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Partial Differentiation, Differentiation and Continuity on n-Dimensional Real Normed Linear Spaces

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Summary. In this article, we aim to prove the characterization of differentiation by means of partial differentiation for vector-valued functions on n-dimensional real normed linear spaces (refer to [15] and [16]).

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The notation and terminology used in this paper have been introduced in the following papers: [2], [7], [1], [3], [4], [5], [17], [11], [13], [6], [9], [14], [10], [8], [12], and [18].

One can prove the following propositions:

(1) Let n, i be elements of $\mathbb{N}$, q be an element of $\mathcal{R}^n$, and p be a point of $\mathcal{E}^n_{\mathrm{T}}$. If $i \in \operatorname{Seg} n$ and $q = p$, then $|p_i| \leq |q|$.

(2) For every real number x and for every element v_1 of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $v_1 = \langle x \rangle$ holds $\|v_1\| = |x|$.

(3) Let n be a non empty element of $\mathbb{N}$, x be a point of $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and i be an element of $\mathbb{N}$. If $1 \leq i \leq n$, then $\|(\mathrm{Proj}(i, n))(x)\| \leq \|x\|$.

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(4) For every non empty element n of $\mathbb{N}$ and for every element x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every element i of $\mathbb{N}$ holds $\|(\mathrm{Proj}(i,n))(x)\| = |(\mathrm{proj}(i,n))(x)|$.

(5) Let n be a non empty element of $\mathbb{N}$, x be an element of $\mathcal{R}^n$, and i be an element of $\mathbb{N}$. If $1 \leq i \leq n$, then $|(\mathrm{proj}(i,n))(x)| \leq |x|$.

(6) Let m, n be non empty elements of $\mathbb{N}$, s be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq n$. Then $\mathrm{Proj}(i,n)$ is a bounded linear operator from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and $(\mathrm{BdLinOpsNorm}(\langle \mathcal{E}^n, \|\cdot\| \rangle, \langle \mathcal{E}^1, \|\cdot\| \rangle))(\mathrm{Proj}(i,n)) \leq 1$.

(7) Let m, n be non empty elements of $\mathbb{N}$, s be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq n$. Then

(i) $\mathrm{Proj}(i,n) \cdot s$ is a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and

(ii) $(\mathrm{BdLinOpsNorm}(\langle \mathcal{E}^m, \|\cdot\| \rangle, \langle \mathcal{E}^1, \|\cdot\| \rangle))(\mathrm{Proj}(i,n) \cdot s) \leq (\mathrm{BdLinOpsNorm}(\langle \mathcal{E}^n, \|\cdot\| \rangle, \langle \mathcal{E}^1, \|\cdot\| \rangle))(\mathrm{Proj}(i,n)) \cdot (\mathrm{BdLinOpsNorm}(\langle \mathcal{E}^m, \|\cdot\| \rangle, \langle \mathcal{E}^n, \|\cdot\| \rangle))(s)$.

(8) For every non empty element n of $\mathbb{N}$ and for every element i of $\mathbb{N}$ holds $\mathrm{Proj}(i,n)$ is homogeneous.

(9) Let n be a non empty element of $\mathbb{N}$, x be an element of $\mathcal{R}^n$, r be a real number, and i be an element of $\mathbb{N}$. Then $(\mathrm{proj}(i,n))(r \cdot x) = r \cdot (\mathrm{proj}(i,n))(x)$.

(10) Let n be a non empty element of $\mathbb{N}$, x, y be elements of $\mathcal{R}^n$, and i be an element of $\mathbb{N}$. Then $(\mathrm{proj}(i,n))(x+y) = (\mathrm{proj}(i,n))(x) + (\mathrm{proj}(i,n))(y)$.

(11) Let n be a non empty element of $\mathbb{N}$, x, y be points of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. Then $(\mathrm{Proj}(i,n))(x-y) = (\mathrm{Proj}(i,n))(x) - (\mathrm{Proj}(i,n))(y)$.

(12) Let n be a non empty element of $\mathbb{N}$, x, y be elements of $\mathcal{R}^n$, and i be an element of $\mathbb{N}$. Then $(\mathrm{proj}(i,n))(x-y) = (\mathrm{proj}(i,n))(x) - (\mathrm{proj}(i,n))(y)$.

