], [18], and [15].

1. Preliminaries

In this paper x denotes a set.

One can verify that every finite 0-sequence of $\mathbb{Z}$ is integer-valued.

Let n be a natural number. Observe that $n!$ is natural.

Let n be a natural number. One can check that $n!$ is positive.

Let c be a real number. One can verify that $\exp c$ is positive.

Let us observe that e is positive.

The following two propositions are true:

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(1) $\mathrm{id}_{\emptyset}$ has no fixpoint.

(2) For every real number c such that $c < 0$ holds $\exp c < 1$.

2. Rounding

Let n be a real number. The functor round n yielding an integer is defined by:

(Def. 1) $\mathrm{round}\, n = \lfloor n + \frac{1}{2} \rfloor$.

One can prove the following two propositions:

(3) For every integer a holds $\mathrm{round}\, a = a$.

(4) For every integer a and for every real number b such that $|a - b| < \frac{1}{2}$ holds $a = \mathrm{round}\, b$.

3. Counting Derangements

Next we state two propositions:

(5) Let n be a natural number and a, b be real numbers. Suppose $a < b$. Then there exists a real number c such that $c \in]a, b[$ and $\exp a = (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(\text{the function } \exp, \Omega_{\mathbb{R}}, b, a))(\alpha))_{\kappa \in \mathbb{N}}(n) + \frac{\exp c \cdot (a-b)^{n+1}}{(n+1)!}$.

(6) For every positive natural number n and for every real number c such that $c < 0$ holds $|-n! \cdot \frac{\exp c \cdot (-1)^{n+1}}{(n+1)!}| < \frac{1}{2}$.

Let s be a set. The functor derangements s is defined as follows:

(Def. 2) derangements $s = \{f; f$ ranges over permutations of s: f has no fixpoint$\}$.

Let s be a finite set. Observe that derangements s is finite.

Next we state several propositions:

(7) Let s be a finite set. Then derangements $s = \{h : s \to s$: h is one-to-one $\wedge \bigwedge_x (x \in s \Rightarrow h(x) \neq x)\}$.

(8) For every non empty finite set s there exists a real number c such that $c \in]-1, 0[$ and $\overline{\overline{\text{derangements}\, s}} - \frac{\overline{\overline{s}}!}{e} = -\overline{\overline{s}}! \cdot \frac{\exp c \cdot (-1)^{\overline{\overline{s}}+1}}{(\overline{\overline{s}}+1)!}$.

(9) For every non empty finite set s holds $|\overline{\overline{\text{derangements}\, s}} - \frac{\overline{\overline{s}}!}{e}| < \frac{1}{2}$.

(10) For every non empty finite set s holds $\overline{\overline{\text{derangements}\, s}} = \mathrm{round}(\frac{\overline{\overline{s}}!}{e})$.

(11) derangements $\emptyset = \{\emptyset\}$.

(12) derangements$\{x\} = \emptyset$.

The function der seq from $\mathbb{N}$ into $\mathbb{Z}$ is defined as follows:

(Def. 3) $(\text{der seq})(0) = 1$ and $(\text{der seq})(1) = 0$ and for every natural number n holds $(\text{der seq})(n + 2) = (n + 1) \cdot ((\text{der seq})(n) + (\text{der seq})(n + 1))$.

Let c be an integer and let F be a finite 0-sequence of $\mathbb{Z}$. Observe that $c\,F$ is finite, integer-valued, and transfinite sequence-like.

Let c be a complex number and let F be an empty function. One can check that $c\,F$ is empty.

Next we state three propositions:

(13) For every finite 0-sequence F of $\mathbb{Z}$ and for every integer c holds $c \cdot \sum F = \sum((c\,F) \restriction (\operatorname{len} F -' 1)) + c \cdot F(\operatorname{len} F -' 1)$.

(14) Let X, N be finite 0-sequences of $\mathbb{Z}$. Suppose $\operatorname{len} N = \operatorname{len} X + 1$. Let c be an integer. If $N \restriction \operatorname{len} X = c\,X$, then $\sum N = c \cdot \sum X + N(\operatorname{len} X)$.

(15) For every finite set s holds $(\text{der seq})(\overline{\overline{s}}) = \overline{\overline{\text{derangements}\, s}}$.

4. Counting not-one-to-one Functions and the Birthday Problem

Let s, t be sets. The functor not-one-to-one(s, t) yields a subset of t^s and is defined by:

(Def. 4) not-one-to-one$(s, t) = \{f : s \to t\colon f$ is not one-to-one$\}$.

Let s, t be finite sets. Observe that not-one-to-one(s, t) is finite.

The scheme *FraenkelDiff* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

$$\{f : \mathcal{A} \to \mathcal{B} : \text{not } \mathcal{P}[f]\} = \mathcal{B}^{\mathcal{A}} \setminus \{f : \mathcal{A} \to \mathcal{B} : \mathcal{P}[f]\}$$

provided the following requirement is met:

- If $\mathcal{B} = \emptyset$, then $\mathcal{A} = \emptyset$.

