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Partial Differentiation of Real Ternary Functions

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Summary. In this article, we shall extend the result of [19] to discuss partial differentiation of real ternary functions (refer to [8] and [16] for partial differentiation).

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The notation and terminology used here have been introduced in the following papers: [7], [12], [13], [14], [1], [2], [3], [4], [5], [8], [19], [15], [9], [18], [6], [11], [10], and [17].

1. Preliminaries

For simplicity, we use the following convention: D denotes a set, x, x_0, y, y_0, z, z_0, r, s, t denote real numbers, p, a, u, u_0 denote elements of $\mathcal{R}^3$, f, f_1, f_2, f_3, g denote partial functions from $\mathcal{R}^3$ to $\mathbb{R}$, R denotes a rest, and L denotes a linear function.

One can prove the following three propositions:

(1) $\operatorname{dom}\operatorname{proj}(1,3) = \mathcal{R}^3$ and $\operatorname{rng}\operatorname{proj}(1,3) = \mathbb{R}$ and for all elements x, y, z of $\mathbb{R}$ holds $(\operatorname{proj}(1,3))(\langle x, y, z\rangle) = x$.

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(2) dom proj(2, 3) = $\mathcal{R}^3$ and rng proj(2, 3) = $\mathbb{R}$ and for all elements x, y, z of $\mathbb{R}$ holds $(\text{proj}(2,3))(\langle x, y, z\rangle) = y$.

(3) dom proj(3, 3) = $\mathcal{R}^3$ and rng proj(3, 3) = $\mathbb{R}$ and for all elements x, y, z of $\mathbb{R}$ holds $(\text{proj}(3,3))(\langle x, y, z\rangle) = z$.

2. Partial Differentiation of Real Ternary Functions

One can prove the following propositions:

(4) If $u = \langle x, y, z\rangle$ and f is partially differentiable in u w.r.t. coordinate number 1, then SVF1$(1, f, u)$ is differentiable in x.

(5) If $u = \langle x, y, z\rangle$ and f is partially differentiable in u w.r.t. coordinate number 2, then SVF1$(2, f, u)$ is differentiable in y.

(6) If $u = \langle x, y, z\rangle$ and f is partially differentiable in u w.r.t. coordinate number 3, then SVF1$(3, f, u)$ is differentiable in z.

(7) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent

(i) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0\rangle$ and SVF1$(1, f, u)$ is differentiable in x_0,

(ii) f is partially differentiable in u w.r.t. coordinate number 1.

(8) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent

(i) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0\rangle$ and SVF1$(2, f, u)$ is differentiable in y_0,

(ii) f is partially differentiable in u w.r.t. coordinate number 2.

(9) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent

(i) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0\rangle$ and SVF1$(3, f, u)$ is differentiable in z_0,

(ii) f is partially differentiable in u w.r.t. coordinate number 3.

(10) Suppose $u = \langle x_0, y_0, z_0\rangle$ and f is partially differentiable in u w.r.t. coordinate number 1. Then there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(1, f, u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\text{SVF1}(1, f, u))(x) - (\text{SVF1}(1, f, u))(x_0) = L(x - x_0) + R(x - x_0)$.

(11) Suppose $u = \langle x_0, y_0, z_0\rangle$ and f is partially differentiable in u w.r.t. coordinate number 2. Then there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF1}(2, f, u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\text{SVF1}(2, f, u))(y) - (\text{SVF1}(2, f, u))(y_0) = L(y - y_0) + R(y - y_0)$.

(12) Suppose $u = \langle x_0, y_0, z_0 \rangle$ and f is partially differentiable in u w.r.t. coordinate number 3. Then there exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, f, u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, f, u))(z) - (\mathrm{SVF1}(3, f, u))(z_0) = L(z - z_0) + R(z - z_0)$.

(13) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent

(i) f is partially differentiable in u w.r.t. coordinate number 1,

(ii) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(1, f, u)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, f, u))(x) - (\mathrm{SVF1}(1, f, u))(x_0) = L(x - x_0) + R(x - x_0)$.

(14) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent

(i) f is partially differentiable in u w.r.t. coordinate number 2,

(ii) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(2, f, u)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, f, u))(y) - (\mathrm{SVF1}(2, f, u))(y_0) = L(y - y_0) + R(y - y_0)$.

(15) Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and u be an element of $\mathcal{R}^3$. Then the following statements are equivalent

(i) f is partially differentiable in u w.r.t. coordinate number 3,

(ii) there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, f, u)$ and there exist L, R such that for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, f, u))(z) - (\mathrm{SVF1}(3, f, u))(z_0) = L(z - z_0) + R(z - z_0)$.