(13) Let m, n be non empty elements of $\mathbb{N}$, s be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, i be an element of $\mathbb{N}$, and s_1 be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$. If $s_1 = \mathrm{Proj}(i,n) \cdot s$ and $1 \leq i \leq n$, then $\|s_1\| \leq \|s\|$.

(14) Let m, n be non empty elements of $\mathbb{N}$, s, t be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, s_1, t_1 be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. If $s_1 = \mathrm{Proj}(i,n) \cdot s$ and $t_1 = \mathrm{Proj}(i,n) \cdot t$ and $1 \leq i \leq n$, then $\|s_1 - t_1\| \leq \|s - t\|$.

(15) Let K be a real number, n be an element of $\mathbb{N}$, and s be an element of $\mathcal{R}^n$. Suppose that for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds

$|s(i)| \leq K$. Then $|s| \leq n \cdot K$.

(16) Let K be a real number, n be a non empty element of $\mathbb{N}$, and s be an element of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose that for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $\|(\mathrm{Proj}(i, n))(s)\| \leq K$. Then $\|s\| \leq n \cdot K$.

(17) Let K be a real number, n be a non empty element of $\mathbb{N}$, and s be an element of $\mathcal{R}^n$. Suppose that for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $|(\mathrm{proj}(i, n))(s)| \leq K$. Then $|s| \leq n \cdot K$.

(18) Let m, n be non empty elements of $\mathbb{N}$, s be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and K be a real number. Suppose that for every element i of $\mathbb{N}$ and for every point s_1 of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $s_1 = \mathrm{Proj}(i, n) \cdot s$ and $1 \leq i \leq n$ holds $\|s_1\| \leq K$. Then $\|s\| \leq n \cdot K$.

(19) Let m, n be non empty elements of $\mathbb{N}$, s, t be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and K be a real number. Suppose that for every element i of $\mathbb{N}$ and for all points s_1, t_1 of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $s_1 = \mathrm{Proj}(i, n) \cdot s$ and $t_1 = \mathrm{Proj}(i, n) \cdot t$ and $1 \leq i \leq n$ holds $\|s_1 - t_1\| \leq K$. Then $\|s - t\| \leq n \cdot K$.

(20) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, X be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq m$ and X is open. Then the following statements are equivalent

(i) f is partially differentiable on X w.r.t. i and $f{\restriction^i}X$ is continuous on X,

(ii) for every element j of $\mathbb{N}$ such that $1 \leq j \leq n$ holds $\mathrm{Proj}(j, n) \cdot f$ is partially differentiable on X w.r.t. i and $\mathrm{Proj}(j, n) \cdot f{\restriction^i}X$ is continuous on X.

(21) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and X be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose X is open. Then f is differentiable on X and $f'_{\restriction X}$ is continuous on X if and only if for every element j of $\mathbb{N}$ such that $1 \leq j \leq n$ holds $\mathrm{Proj}(j, n) \cdot f$ is differentiable on X and $(\mathrm{Proj}(j, n) \cdot f)'_{\restriction X}$ is continuous on X.

(22) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and X be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose X is open. Then for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\restriction^i}X$ is continuous on X if and only if f is differentiable on X and $f'_{\restriction X}$ is continuous on X.

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Differentiable Functions into Real Normed Spaces

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Summary. In this article, we formalize the differentiability of functions from the set of real numbers into a normed vector space [14].

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The notation and terminology used here have been introduced in the following papers: [12], [2], [3], [7], [9], [11], [1], [4], [10], [13], [6], [17], [18], [15], [8], [16], [19], and [5].

For simplicity, we adopt the following rules: F denotes a non trivial real normed space, G denotes a real normed space, X denotes a set, x, x_0, r, p denote real numbers, n, k denote elements of $\mathbb{N}$, Y denotes a subset of $\mathbb{R}$, Z denotes an open subset of $\mathbb{R}$, s_1 denotes a sequence of real numbers, s_2 denotes a sequence of G, f, f_1, f_2 denote partial functions from $\mathbb{R}$ to the carrier of F, h denotes a convergent to 0 sequence of real numbers, and c denotes a constant sequence of real numbers.

We now state two propositions:

(1) If for every n holds $\|s_2(n)\| \leq s_1(n)$ and s_1 is convergent and $\lim s_1 = 0$, then s_2 is convergent and $\lim s_2 = 0_G$.