We now state three propositions:

(16) For all finite sets s, t such that $\overline{\overline{s}} \leq \overline{\overline{t}}$ holds $\overline{\overline{\text{not-one-to-one}(s, t)}} = \overline{\overline{t}}^{\overline{\overline{s}}} - \frac{\overline{\overline{t}}!}{(\overline{\overline{t}} -' \overline{\overline{s}})!}$.

(17) For every finite set s and for every non empty finite set t such that $\overline{\overline{s}} = 23$ and $\overline{\overline{t}} = 365$ holds $2 \cdot \overline{\overline{\text{not-one-to-one}(s, t)}} > \overline{\overline{t^s}}$.

(18) For all non empty finite sets s, t such that $\overline{\overline{s}} = 23$ and $\overline{\overline{t}} = 365$ holds $\mathrm{P}(\text{not-one-to-one}(s, t)) > \frac{1}{2}$.

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Riemann Integral of Functions $\mathbb{R}$ into $\mathbb{C}$

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Summary. In this article, we define the Riemann Integral on functions $\mathbb{R}$ into $\mathbb{C}$ and proof the linearity of this operator. Especially, the Riemann integral of complex functions is constituted by the redefinition about the Riemann sum of complex numbers. Our method refers to the [19].

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The terminology and notation used here have been introduced in the following articles: [5], [1], [16], [18], [4], [6], [7], [15], [10], [13], [11], [12], [2], [3], [8], [17], [21], [9], [14], and [20].

1. Preliminaries

One can prove the following proposition

(1) For every complex number z and for every real number r holds $\Re(r \cdot z) = r \cdot \Re(z)$ and $\Im(r \cdot z) = r \cdot \Im(z)$.

Let S be a finite sequence of elements of $\mathbb{C}$. The functor $\Re(S)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 1) $\Re(S) = \Re(S$ **qua** partial function from $\mathbb{N}$ to $\mathbb{C})$.

The functor $\Im(S)$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 2) $\Im(S) = \Im(S$ **qua** partial function from $\mathbb{N}$ to $\mathbb{C})$.

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Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{C}$, let S be a non empty Division of A, and let D be an element of S. A finite sequence of elements of $\mathbb{C}$ is said to be a middle volume of f and D if it satisfies the conditions (Def. 3).

(Def. 3)(i) $\operatorname{len} \mathrm{it} = \operatorname{len} D$, and

(ii) for every natural number i such that $i \in \operatorname{dom} D$ there exists an element c of $\mathbb{C}$ such that $c \in \operatorname{rng}(f \restriction \operatorname{divset}(D, i))$ and $\mathrm{it}(i) = c \cdot \operatorname{vol}(\operatorname{divset}(D, i))$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{C}$, let S be a non empty Division of A, let D be an element of S, and let F be a middle volume of f and D. The functor middle sum(f, F) yields an element of $\mathbb{C}$ and is defined by:

(Def. 4) middle sum$(f, F) = \sum F$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{C}$, and let T be a DivSequence of A. A function from $\mathbb{N}$ into $\mathbb{C}^*$ is said to be a middle volume sequence of f and T if:

(Def. 5) For every element k of $\mathbb{N}$ holds $\mathrm{it}(k)$ is a middle volume of f and $T(k)$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{C}$, let T be a DivSequence of A, let S be a middle volume sequence of f and T, and let k be an element of $\mathbb{N}$. Then $S(k)$ is a middle volume of f and $T(k)$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{C}$, let T be a DivSequence of A, and let S be a middle volume sequence of f and T. The functor middle sum(f, S) yields a complex sequence and is defined as follows:

(Def. 6) For every element i of $\mathbb{N}$ holds (middle sum$(f, S))(i) =$ middle sum$(f, S(i))$.

2. Definition of Riemann Integral of Functions $\mathbb{R}$ into $\mathbb{C}$

Next we state two propositions:

(2) For every partial function f from $\mathbb{R}$ to $\mathbb{C}$ and for every subset A of $\mathbb{R}$ holds $\Re(f \restriction A) = \Re(f) \restriction A$.

(3) For every partial function f from $\mathbb{R}$ to $\mathbb{C}$ and for every subset A of $\mathbb{R}$ holds $\Im(f \restriction A) = \Im(f) \restriction A$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a function from A into $\mathbb{C}$. Observe that $\Re(f)$ is quasi total and $\Im(f)$ is quasi total.

We now state several propositions:

(4) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{C}$, s be a non empty Division of A, D be an element of s, and S be a middle volume of f and D. Then $\Re(S)$ is a middle volume of $\Re(f)$ and D and $\Im(S)$ is a middle volume of $\Im(f)$ and D.

(5) For every finite sequence F of elements of $\mathbb{C}$ and for every element x of $\mathbb{C}$ holds $\Re(F \frown \langle x \rangle) = \Re(F) \frown \langle \Re(x) \rangle$.

(6) For every finite sequence F of elements of $\mathbb{C}$ and for every element x of $\mathbb{C}$ holds $\Im(F \frown \langle x \rangle) = \Im(F) \frown \langle \Im(x) \rangle$.

