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The Correspondence Between n-dimensional Euclidean Space and the Product of n Real Lines

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Summary. In the article we prove that a family of open n-hypercubes is a basis of n-dimensional Euclidean space. The equality of the space and the product of n real lines has been proven.

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The terminology and notation used in this paper have been introduced in the following papers: [2], [6], [10], [4], [7], [18], [8], [13], [1], [3], [5], [15], [16], [17], [21], [22], [9], [19], [20], [11], [14], and [12].

For simplicity, we use the following convention: x, y are sets, i, n are natural numbers, r, s are real numbers, and f_1, f_2 are n-long real-valued finite sequences.

Let s be a real number and let r be a non positive real number. One can check the following observations:

* $]s-r, s+r[$ is empty,
* $[s-r, s+r[$ is empty, and
* $]s-r, s+r]$ is empty.

Let s be a real number and let r be a negative real number. Observe that $[s-r, s+r]$ is empty.

Let f be an empty yielding function and let us consider x. Observe that $f(x)$ is empty.

Let us consider i. Observe that $i \mapsto 0$ is empty yielding.

Let f be an n-long complex-valued finite sequence. One can check the following observations:

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* $-f$ is n-long,
* f^{-1} is n-long,
* $f^{\mathbf{2}}$ is n-long, and
* $|f|$ is n-long.

Let g be an n-long complex-valued finite sequence. One can verify the following observations:

* $f+g$ is n-long,
* $f-g$ is n-long,
* $f\,g$ is n-long, and
* f/g is n-long.

Let c be a complex number and let f be an n-long complex-valued finite sequence. One can check the following observations:

* $c+f$ is n-long,
* $f-c$ is n-long, and
* $c\,f$ is n-long.

Let f be a real-valued function. Note that $\{f\}$ is real-functions-membered. Let g be a real-valued function. One can verify that $\{f,g\}$ is real-functions-membered.

Let D be a set and let us consider n. Note that D^n is finite sequence-membered.

Let us consider n. Note that $\mathcal{R}^n$ is finite sequence-membered.

Let us consider n. Observe that $\mathcal{R}^n$ is real-functions-membered.

Let us consider x, y and let f be an n-long finite sequence. Observe that $f +\cdot (x,y)$ is n-long.

One can prove the following three propositions:

(1) For every n-long finite sequence f such that f is empty holds $n=0$.

(2) For every n-long real-valued finite sequence f holds $f \in \mathcal{R}^n$.

(3) For all complex-valued functions f, g holds $|f-g| = |g-f|$.

Let us consider f_1, f_2. The functor max-diff-index(f_1,f_2) yields a natural number and is defined as follows:

(Def. 1) max-diff-index(f_1,f_2) is the element of $|f_1-f_2|^{-1}(\{\sup \operatorname{rng}|f_1-f_2|\})$.

Let us note that the functor max-diff-index(f_1,f_2) is commutative.

One can prove the following propositions:

(4) If $n \neq 0$, then max-diff-index$(f_1,f_2) \in \operatorname{dom} f_1$.

(5) $|f_1-f_2|(x) \leq |f_1-f_2|(\text{max-diff-index}(f_1,f_2))$.

One can verify that the metric space of real numbers is real-membered.

Let us observe that $(\mathcal{E}^0)_{\rm top}$ is trivial.

Let us consider n. Observe that $\mathcal{E}^n$ is constituted finite sequences.

Let us consider n. One can verify that every point of $\mathcal{E}^n$ is real-valued.

Let us consider n. One can check that every point of $\mathcal{E}^n$ is n-long.

The following two propositions are true:

(6) The open set family of $\mathcal{E}^0 = \{\emptyset, \{\emptyset\}\}$.

(7) For every subset B of $\mathcal{E}^0$ holds $B = \emptyset$ or $B = \{\emptyset\}$.

In the sequel e, e_1 are points of $\mathcal{E}^n$.

Let us consider n, e. The functor ${}^{@}e$ yields a point of $(\mathcal{E}^n)_{\rm top}$ and is defined by:

(Def. 2) ${}^{@}e = e$.

Let us consider n, e and let r be a non positive real number. Observe that $\mathrm{Ball}(e, r)$ is empty.

Let us consider n, e and let r be a positive real number. Note that $\mathrm{Ball}(e, r)$ is non empty.

We now state three propositions:

(8) For all points p_1, p_2 of $\mathcal{E}^n_{\rm T}$ such that $i \in \mathrm{dom}\, p_1$ holds $(p_1(i) - p_2(i))^{\mathbf{2}} \leq \sum^2(p_1 - p_2)$.

(9) Let n be an element of $\mathbb{N}$ and a, o, p be elements of $\mathcal{E}^n_{\rm T}$. If $a \in \mathrm{Ball}(o, r)$, then for every set x holds $|(a - o)(x)| < r$ and $|a(x) - o(x)| < r$.

(10) For all points a, o of $\mathcal{E}^n$ such that $a \in \mathrm{Ball}(o, r)$ and for every set x holds $|(a - o)(x)| < r$ and $|a(x) - o(x)| < r$.

Let f be a real-valued function and let r be a real number. The functor $\mathrm{Intervals}(f, r)$ yields a function and is defined as follows:

(Def. 3) $\mathrm{dom}\,\mathrm{Intervals}(f, r) = \mathrm{dom}\, f$ and for every set x such that $x \in \mathrm{dom}\, f$ holds $(\mathrm{Intervals}(f, r))(x) =]f(x) - r, f(x) + r[$.

Let us consider r. Note that $\mathrm{Intervals}(\emptyset, r)$ is empty.