(2) $(s_1 \uparrow k)\,(s_2 \uparrow k) = (s_1\, s_2) \uparrow k$.

Let us consider F and let I_1 be a partial function from $\mathbb{R}$ to the carrier of F. We say that I_1 is rest-like if and only if:

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(Def. 1) I_1 is total and for every h holds $h^{-1}\,(I_{1*}h)$ is convergent and $\lim(h^{-1}\,(I_{1*}h)) = 0_F$.

Let us consider F. One can check that there exists a partial function from $\mathbb{R}$ to the carrier of F which is rest-like. Let us consider F. A rest of F is a rest-like partial function from $\mathbb{R}$ to the carrier of F. Let us consider F and let I_1 be a function from $\mathbb{R}$ into the carrier of F. We say that I_1 is linear if and only if:

(Def. 2) There exists a point r of F such that for every real number p holds $I_1(p) = p \cdot r$.

Let us consider F. Note that there exists a function from $\mathbb{R}$ into the carrier of F which is linear. Let us consider F. A linear of F is a linear function from $\mathbb{R}$ into the carrier of F.

We use the following convention: R, R_1, R_2 denote rests of F and L, L_1, L_2 denote linears of F.

The following propositions are true:

(3) $L_1 + L_2$ is a linear of F and $L_1 - L_2$ is a linear of F.

(4) $r\,L$ is a linear of F.

(5) Let h_1, h_2 be partial functions from $\mathbb{R}$ to the carrier of F and s_2 be a sequence of real numbers. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1 + h_2)_* s_2 = (h_{1*}s_2) + (h_{2*}s_2)$ and $(h_1 - h_2)_* s_2 = (h_{1*}s_2) - (h_{2*}s_2)$.

(6) Let h_1, h_2 be partial functions from $\mathbb{R}$ to the carrier of F and s_2 be a sequence of real numbers. If h_1 is total and h_2 is total, then $(h_1 + h_2)_* s_2 = (h_{1*}s_2) + (h_{2*}s_2)$ and $(h_1 - h_2)_* s_2 = (h_{1*}s_2) - (h_{2*}s_2)$.

(7) $R_1 + R_2$ is a rest of F and $R_1 - R_2$ is a rest of F.

(8) $r\,R$ is a rest of F.

Let us consider F, f and let x_0 be a real number. We say that f is differentiable in x_0 if and only if:

(Def. 3) There exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and there exist L, R such that for every x such that $x \in N$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x - x_0}$.

Let us consider F, f and let x_0 be a real number. Let us assume that f is differentiable in x_0. The functor $f'(x_0)$ yielding a point of F is defined by the condition (Def. 4).

(Def. 4) There exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and there exist L, R such that $f'(x_0) = L(1)$ and for every x such that $x \in N$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x - x_0}$.

Let us consider F, f, X. We say that f is differentiable on X if and only if:

(Def. 5) $X \subseteq \operatorname{dom} f$ and for every x such that $x \in X$ holds $f{\restriction}X$ is differentiable in x.

The following propositions are true:

(9) If f is differentiable on X, then X is a subset of $\mathbb{R}$.

(10) f is differentiable on Z iff $Z \subseteq \operatorname{dom} f$ and for every x such that $x \in Z$ holds f is differentiable in x.

(11) If f is differentiable on Y, then Y is open.

Let us consider F, f, X. Let us assume that f is differentiable on X. The functor $f'_{\restriction X}$ yields a partial function from $\mathbb{R}$ to the carrier of F and is defined by:

(Def. 6) $\operatorname{dom}(f'_{\restriction X}) = X$ and for every x such that $x \in X$ holds $f'_{\restriction X}(x) = f'(x)$.

Next we state a number of propositions:

(12) Suppose $Z \subseteq \operatorname{dom} f$ and there exists a point r of F such that $\operatorname{rng} f = \{r\}$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})_x = 0_F$.

(13) Let x_0 be a real number and N be a neighbourhood of x_0. Suppose f is differentiable in x_0 and $N \subseteq \operatorname{dom} f$. Let given h, c. Suppose $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}\left((f_*(h + c)) - (f_*c)\right)$ is convergent and $f'(x_0) = \lim(h^{-1}\left((f_*(h + c)) - (f_*c)\right))$.

(14) If f_1 is differentiable in x_0 and f_2 is differentiable in x_0, then $f_1 + f_2$ is differentiable in x_0 and $(f_1 + f_2)'(x_0) = {f_1}'(x_0) + {f_2}'(x_0)$.