(7) Let F be a finite sequence of elements of $\mathbb{C}$ and F_1 be a finite sequence of elements of $\mathbb{R}$. If $F_1 = \Re(F)$, then $\sum F_1 = \Re(\sum F)$.

(8) Let F be a finite sequence of elements of $\mathbb{C}$ and F_2 be a finite sequence of elements of $\mathbb{R}$. If $F_2 = \Im(F)$, then $\sum F_2 = \Im(\sum F)$.

(9) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{C}$, S be a non empty Division of A, D be an element of S, F be a middle volume of f and D, and F_1 be a middle volume of $\Re(f)$ and D. If $F_1 = \Re(F)$, then $\Re(\text{middle sum}(f, F)) = \text{middle sum}(\Re(f), F_1)$.

(10) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{C}$, S be a non empty Division of A, D be an element of S, F be a middle volume of f and D, and F_2 be a middle volume of $\Im(f)$ and D. If $F_2 = \Im(F)$, then $\Im(\text{middle sum}(f, F)) = \text{middle sum}(\Im(f), F_2)$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a function from A into $\mathbb{C}$. We say that f is integrable if and only if:

(Def. 7) $\Re(f)$ is integrable and $\Im(f)$ is integrable.

We now state three propositions:

(11) For every partial function f from $\mathbb{R}$ to $\mathbb{C}$ holds f is bounded iff $\Re(f)$ is bounded and $\Im(f)$ is bounded.

(12) Let A be a non empty subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to $\mathbb{C}$, and g be a function from A into $\mathbb{C}$. If $f = g$, then $\Re(f) = \Re(g)$ and $\Im(f) = \Im(g)$.

(13) Let A be a closed-interval subset of $\mathbb{R}$ and f be a function from A into $\mathbb{C}$. Then f is bounded if and only if $\Re(f)$ is bounded and $\Im(f)$ is bounded.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a function from A into $\mathbb{C}$. The functor integral f yielding an element of $\mathbb{C}$ is defined as follows:

(Def. 8) $\text{integral}\, f = \text{integral}\, \Re(f) + \text{integral}\, \Im(f) \cdot i$.

Next we state two propositions:

(14) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{C}$, T be a DivSequence of A, and S be a middle volume sequence of f and T. Suppose f is bounded and integrable and δ_T is convergent and $\lim(\delta_T) = 0$. Then middle $\text{sum}(f, S)$ is convergent and $\lim \text{middle sum}(f, S) = \text{integral}\, f$.

(15) Let A be a closed-interval subset of $\mathbb{R}$ and f be a function from A into $\mathbb{C}$. Suppose f is bounded. Then f is integrable if and only if there exists an element I of $\mathbb{C}$ such that for every DivSequence T of A and for every middle volume sequence S of f and T such that δ_T is convergent and $\lim(\delta_T) = 0$

holds middle sum(f,S) is convergent and lim middle sum$(f,S) = I$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from $\mathbb{R}$ to $\mathbb{C}$. We say that f is integrable on A if and only if:

(Def. 9) $\Re(f)$ is integrable on A and $\Im(f)$ is integrable on A.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from $\mathbb{R}$ to $\mathbb{C}$. The functor $\int_A f(x)dx$ yields an element of $\mathbb{C}$ and is defined by:

(Def. 10) $\int_A f(x)dx = \int_A \Re(f)(x)dx + \int_A \Im(f)(x)dx \cdot i$.

We now state two propositions:

(16) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to $\mathbb{C}$, and g be a function from A into $\mathbb{C}$. Suppose $f \restriction A = g$. Then f is integrable on A if and only if g is integrable.

(17) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to $\mathbb{C}$, and g be a function from A into $\mathbb{C}$. If $f \restriction A = g$, then $\int_A f(x)dx =$ integral g.

Let a, b be real numbers and let f be a partial function from $\mathbb{R}$ to $\mathbb{C}$. The functor $\int_a^b f(x)dx$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 11) $\int_a^b f(x)dx = \int_a^b \Re(f)(x)dx + \int_a^b \Im(f)(x)dx \cdot i$.

3. Linearity of the Integration Operator

Next we state several propositions:

(18) Let c be a complex number and f be a partial function from $\mathbb{R}$ to $\mathbb{C}$. Then $\Re(c\,f) = \Re(c)\,\Re(f) - \Im(c)\,\Im(f)$ and $\Im(c\,f) = \Re(c)\,\Im(f) + \Im(c)\,\Re(f)$.

(19) Let A be a closed-interval subset of $\mathbb{R}$ and f_1, f_2 be partial functions from $\mathbb{R}$ to $\mathbb{C}$. Suppose f_1 is integrable on A and f_2 is integrable on A and $A \subseteq \operatorname{dom} f_1$ and $A \subseteq \operatorname{dom} f_2$ and $f_1 \restriction A$ is bounded and $f_2 \restriction A$ is bounded. Then $f_1 + f_2$ is integrable on A and $f_1 - f_2$ is integrable on A and $\int_A (f_1 + f_2)(x)dx = \int_A f_1(x)dx + \int_A f_2(x)dx$ and $\int_A (f_1 - f_2)(x)dx = \int_A f_1(x)dx - \int_A f_2(x)dx$.