(16) Suppose $u = \langle x_0, y_0, z_0 \rangle$ and f is partially differentiable in u w.r.t. coordinate number 1. Then $r = \mathrm{partdiff}(f, u, 1)$ if and only if there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(1, f, u)$ and there exist L, R such that $r = L(1)$ and for every x such that $x \in N$ holds $(\mathrm{SVF1}(1, f, u))(x) - (\mathrm{SVF1}(1, f, u))(x_0) = L(x - x_0) + R(x - x_0)$.

(17) Suppose $u = \langle x_0, y_0, z_0 \rangle$ and f is partially differentiable in u w.r.t. coordinate number 2. Then $r = \mathrm{partdiff}(f, u, 2)$ if and only if there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(2, f, u)$ and there exist L, R such that $r = L(1)$ and for every y such that $y \in N$ holds $(\mathrm{SVF1}(2, f, u))(y) - (\mathrm{SVF1}(2, f, u))(y_0) = L(y - y_0) + R(y - y_0)$.

(18) Suppose $u = \langle x_0, y_0, z_0 \rangle$ and f is partially differentiable in u w.r.t. coordinate number 3. Then $r = \mathrm{partdiff}(f, u, 3)$ if and only if there exist real numbers x_0, y_0, z_0 such that $u = \langle x_0, y_0, z_0 \rangle$ and there exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(3, f, u)$ and there

exist L, R such that $r = L(1)$ and for every z such that $z \in N$ holds $(\mathrm{SVF1}(3, f, u))(z) - (\mathrm{SVF1}(3, f, u))(z_0) = L(z - z_0) + R(z - z_0)$.

(19) If $u = \langle x_0, y_0, z_0 \rangle$, then $\mathrm{partdiff}(f, u, 1) = (\mathrm{SVF1}(1, f, u))'(x_0)$.

(20) If $u = \langle x_0, y_0, z_0 \rangle$, then $\mathrm{partdiff}(f, u, 2) = (\mathrm{SVF1}(2, f, u))'(y_0)$.

(21) If $u = \langle x_0, y_0, z_0 \rangle$, then $\mathrm{partdiff}(f, u, 3) = (\mathrm{SVF1}(3, f, u))'(z_0)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. We say that f is partially differentiable w.r.t. 1st coordinate on D if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $D \subseteq \mathrm{dom}\, f$, and

(ii) for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f \restriction D$ is partially differentiable in u w.r.t. coordinate number 1.

We say that f is partially differentiable w.r.t. 2nd coordinate on D if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) $D \subseteq \mathrm{dom}\, f$, and

(ii) for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f \restriction D$ is partially differentiable in u w.r.t. coordinate number 2.

We say that f is partially differentiable w.r.t. 3rd coordinate on D if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $D \subseteq \mathrm{dom}\, f$, and

(ii) for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f \restriction D$ is partially differentiable in u w.r.t. coordinate number 3.

The following three propositions are true:

(22) Suppose f is partially differentiable w.r.t. 1st coordinate on D. Then $D \subseteq \mathrm{dom}\, f$ and for every u such that $u \in D$ holds f is partially differentiable in u w.r.t. coordinate number 1.

(23) Suppose f is partially differentiable w.r.t. 2nd coordinate on D. Then $D \subseteq \mathrm{dom}\, f$ and for every u such that $u \in D$ holds f is partially differentiable in u w.r.t. coordinate number 2.

(24) Suppose f is partially differentiable w.r.t. 3rd coordinate on D. Then $D \subseteq \mathrm{dom}\, f$ and for every u such that $u \in D$ holds f is partially differentiable in u w.r.t. coordinate number 3.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partially differentiable w.r.t. 1st coordinate on D. The functor $f^{\mathrm{1st}}_{\restriction D}$ yielding a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ is defined as follows:

(Def. 4) $\mathrm{dom}(f^{\mathrm{1st}}_{\restriction D}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f^{\mathrm{1st}}_{\restriction D}(u) = \mathrm{partdiff}(f, u, 1)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partially differentiable w.r.t. 2nd coordinate on D. The functor $f^{\mathrm{2nd}}_{\restriction D}$ yields a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and is defined as follows:

(Def. 5) $\operatorname{dom}(f^{\rm 2nd}_{\restriction D}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f^{\rm 2nd}_{\restriction D}(u) = \operatorname{partdiff}(f, u, 2)$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let D be a set. Let us assume that f is partially differentiable w.r.t. 3rd coordinate on D. The functor $f^{\rm 3rd}_{\restriction D}$ yielding a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ is defined as follows:

(Def. 6) $\operatorname{dom}(f^{\rm 3rd}_{\restriction D}) = D$ and for every element u of $\mathcal{R}^3$ such that $u \in D$ holds $f^{\rm 3rd}_{\restriction D}(u) = \operatorname{partdiff}(f, u, 3)$.