(15) If f_1 is differentiable in x_0 and f_2 is differentiable in x_0, then $f_1 - f_2$ is differentiable in x_0 and $(f_1 - f_2)'(x_0) = {f_1}'(x_0) - {f_2}'(x_0)$.

(16) For every real number r such that f is differentiable in x_0 holds $r\,f$ is differentiable in x_0 and $(r\,f)'(x_0) = r \cdot f'(x_0)$.

(17) Suppose $Z \subseteq \operatorname{dom}(f_1 + f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 + f_2)'_{\restriction Z}(x) = {f_1}'(x) + {f_2}'(x)$.

(18) Suppose $Z \subseteq \operatorname{dom}(f_1 - f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 - f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 - f_2)'_{\restriction Z}(x) = {f_1}'(x) - {f_2}'(x)$.

(19) Suppose $Z \subseteq \operatorname{dom}(r\,f)$ and f is differentiable on Z. Then $r\,f$ is differentiable on Z and for every x such that $x \in Z$ holds $(r\,f)'_{\restriction Z}(x) = r \cdot f'(x)$.

(20) If $Z \subseteq \operatorname{dom} f$ and $f{\restriction}Z$ is constant, then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\restriction Z}(x) = 0_F$.

(21) Let r, p be points of F and given Z, f. Suppose $Z \subseteq \operatorname{dom} f$ and for every x such that $x \in Z$ holds $f_x = x \cdot r + p$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\restriction Z}(x) = r$.

(22) For every real number x_0 such that f is differentiable in x_0 holds f is continuous in x_0.

(23) If f is differentiable on X, then $f{\restriction}X$ is continuous.

(24) If f is differentiable on X and $Z \subseteq X$, then f is differentiable on Z.

(25) There exists a rest R of F such that $R_0 = 0_F$ and R is continuous in 0.

Let us consider F and let f be a partial function from $\mathbb{R}$ to the carrier of F. We say that f is differentiable if and only if:

(Def. 7) f is differentiable on $\operatorname{dom} f$.

Let us consider F. One can check that there exists a function from $\mathbb{R}$ into the carrier of F which is differentiable. We now state the proposition

(26) Let f be a differentiable partial function from $\mathbb{R}$ to the carrier of F. If $Z \subseteq \operatorname{dom} f$, then f is differentiable on Z.

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Conway's Games and Some of their Basic Properties

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Summary. We formulate a few basic concepts of J. H. Conway's theory of games based on his book [6]. This is a first step towards formalizing Conway's theory of numbers into Mizar, which is an approach to proving the existence of a FIELD (i.e., a proper class that satisfies the axioms of a real-closed field) that includes the reals and ordinals, thus providing a uniform, independent and simple approach to these two constructions that does not go via the rational numbers and hence does for example not need the notion of a quotient field.

In this first article on Conway's games, we provide a definition of games, their birthdays (or ranks), their trees (a notion which is not in Conway's book, but is useful as a tool), their negates and their signs, together with some elementary properties of these notions. If one is interested only in Conway's numbers, it would have been easier to define them directly, but going via the notion of a game is a more general approach in the sense that a number is a special instance of a game and that there is a rich theory of games that are not numbers.

The main obstacle in formulating these topics in Mizar is that all definitions are highly recursive, which is not entirely simple to translate into the Mizar language. For example, according to Conway's definition, a game is an object consisting of left and right options which are themselves games, and this is by definition the only way to construct a game. This cannot directly be translated into Mizar, but a theorem is included in the article which proves that our definition is equivalent to Conway's.

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The terminology and notation used here have been introduced in the following articles: [1], [4], [7], [5], [2], [3], [9], and [8].

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1. Construction of Days

We follow the rules: x, z, s are sets, α, β are ordinal numbers, and n is a natural number.

We introduce lefts-rights which are systems

$\langle$ left options, right options $\rangle$,

where the left options and the right options constitute sets.

The functor $\mathbf{0}$ is defined by:

(Def. 1) $\mathbf{0} = \langle \emptyset, \emptyset \rangle$.

One can verify that there exists a left-right which is strict.

Let us consider α. The functor ConwayDay α yields a set and is defined by the condition (Def. 2).