(20) Let r be a real number, A be a closed-interval subset of $\mathbb{R}$, and f be a partial function from $\mathbb{R}$ to $\mathbb{C}$. Suppose $A \subseteq \operatorname{dom} f$ and f is integrable on

A and $f{\restriction}A$ is bounded. Then $r\,f$ is integrable on A and $\int\limits_{A}(r\,f)(x)dx = r \cdot \int\limits_{A} f(x)dx$.

(21) Let c be a complex number, A be a closed-interval subset of $\mathbb{R}$, and f be a partial function from $\mathbb{R}$ to $\mathbb{C}$. Suppose $A \subseteq \operatorname{dom} f$ and f is integrable on A and $f{\restriction}A$ is bounded. Then $c\,f$ is integrable on A and $\int\limits_{A}(c\,f)(x)dx = c \cdot \int\limits_{A} f(x)dx$.

(22) Let f be a partial function from $\mathbb{R}$ to $\mathbb{C}$, A be a closed-interval subset of $\mathbb{R}$, and a, b be real numbers. If $A = [a, b]$, then $\int\limits_{A} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

(23) Let f be a partial function from $\mathbb{R}$ to $\mathbb{C}$, A be a closed-interval subset of $\mathbb{R}$, and a, b be real numbers. If $A = [b, a]$, then $-\int\limits_{A} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

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Differentiation of Vector-Valued Functions on n-Dimensional Real Normed Linear Spaces

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Summary. In this article, we define and develop differentiation of vector-valued functions on n-dimensional real normed linear spaces (refer to [16] and [17]).

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The papers [8], [14], [2], [3], [4], [5], [13], [18], [1], [12], [6], [10], [15], [11], [9], [21], [19], [20], and [7] provide the terminology and notation for this paper.

1. The Basic Properties of Differentiation of Functions from $\mathcal{R}^m$ to $\mathcal{R}^n$

In this paper i, n, m are elements of $\mathbb{N}$.

The following propositions are true:

(1) Let f be a set. Then f is a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$ if and only if f is a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$.

(2) Let n, m be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, x be an element of $\mathcal{R}^m$, and y be a point of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Suppose $f = g$ and $x = y$. Then f is differentiable in x if and only if g is differentiable in y.

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(3) Let n, m be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x be an element of $\mathcal{R}^m$, and y be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. If $f = g$ and $x = y$ and f is differentiable in x, then $f'(x) = g'(y)$.

(4) Let f_1, f_2 be partial functions from $\mathcal{R}^m$ to $\mathcal{R}^n$ and g_1, g_2 be partial functions from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 + f_2 = g_1 + g_2$.

(5) Let f_1, f_2 be partial functions from $\mathcal{R}^m$ to $\mathcal{R}^n$ and g_1, g_2 be partial functions from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 - f_2 = g_1 - g_2$.

(6) Let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a be a real number. If $f = g$, then $a\,f = a\,g$.

(7) Let f_1, f_2 be functions from $\mathcal{R}^m$ into $\mathcal{R}^n$ and g_1, g_2 be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 + f_2 = g_1 + g_2$.

(8) Let f_1, f_2 be functions from $\mathcal{R}^m$ into $\mathcal{R}^n$ and g_1, g_2 be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 - f_2 = g_1 - g_2$.

(9) Let f be a function from $\mathcal{R}^m$ into $\mathcal{R}^n$, g be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and r be a real number. If $f = g$, then $r\,f = r \cdot g$.

(10) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x be an element of $\mathcal{R}^m$. Suppose f is differentiable in x. Then $f'(x)$ is a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

Let n, m be natural numbers and let I_1 be a function from $\mathcal{R}^m$ into $\mathcal{R}^n$. We say that I_1 is additive if and only if:

(Def. 1) For all elements x, y of $\mathcal{R}^m$ holds $I_1(x + y) = I_1(x) + I_1(y)$.

We say that I_1 is homogeneous if and only if:

(Def. 2) For every element x of $\mathcal{R}^m$ and for every real number r holds $I_1(r \cdot x) = r \cdot I_1(x)$.

The following three propositions are true:

(11) For every function I_1 from $\mathcal{R}^m$ into $\mathcal{R}^n$ such that I_1 is additive holds $I_1(\langle \underbrace{0, \ldots, 0}_{m} \rangle) = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(12) Let f be a function from $\mathcal{R}^m$ into $\mathcal{R}^n$ and g be a function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f = g$, then f is additive iff g is additive.

(13) Let f be a function from $\mathcal{R}^m$ into $\mathcal{R}^n$ and g be a function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f = g$, then f is homogeneous iff g is homogeneous.

Let n, m be natural numbers. One can verify that the function $\mathcal{R}^m \longmapsto \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ is additive and homogeneous.