Let f be a real-valued finite sequence and let us consider r. One can check that $\mathrm{Intervals}(f, r)$ is finite sequence-like.

Let us consider n, e, r. The functor $\mathrm{OpenHypercube}(e, r)$ yielding a subset of $(\mathcal{E}^n)_{\rm top}$ is defined by:

(Def. 4) $\mathrm{OpenHypercube}(e, r) = \prod \mathrm{Intervals}(e, r)$.

Next we state the proposition

(11) If $0 < r$, then $e \in \mathrm{OpenHypercube}(e, r)$.

Let n be a non zero natural number, let e be a point of $\mathcal{E}^n$, and let r be a non positive real number. Observe that $\mathrm{OpenHypercube}(e, r)$ is empty.

One can prove the following proposition

(12) For every point e of $\mathcal{E}^0$ holds $\mathrm{OpenHypercube}(e, r) = \{\emptyset\}$.

Let e be a point of $\mathcal{E}^0$ and let us consider r. Note that $\mathrm{OpenHypercube}(e, r)$ is non empty.

Let us consider n, e and let r be a positive real number. One can check that $\mathrm{OpenHypercube}(e, r)$ is non empty.

One can prove the following propositions:

(13) If $r \leq s$, then $\mathrm{OpenHypercube}(e, r) \subseteq \mathrm{OpenHypercube}(e, s)$.

(14) If $n \neq 0$ or $0 < r$ and if $e_1 \in \mathrm{OpenHypercube}(e, r)$, then for every set x holds $|(e_1 - e)(x)| < r$ and $|e_1(x) - e(x)| < r$.

(15) If $n \neq 0$ and $e_1 \in \mathrm{OpenHypercube}(e, r)$, then $\sum^2(e_1 - e) < n \cdot r^{\mathbf{2}}$.

(16) If $n \neq 0$ and $e_1 \in \mathrm{OpenHypercube}(e, r)$, then $\rho(e_1, e) < r \cdot \sqrt{n}$.

(17) If $n \neq 0$, then $\mathrm{OpenHypercube}(e, \frac{r}{\sqrt{n}}) \subseteq \mathrm{Ball}(e, r)$.

(18) If $n \neq 0$, then $\mathrm{OpenHypercube}(e, r) \subseteq \mathrm{Ball}(e, r \cdot \sqrt{n})$.

(19) If $e_1 \in \mathrm{Ball}(e, r)$, then there exists a non zero element m of $\mathbb{N}$ such that $\mathrm{OpenHypercube}(e_1, \frac{1}{m}) \subseteq \mathrm{Ball}(e, r)$.

(20) If $n \neq 0$ and $e_1 \in \mathrm{OpenHypercube}(e, r)$, then $r > |e_1 - e|(\text{max-diff-index}(e_1, e))$.

(21) $\mathrm{OpenHypercube}(e_1, r - |e_1 - e|(\text{max-diff-index}(e_1, e))) \subseteq \mathrm{OpenHypercube}(e, r)$.

(22) $\mathrm{Ball}(e, r) \subseteq \mathrm{OpenHypercube}(e, r)$.

Let us consider n, e, r. Observe that $\mathrm{OpenHypercube}(e, r)$ is open.

We now state two propositions:

(23) Let V be a subset of $(\mathcal{E}^n)_{\mathrm{top}}$. Suppose V is open. Let e be a point of $\mathcal{E}^n$. If $e \in V$, then there exists a non zero element m of $\mathbb{N}$ such that $\mathrm{OpenHypercube}(e, \frac{1}{m}) \subseteq V$.

(24) Let V be a subset of $(\mathcal{E}^n)_{\mathrm{top}}$. Suppose that for every point e of $\mathcal{E}^n$ such that $e \in V$ there exists a real number r such that $r > 0$ and $\mathrm{OpenHypercube}(e, r) \subseteq V$. Then V is open.

Let us consider n, e. The functor $\mathrm{OpenHypercubes}\, e$ yields a family of subsets of $(\mathcal{E}^n)_{\mathrm{top}}$ and is defined by:

(Def. 5) $\mathrm{OpenHypercubes}\, e = \{\mathrm{OpenHypercube}(e, \frac{1}{m}) : m$ ranges over non zero elements of $\mathbb{N}\}$.

Let us consider n, e. Observe that $\mathrm{OpenHypercubes}\, e$ is non empty, open, and e-quasi-basis.

Next we state four propositions:

(25) For every 1-sorted yielding many sorted set J indexed by $\mathrm{Seg}\, n$ such that $J = \mathrm{Seg}\, n \longmapsto \mathbb{R}^{\mathbf{1}}$ holds $\mathbb{R}^{\mathrm{Seg}\, n} = \prod(\text{the support of } J)$.

(26) Let J be a topological space yielding many sorted set indexed by $\mathrm{Seg}\, n$. Suppose $n \neq 0$ and $J = \mathrm{Seg}\, n \longmapsto \mathbb{R}^{\mathbf{1}}$. Let P_1 be a family of subsets of $(\mathcal{E}^n)_{\mathrm{top}}$. If $P_1 =$ the product prebasis for J, then P_1 is quasi-prebasis.

(27) Let J be a topological space yielding many sorted set indexed by $\mathrm{Seg}\, n$. Suppose $J = \mathrm{Seg}\, n \longmapsto \mathbb{R}^{\mathbf{1}}$. Let P_1 be a family of subsets of $(\mathcal{E}^n)_{\mathrm{top}}$. If $P_1 =$ the product prebasis for J, then P_1 is open.

(28) $(\mathcal{E}^n)_{\mathrm{top}} = \prod(\mathrm{Seg}\, n \longmapsto \mathbb{R}^{\mathbf{1}})$.

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