3. Main Properties of Partial Differentiation of Real Ternary Functions

We now state a number of propositions:

(25) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\operatorname{proj}(1,3))(u_0)$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 1 and $N \subseteq \operatorname{dom} \operatorname{SVF1}(1, f, u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\operatorname{proj}(1,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}\,(\operatorname{SVF1}(1, f, u_0) \cdot (h + c) - \operatorname{SVF1}(1, f, u_0) \cdot c)$ is convergent and $\operatorname{partdiff}(f, u_0, 1) = \lim(h^{-1}\,(\operatorname{SVF1}(1, f, u_0) \cdot (h + c) - \operatorname{SVF1}(1, f, u_0) \cdot c))$.

(26) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\operatorname{proj}(2,3))(u_0)$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 2 and $N \subseteq \operatorname{dom} \operatorname{SVF1}(2, f, u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\operatorname{proj}(2,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}\,(\operatorname{SVF1}(2, f, u_0) \cdot (h + c) - \operatorname{SVF1}(2, f, u_0) \cdot c)$ is convergent and $\operatorname{partdiff}(f, u_0, 2) = \lim(h^{-1}\,(\operatorname{SVF1}(2, f, u_0) \cdot (h + c) - \operatorname{SVF1}(2, f, u_0) \cdot c))$.

(27) Let u_0 be an element of $\mathcal{R}^3$ and N be a neighbourhood of $(\operatorname{proj}(3,3))(u_0)$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 3 and $N \subseteq \operatorname{dom} \operatorname{SVF1}(3, f, u_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\operatorname{rng} c = \{(\operatorname{proj}(3,3))(u_0)\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}\,(\operatorname{SVF1}(3, f, u_0) \cdot (h + c) - \operatorname{SVF1}(3, f, u_0) \cdot c)$ is convergent and $\operatorname{partdiff}(f, u_0, 3) = \lim(h^{-1}\,(\operatorname{SVF1}(3, f, u_0) \cdot (h + c) - \operatorname{SVF1}(3, f, u_0) \cdot c))$.

(28) Suppose that

(i) f_1 is partially differentiable in u_0 w.r.t. coordinate number 1, and

(ii) f_2 is partially differentiable in u_0 w.r.t. coordinate number 1.

Then $f_1\,f_2$ is partially differentiable in u_0 w.r.t. coordinate number 1.

(29) Suppose that

(i) f_1 is partially differentiable in u_0 w.r.t. coordinate number 2, and

(ii) f_2 is partially differentiable in u_0 w.r.t. coordinate number 2.

Then $f_1\,f_2$ is partially differentiable in u_0 w.r.t. coordinate number 2.

(30) Suppose that

(i) f_1 is partially differentiable in u_0 w.r.t. coordinate number 3, and

(ii) f_2 is partially differentiable in u_0 w.r.t. coordinate number 3.

Then $f_1\,f_2$ is partially differentiable in u_0 w.r.t. coordinate number 3.

(31) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 1. Then $\mathrm{SVF1}(1, f, u_0)$ is continuous in $(\mathrm{proj}(1,3))(u_0)$.

(32) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 2. Then $\mathrm{SVF1}(2, f, u_0)$ is continuous in $(\mathrm{proj}(2,3))(u_0)$.

(33) Let u_0 be an element of $\mathcal{R}^3$. Suppose f is partially differentiable in u_0 w.r.t. coordinate number 3. Then $\mathrm{SVF1}(3, f, u_0)$ is continuous in $(\mathrm{proj}(3,3))(u_0)$.

(34) Suppose f is partially differentiable in u_0 w.r.t. coordinate number 1. Then there exists R such that $R(0) = 0$ and R is continuous in 0.

(35) Suppose f is partially differentiable in u_0 w.r.t. coordinate number 2. Then there exists R such that $R(0) = 0$ and R is continuous in 0.

(36) Suppose f is partially differentiable in u_0 w.r.t. coordinate number 3. Then there exists R such that $R(0) = 0$ and R is continuous in 0.

4. Grads and Curl

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let p be an element of $\mathcal{R}^3$. The functor $\mathrm{grad}(f,p)$ yields an element of $\mathcal{R}^3$ and is defined as follows:

(Def. 7) $\mathrm{grad}(f,p) = \mathrm{partdiff}(f,p,1)\cdot e_1 + \mathrm{partdiff}(f,p,2)\cdot e_2 + \mathrm{partdiff}(f,p,3)\cdot e_3$.