Let n, m be natural numbers. Note that there exists a function from $\mathcal{R}^m$ into $\mathcal{R}^n$ which is additive and homogeneous.

Let m, n be natural numbers. A linear operator from m into n is defined by an additive homogeneous function from $\mathcal{R}^m$ into $\mathcal{R}^n$.

We now state the proposition

(14) Let f be a set. Then f is a linear operator from m into n if and only if f is a linear operator from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

Let m, n be natural numbers, let I_1 be a function from $\mathcal{R}^m$ into $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. Then $I_1(x)$ is an element of $\mathcal{R}^n$.

Let m, n be natural numbers and let I_1 be a function from $\mathcal{R}^m$ into $\mathcal{R}^n$. We say that I_1 is bounded if and only if:

(Def. 3) There exists a real number K such that $0 \leq K$ and for every element x of $\mathcal{R}^m$ holds $|I_1(x)| \leq K \cdot |x|$.

Next we state three propositions:

(15) Let x_1, y_1 be finite sequences of elements of $\mathcal{R}^m$. Suppose $\operatorname{len} x_1 = \operatorname{len} y_1 + 1$ and $x_1 \restriction \operatorname{len} y_1 = y_1$. Then there exists an element v of $\mathcal{R}^m$ such that $v = x_1(\operatorname{len} x_1)$ and $\sum x_1 = \sum y_1 + v$.

(16) Let f be a linear operator from m into n, x_1 be a finite sequence of elements of $\mathcal{R}^m$, and y_1 be a finite sequence of elements of $\mathcal{R}^n$. Suppose $\operatorname{len} x_1 = \operatorname{len} y_1$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} x_1$ holds $y_1(i) = f(x_1(i))$. Then $\sum y_1 = f(\sum x_1)$.

(17) Let x_1 be a finite sequence of elements of $\mathcal{R}^m$ and y_1 be a finite sequence of elements of $\mathbb{R}$. Suppose $\operatorname{len} x_1 = \operatorname{len} y_1$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} x_1$ there exists an element v of $\mathcal{R}^m$ such that $v = x_1(i)$ and $y_1(i) = |v|$. Then $|\sum x_1| \leq \sum y_1$.

Let m, n be natural numbers. Note that every linear operator from m into n is bounded.

Let us consider m, n. Observe that every linear operator from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is bounded.

Next we state several propositions:

(18) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x be an element of $\mathcal{R}^m$. Suppose f is differentiable in x. Then $f'(x)$ is a linear operator from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

(19) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x be an element of $\mathcal{R}^m$. Suppose f is differentiable in x. Then $f'(x)$ is a linear operator from m into n.

(20) Let n, m be non empty elements of $\mathbb{N}$, g_1, g_2 be partial functions from

$\mathcal{R}^m$ to $\mathcal{R}^n$, and y_0 be an element of $\mathcal{R}^m$. Suppose g_1 is differentiable in y_0 and g_2 is differentiable in y_0. Then $g_1 + g_2$ is differentiable in y_0 and $(g_1 + g_2)'(y_0) = {g_1}'(y_0) + {g_2}'(y_0)$.

(21) Let n, m be non empty elements of $\mathbb{N}$, g_1, g_2 be partial functions from $\mathcal{R}^m$ to $\mathcal{R}^n$, and y_0 be an element of $\mathcal{R}^m$. Suppose g_1 is differentiable in y_0 and g_2 is differentiable in y_0. Then $g_1 - g_2$ is differentiable in y_0 and $(g_1 - g_2)'(y_0) = {g_1}'(y_0) - {g_2}'(y_0)$.

(22) Let n, m be non empty elements of $\mathbb{N}$, g be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, y_0 be an element of $\mathcal{R}^m$, and r be a real number. Suppose g is differentiable in y_0. Then $r\,g$ is differentiable in y_0 and $(r\,g)'(y_0) = r\,g'(y_0)$.

(23) Let x_0 be an element of $\mathcal{R}^m$, y_0 be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and r be a real number. Suppose $x_0 = y_0$. Then $\{y \in \mathcal{R}^m\colon |y - x_0| < r\} = \{z; z$ ranges over points of $\langle \mathcal{E}^m, \|\cdot\| \rangle\colon \|z - y_0\| < r\}$.

(24) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x_0 be an element of $\mathcal{R}^m$, and L, R be functions from $\mathcal{R}^m$ into $\mathcal{R}^n$. Suppose that

(i) L is a linear operator from m into n, and

(ii) there exists a real number r_0 such that $0 < r_0$ and $\{y \in \mathcal{R}^m\colon |y - x_0| < r_0\} \subseteq \operatorname{dom} f$ and for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every element z of $\mathcal{R}^m$ and for every element w of $\mathcal{R}^n$ such that $z \neq \langle\underbrace{0, \ldots, 0}_{m}\rangle$ and $|z| < d$ and $w = R(z)$ holds $|z|^{-1} \cdot |w| < r$ and for every element x of $\mathcal{R}^m$ such that $|x - x_0| < r_0$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

Then f is differentiable in x_0 and $f'(x_0) = L$.