(Def. 2) There exists a transfinite sequence f such that $\alpha \in \operatorname{dom} f$ and $f(\alpha) =$ ConwayDay α and for every β such that $\beta \in \operatorname{dom} f$ holds $f(\beta) = \{\langle x, y \rangle :$ x ranges over subsets of $\bigcup \operatorname{rng}(f \restriction \beta)$, y ranges over subsets of $\bigcup \operatorname{rng}(f \restriction \beta)\}$.

We now state three propositions:

(1) $z \in$ ConwayDay α if and only if there exists a strict left-right w such that $z = w$ and for every x such that $x \in$ (the left options of w) $\cup$ (the right options of w) there exists β such that $\beta \in \alpha$ and $x \in$ ConwayDay β.

(2) ConwayDay $0 = \{\mathbf{0}\}$.

(3) If $\alpha \subseteq \beta$, then ConwayDay $\alpha \subseteq$ ConwayDay β.

Let us consider α. Note that ConwayDay α is non empty.

2. Games

Let us consider x. We say that x is Conway game-like if and only if:

(Def. 3) There exists α such that $x \in$ ConwayDay α.

Let us consider α. Note that every element of ConwayDay α is Conway game-like.

Let us observe that $\mathbf{0}$ is Conway game-like.

One can check that there exists a left-right which is Conway game-like and strict and there exists a set which is Conway game-like.

A Conway game is a Conway game-like set.

$\mathbf{0}$ is an element of ConwayDay 0.

The element $\mathbf{1}$ of ConwayDay 1 is defined by:

(Def. 4) $\mathbf{1} = \langle \{\mathbf{0}\}, \emptyset \rangle$.

The element $*$ of ConwayDay 1 is defined as follows:

(Def. 5) $* = \langle \{\mathbf{0}\}, \{\mathbf{0}\} \rangle$.

In the sequel g, g_0, g_1, g_2, g_3, g_4, g_5, g_6 are Conway games.

We now state the proposition

(4) g is a strict left-right.

One can verify that every left-right which is Conway game-like is also strict.

Let us consider g. The left options of g is defined as follows:

(Def. 6) There exists a left-right w such that $g = w$ and the left options of $g =$ the left options of w.

The right options of g is defined by:

(Def. 7) There exists a left-right w such that $g = w$ and the right options of $g =$ the right options of w.

Let us consider g. The options of g is defined by:

(Def. 8) The options of $g =$ (the left options of g) $\cup$ (the right options of g).

Next we state the proposition

(5) $g_1 = g_2$ if and only if the following conditions are satisfied:

(i) the left options of $g_1 =$ the left options of g_2, and

(ii) the right options of $g_1 =$ the right options of g_2.

One can verify the following observations:

* the left options of $\mathbf{0}$ is empty,
* the right options of $\mathbf{0}$ is empty, and
* the right options of $\mathbf{1}$ is empty.

Next we state four propositions:

(6) $g = \mathbf{0}$ iff the options of $g = \emptyset$.

(7) $x \in$ the left options of $\mathbf{1}$ iff $x = \mathbf{0}$.

(8)(i) $x \in$ the options of $*$ iff $x = \mathbf{0}$,

(ii) $x \in$ the left options of $*$ iff $x = \mathbf{0}$, and

(iii) $x \in$ the right options of $*$ iff $x = \mathbf{0}$.

(9) $g \in \text{ConwayDay}\,\alpha$ iff for every x such that $x \in$ the options of g there exists β such that $\beta \in \alpha$ and $x \in \text{ConwayDay}\,\beta$.

Let g be a set. Let us assume that g is a Conway game. The functor ConwayRank g yields an ordinal number and is defined as follows:

(Def. 9) $g \in \text{ConwayDay}\,\text{ConwayRank}\,g$ and for every β such that $\beta \in \text{ConwayRank}\,g$ holds $g \notin \text{ConwayDay}\,\beta$.

One can prove the following propositions:

(10) If $g \in \text{ConwayDay}\,\alpha$ and $x \in$ the options of g, then $x \in \text{ConwayDay}\,\alpha$.

(11) If $g \in \text{ConwayDay}\,\alpha$ and if $x \in$ the left options of g or $x \in$ the right options of g, then $x \in \text{ConwayDay}\,\alpha$.

(12) $g \in \text{ConwayDay}\,\alpha$ iff $\text{ConwayRank}\,g \subseteq \alpha$.