We now state several propositions:

(37) $\mathrm{grad}(f,p) = [\mathrm{partdiff}(f,p,1), \mathrm{partdiff}(f,p,2), \mathrm{partdiff}(f,p,3)]$.

(38) Suppose that

(i) f is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3, and

(ii) g is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3.

Then $\mathrm{grad}(f+g,p) = \mathrm{grad}(f,p) + \mathrm{grad}(g,p)$.

(39) Suppose that

(i) f is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3, and

(ii) g is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3.

Then $\mathrm{grad}(f - g, p) = \mathrm{grad}(f, p) - \mathrm{grad}(g, p)$.

(40) Suppose that

(i) f is partially differentiable in p w.r.t. coordinate number 1,

(ii) f is partially differentiable in p w.r.t. coordinate number 2, and

(iii) f is partially differentiable in p w.r.t. coordinate number 3.

Then $\mathrm{grad}(r\, f, p) = r \cdot \mathrm{grad}(f, p)$.

(41) Suppose that

(i) f is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3, and

(ii) g is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3.

Then $\mathrm{grad}(s\, f + t\, g, p) = s \cdot \mathrm{grad}(f, p) + t \cdot \mathrm{grad}(g, p)$.

(42) Suppose that

(i) f is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3, and

(ii) g is partially differentiable in p w.r.t. coordinate number 1, partially differentiable in p w.r.t. coordinate number 2, and partially differentiable in p w.r.t. coordinate number 3.

Then $\mathrm{grad}(s\, f - t\, g, p) = s \cdot \mathrm{grad}(f, p) - t \cdot \mathrm{grad}(g, p)$.

(43) If f is total and constant, then $\mathrm{grad}(f, p) = 0_{\mathcal{E}^3_{\mathrm{T}}}$.

Let a be an element of $\mathcal{R}^3$. The functor unitvector a yields an element of $\mathcal{R}^3$ and is defined as follows:

(Def. 8) $\mathrm{unitvector}\, a = [\frac{a(1)}{\sqrt{a(1)^2+a(2)^2+a(3)^2}}, \frac{a(2)}{\sqrt{a(1)^2+a(2)^2+a(3)^2}}, \frac{a(3)}{\sqrt{a(1)^2+a(2)^2+a(3)^2}}]$.

Let f be a partial function from $\mathcal{R}^3$ to $\mathbb{R}$ and let p, a be elements of $\mathcal{R}^3$. The functor Directiondiff(f, p, a) yielding a real number is defined by:

(Def. 9) $\mathrm{Directiondiff}(f, p, a) = \mathrm{partdiff}(f, p, 1) \cdot (\mathrm{unitvector}\, a)(1) + \mathrm{partdiff}(f, p, 2) \cdot (\mathrm{unitvector}\, a)(2) + \mathrm{partdiff}(f, p, 3) \cdot (\mathrm{unitvector}\, a)(3)$.

The following propositions are true:

(44) If $a = \langle x_0, y_0, z_0 \rangle$, then $\mathrm{Directiondiff}(f, p, a) = \frac{\mathrm{partdiff}(f,p,1) \cdot x_0}{\sqrt{{x_0}^2+{y_0}^2+{z_0}^2}} + \frac{\mathrm{partdiff}(f,p,2) \cdot y_0}{\sqrt{{x_0}^2+{y_0}^2+{z_0}^2}} + \frac{\mathrm{partdiff}(f,p,3) \cdot z_0}{\sqrt{{x_0}^2+{y_0}^2+{z_0}^2}}$.

(45) $\mathrm{Directiondiff}(f, p, a) = |(\mathrm{grad}(f, p), \mathrm{unitvector}\, a)|$.

Let f_1, f_2, f_3 be partial functions from $\mathcal{R}^3$ to $\mathbb{R}$ and let p be an element of $\mathcal{R}^3$. The functor $\mathrm{curl}(f_1, f_2, f_3, p)$ yields an element of $\mathcal{R}^3$ and is defined by:

(Def. 10) $\text{curl}(f_1, f_2, f_3, p) = (\text{partdiff}(f_3, p, 2) - \text{partdiff}(f_2, p, 3)) \cdot e_1 + (\text{partdiff}(f_1, p, 3) - \text{partdiff}(f_3, p, 1)) \cdot e_2 + (\text{partdiff}(f_2, p, 1) - \text{partdiff}(f_1, p, 2)) \cdot e_3$.

Next we state the proposition

(46) $\text{curl}(f_1, f_2, f_3, p) = [\text{partdiff}(f_3, p, 2) - \text{partdiff}(f_2, p, 3), \text{partdiff}(f_1, p, 3) - \text{partdiff}(f_3, p, 1), \text{partdiff}(f_2, p, 1) - \text{partdiff}(f_1, p, 2)]$.

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