(25) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x_0 be an element of $\mathcal{R}^m$. Then f is differentiable in x_0 if and only if there exists a real number r_0 such that $0 < r_0$ and $\{y \in \mathcal{R}^m\colon |y - x_0| < r_0\} \subseteq \operatorname{dom} f$ and there exist functions L, R from $\mathcal{R}^m$ into $\mathcal{R}^n$ such that L is a linear operator from m into n and for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every element z of $\mathcal{R}^m$ and for every element w of $\mathcal{R}^n$ such that $z \neq \langle\underbrace{0, \ldots, 0}_{m}\rangle$ and $|z| < d$ and $w = R(z)$ holds $|z|^{-1} \cdot |w| < r$ and for every element x of $\mathcal{R}^m$ such that $|x - x_0| < r_0$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

2. Differentiation of Functions from Normed Linear Spaces $\mathcal{R}^m$ to Normed Linear Spaces $\mathcal{R}^n$

One can prove the following propositions:

(26) For all points y_2, y_3 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ holds $(\mathrm{Proj}(i,n))(y_2 + y_3) = (\mathrm{Proj}(i,n))(y_2) + (\mathrm{Proj}(i,n))(y_3)$.

(27) For every point y_2 of $\langle \mathcal{E}^n, \| \cdot \| \rangle$ and for every real number r holds $(\mathrm{Proj}(i,n))(r \cdot y_2) = r \cdot (\mathrm{Proj}(i,n))(y_2)$.

(28) Let m, n be non empty elements of $\mathbb{N}$, g be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, x_0 be a point of $\langle \mathcal{E}^m, \| \cdot \| \rangle$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq n$ and g is differentiable in x_0. Then $\mathrm{Proj}(i,n) \cdot g$ is differentiable in x_0 and $\mathrm{Proj}(i,n) \cdot g'(x_0) = (\mathrm{Proj}(i,n) \cdot g)'(x_0)$.

(29) Let m, n be non empty elements of $\mathbb{N}$, g be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and x_0 be a point of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Then g is differentiable in x_0 if and only if for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $\mathrm{Proj}(i,n) \cdot g$ is differentiable in x_0.

Let X be a set, let n, m be non empty elements of $\mathbb{N}$, and let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$. We say that f is differentiable on X if and only if:

(Def. 4) $X \subseteq \mathrm{dom}\, f$ and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $f{\restriction}X$ is differentiable in x.

The following four propositions are true:

(30) Let X be a set, m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and g be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose $f = g$. Then f is differentiable on X if and only if g is differentiable on X.

(31) Let X be a set, m, n be non empty elements of $\mathbb{N}$, and f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$. If f is differentiable on X, then X is a subset of $\mathcal{R}^m$.

(32) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and Z be a subset of $\mathcal{R}^m$. Given a subset Z_0 of $\langle \mathcal{E}^m, \| \cdot \| \rangle$ such that $Z = Z_0$ and Z_0 is open. Then f is differentiable on Z if and only if the following conditions are satisfied:

(i) $Z \subseteq \mathrm{dom}\, f$, and

(ii) for every element x of $\mathcal{R}^m$ such that $x \in Z$ holds f is differentiable in x.

(33) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and Z be a subset of $\mathcal{R}^m$. Suppose f is differentiable on Z. Then there exists a subset Z_0 of $\langle \mathcal{E}^m, \| \cdot \| \rangle$ such that $Z = Z_0$ and Z_0 is open.

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Probability Measure on Discrete Spaces and Algebra of Real-Valued Random Variables

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Summary. In this article we continue formalizing probability and randomness started in [13], where we formalized some theorems concerning the probability and real-valued random variables. In this paper we formalize the variance of a random variable and prove Chebyshev's inequality. Next we formalize the product probability measure on the Cartesian product of discrete spaces. In the final part of this article we define the algebra of real-valued random variables.

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The notation and terminology used here have been introduced in the following papers: [21], [3], [16], [1], [9], [17], [14], [4], [5], [11], [15], [6], [12], [22], [13], [19], [20], [8], [10], [18], [2], and [7].

1. Variance

In this paper O_1 denotes a non empty set, r denotes a real number, S_1 denotes a σ-field of subsets of O_1, and P denotes a probability on S_1.

One can prove the following two propositions:

(1) For every one-to-one function f and for all subsets A, B of $\operatorname{dom} f$ such that A misses B holds $\operatorname{rng}(f \restriction A)$ misses $\operatorname{rng}(f \restriction B)$.

(2) For all functions f, g holds $\operatorname{rng}(f \cdot g) \subseteq \operatorname{rng}(f \restriction \operatorname{rng} g)$.

Let us consider O_1, S_1. Observe that there exists a real-valued random variable of S_1 which is non-negative.

Let us consider O_1, S_1 and let X be a real-valued random variable of S_1. Note that $|X|$ is non-negative.

The following propositions are true:

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(3) $O_1 \longmapsto 1 = \chi_{(O_1),O_1}$.

(4) $O_1 \longmapsto r$ is a real-valued random variable of S_1.

(5) For every non empty set X and for every partial function f from X to $\mathbb{R}$ holds $f^2 = (-f)^2$ and $f^2 = |f|^2$.

(6) Let X be a non empty set and f, g be partial functions from X to $\mathbb{R}$. Then $(f+g)^2 = f^2 + 2\,(f\,g) + g^2$ and $(f-g)^2 = (f^2 - 2\,(f\,g)) + g^2$.

Let us consider O_1, S_1, P and let X be a real-valued random variable of S_1. Let us assume that X is integrable on P and $|X|^2$ is integrable on P2M P. The functor variance(X, P) yielding an element of $\mathbb{R}$ is defined by the condition (Def. 1).

(Def. 1) There exists a real-valued random variable Y of S_1 and there exists a real-valued random variable E of S_1 such that $E = O_1 \longmapsto E_P\{X\}$ and $Y = X - E$ and Y is integrable on P and $|Y|^2$ is integrable on P2M P and variance$(X, P) = \int |Y|^2 \,\mathrm{d}\,\mathrm{P2M}\, P$.

2. Chebyshev's Inequality

One can prove the following proposition

(7) Let given O_1, S_1, P, r and X be a real-valued random variable of S_1. Suppose $0 < r$ and X is non-negative and X is integrable on P and $|X|^2$ is integrable on P2M P. Then $P(\{t \in O_1\colon r \leq |X(t) - E_P\{X\}|\}) \leq \frac{\mathrm{variance}(X,P)}{r^2}$.

3. Product Probability Measure

The following propositions are true:

(8) Let O_1 be a non empty finite set, f be a function from O_1 into $\mathbb{R}$, and P be a function from 2^{O_1} into $\mathbb{R}$. Suppose that

(i) for every set x such that $x \subseteq O_1$ holds $0 \leq P(x) \leq 1$,

(ii) $P(O_1) = 1$, and

(iii) for every finite subset z of O_1 holds $P(z) = \mathrm{setopfunc}(z, O_1, \mathbb{R}, f, +_{\mathbb{R}})$.

Then P is a probability on the trivial σ-field of O_1.

(9) Let D_1 be a non empty set, F be a function from D_1 into $\mathbb{R}$, and Y be a finite subset of D_1. Then there exists a finite sequence p of elements of D_1 such that p is one-to-one and rng $p = Y$ and $\mathrm{setopfunc}(Y, D_1, \mathbb{R}, F, +_{\mathbb{R}}) = \sum \mathrm{FuncSeq}(F, p)$.

(10) Let D_1 be a non empty set, F be a function from D_1 into $\mathbb{R}$, Y be a finite subset of D_1, and p be a finite sequence of elements of D_1. If p is one-to-one and rng $p = Y$, then $\mathrm{setopfunc}(Y, D_1, \mathbb{R}, F, +_{\mathbb{R}}) = \sum \mathrm{FuncSeq}(F, p)$.

(11) Let D_2, D_3 be non empty sets, F_1 be a function from D_2 into $\mathbb{R}$, F_2 be a function from D_3 into $\mathbb{R}$, G be a function from $D_2 \times D_3$ into $\mathbb{R}$, Y_1 be a non empty finite subset of D_2, and p_1 be a finite sequence of elements of D_2. Suppose p_1 is one-to-one and $\operatorname{rng} p_1 = Y_1$. Let p_2 be a finite sequence of elements of D_3, p_3 be a finite sequence of elements of $D_2 \times D_3$, Y_2 be a non empty finite subset of D_3, and Y_3 be a finite subset of $D_2 \times D_3$. Suppose that

(i) p_2 is one-to-one,

(ii) $\operatorname{rng} p_2 = Y_2$,

(iii) p_3 is one-to-one,

(iv) $\operatorname{rng} p_3 = Y_3$,

(v) $Y_3 = Y_1 \times Y_2$, and

(vi) for all sets x, y such that $x \in Y_1$ and $y \in Y_2$ holds $G(x, y) = F_1(x) \cdot F_2(y)$.

Then $\sum \mathrm{FuncSeq}(G, p_3) = \sum \mathrm{FuncSeq}(F_1, p_1) \cdot \sum \mathrm{FuncSeq}(F_2, p_2)$.

(12) Let D_2, D_3 be non empty sets, F_1 be a function from D_2 into $\mathbb{R}$, F_2 be a function from D_3 into $\mathbb{R}$, G be a function from $D_2 \times D_3$ into $\mathbb{R}$, Y_1 be a non empty finite subset of D_2, Y_2 be a non empty finite subset of D_3, and Y_3 be a finite subset of $D_2 \times D_3$. Suppose $Y_3 = Y_1 \times Y_2$ and for all sets x, y such that $x \in Y_1$ and $y \in Y_2$ holds $G(x, y) = F_1(x) \cdot F_2(y)$. Then $\mathrm{setopfunc}(Y_3, D_2 \times D_3, \mathbb{R}, G, +_{\mathbb{R}}) = \mathrm{setopfunc}(Y_1, D_2, \mathbb{R}, F_1, +_{\mathbb{R}}) \cdot \mathrm{setopfunc}(Y_2, D_3, \mathbb{R}, F_2, +_{\mathbb{R}})$.

(13) Let D_1 be a non empty set, F be a function from D_1 into $\mathbb{R}$, and Y be a finite subset of D_1. If for every set x such that $x \in Y$ holds $0 \leq F(x)$, then $0 \leq \mathrm{setopfunc}(Y, D_1, \mathbb{R}, F, +_{\mathbb{R}})$.

(14) Let D_1 be a non empty set, F be a function from D_1 into $\mathbb{R}$, and Y_1, Y_2 be finite subsets of D_1. Suppose $Y_1 \subseteq Y_2$ and for every set x such that $x \in Y_2$ holds $0 \leq F(x)$. Then $\mathrm{setopfunc}(Y_1, D_1, \mathbb{R}, F, +_{\mathbb{R}}) \leq \mathrm{setopfunc}(Y_2, D_1, \mathbb{R}, F, +_{\mathbb{R}})$.

(15) Let O_1 be a non empty finite set, P be a probability on the trivial σ-field of O_1, Y be a non empty finite subset of O_1, and f be a function from O_1 into $\mathbb{R}$. Then there exists a finite sequence G of elements of $\mathbb{R}$ and there exists a finite sequence s of elements of Y such that

(i) $\operatorname{len} G = \overline{\overline{Y}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = Y$,

(iv) $\operatorname{len} s = \overline{\overline{Y}}$,

(v) for every natural number n such that $n \in \operatorname{dom} G$ holds $G(n) = f(s(n)) \cdot P(\{s(n)\})$, and

(vi) $\int f \restriction Y \, \mathrm{d}\, \mathrm{P2M}\, P = \sum G$.

Let O_2, O_3 be non empty finite sets, let P_1 be a probability on the trivial

σ-field of O_2, and let P_2 be a probability on the trivial σ-field of O_3. The functor Product-Probability(O_2, O_3, P_1, P_2) yielding a probability on the trivial σ-field of $O_2 \times O_3$ is defined by the condition (Def. 2).

(Def. 2) There exists a function Q from $O_2 \times O_3$ into $\mathbb{R}$ such that

(i) for all sets x, y such that $x \in O_2$ and $y \in O_3$ holds $Q(x, y) = P_1(\{x\}) \cdot P_2(\{y\})$, and

(ii) for every finite subset z of $O_2 \times O_3$ holds (Product-Probability(O_2, O_3, P_1, P_2))$(z) =$ setopfunc$(z, O_2 \times O_3, \mathbb{R}, Q, +_{\mathbb{R}})$.

Next we state two propositions:

(16) Let O_2, O_3 be non empty finite sets, P_1 be a probability on the trivial σ-field of O_2, P_2 be a probability on the trivial σ-field of O_3, Y_1 be a non empty finite subset of O_2, and Y_2 be a non empty finite subset of O_3. Then (Product-Probability(O_2, O_3, P_1, P_2))$(Y_1 \times Y_2) = P_1(Y_1) \cdot P_2(Y_2)$.

(17) Let O_2, O_3 be non empty finite sets, P_1 be a probability on the trivial σ-field of O_2, P_2 be a probability on the trivial σ-field of O_3, and y_1, y_2 be sets. If $y_1 \in O_2$ and $y_2 \in O_3$, then (Product-Probability(O_2, O_3, P_1, P_2))$(\{\langle y_1, y_2 \rangle\}) = P_1(\{y_1\}) \cdot P_2(\{y_2\})$.

4. Algebra of Real-valued Random Variables

Let O_1 be a non empty set and let S_1 be a σ-field of subsets of O_1. The $\mathbb{R}$-valued random variables set of S_1 yields a non empty subset of RAlgebra O_1 and is defined as follows:

(Def. 3) The $\mathbb{R}$-valued random variables set of $S_1 = \{f : f$ ranges over real-valued random variables of $S_1\}$.

Let us consider O_1, S_1. Note that the $\mathbb{R}$-valued random variables set of S_1 is additively-linearly-closed and multiplicatively-closed.

Let us consider O_1, S_1. The $\mathbb{R}$ algebra of real-valued-random-variables of S_1 yielding an algebra is defined by the condition (Def. 4).

(Def. 4) The $\mathbb{R}$ algebra of real-valued-random-variables of S_1 = $\langle$the $\mathbb{R}$-valued random variables set of S_1, mult(the $\mathbb{R}$-valued random variables set of S_1, RAlgebra O_1), Add(the $\mathbb{R}$-valued random variables set of S_1, RAlgebra O_1), Mult(the $\mathbb{R}$-valued random variables set of S_1, RAlgebra O_1), One(the $\mathbb{R}$-valued random variables set of S_1, RAlgebra O_1), Zero(the $\mathbb{R}$-valued random variables set of S_1, RAlgebra O_1)$\rangle$.

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