(13) $\text{ConwayRank}\,g \in \alpha$ iff there exists β such that $\beta \in \alpha$ and $g \in \text{ConwayDay}\,\beta$.

(14) If $g_3 \in$ the options of g, then $\text{ConwayRank}\,g_3 \in \text{ConwayRank}\,g$.

(15) If $g_3 \in$ the left options of g or $g_3 \in$ the right options of g, then ConwayRank $g_3 \in$ ConwayRank g.

(16) $g \notin$ the options of g.

(17) If $x \in$ the options of g, then x is a Conway game-like left-right.

(18) If $x \in$ the left options of g or $x \in$ the right options of g, then x is a Conway game-like left-right.

(19) Let w be a strict left-right. Then w is a Conway game if and only if for every z such that $z \in$ (the left options of w) $\cup$ (the right options of w) holds z is a Conway game.

3. Schemes of Induction

In this article we present several logical schemes. The scheme *ConwayGameMinTot* concerns a unary predicate $\mathcal{P}$, and states that:

There exists g such that $\mathcal{P}[g]$ and for every g_1 such that ConwayRank $g_1 \in$ ConwayRank g holds not $\mathcal{P}[g_1]$

provided the following condition is met:

- There exists g such that $\mathcal{P}[g]$.

The scheme *ConwayGameMin* concerns a unary predicate $\mathcal{P}$, and states that:

There exists g such that $\mathcal{P}[g]$ and for every g_3 such that $g_3 \in$ the options of g holds not $\mathcal{P}[g_3]$

provided the parameters satisfy the following condition:

- There exists g such that $\mathcal{P}[g]$.

The scheme *ConwayGameInd* concerns a unary predicate $\mathcal{P}$, and states that:

For every g holds $\mathcal{P}[g]$

provided the following condition is met:

- For every g such that for every g_3 such that $g_3 \in$ the options of g holds $\mathcal{P}[g_3]$ holds $\mathcal{P}[g]$.

4. Tree of a Game

Let f be a function. We say that f is Conway game-valued if and only if:

(Def. 10) For every x such that $x \in \operatorname{dom} f$ holds $f(x)$ is a Conway game.

Let us consider g. One can verify that $\langle g \rangle$ is Conway game-valued.

Let us mention that there exists a finite sequence which is Conway game-valued and non empty.

Let f be a non empty finite sequence. Observe that every element of dom f is natural and non empty.

Let f be a Conway game-valued non empty function and let x be an element of dom f. Note that $f(x)$ is Conway game-like.

Let f be a Conway game-valued non empty finite sequence. We say that f is Conway game chain-like if and only if:

(Def. 11) For every element n of $\operatorname{dom} f$ such that $n > 1$ holds $f(n-1) \in$ the options of $f(n)$.

One can prove the following proposition

(20) For every finite sequence f and for every n such that $n \in \operatorname{dom} f$ and $n > 1$ holds $n - 1 \in \operatorname{dom} f$.

Let us consider g. Observe that $\langle g \rangle$ is Conway game chain-like.

Let us observe that there exists a Conway game-valued non empty finite sequence which is Conway game chain-like.

A Conway game chain is a Conway game chain-like Conway game-valued non empty finite sequence.

Next we state three propositions:

(21) For every Conway game chain f and for all elements n, m of $\operatorname{dom} f$ such that $n < m$ holds ConwayRank $f(n) \in$ ConwayRank $f(m)$.

(22) For every Conway game chain f and for all elements n, m of $\operatorname{dom} f$ such that $n \leq m$ holds ConwayRank $f(n) \subseteq$ ConwayRank $f(m)$.

(23) For every Conway game chain f such that $f(\operatorname{len} f) \in$ ConwayDay α holds $f(1) \in$ ConwayDay α.

Let us consider g. The tree of g yields a set and is defined as follows:

(Def. 12) $z \in$ the tree of g iff there exists a Conway game chain f such that $f(1) = z$ and $f(\operatorname{len} f) = g$.

Let us consider g. Observe that the tree of g is non empty.

Let us consider g. The proper tree of g yielding a subset of the tree of g is defined by:

(Def. 13) The proper tree of $g =$ (the tree of g) $\setminus \{g\}$.

We now state the proposition

(24) $g \in$ the tree of g.

Let us consider α and let g be an element of ConwayDay α. Then the tree of g is a subset of ConwayDay α.

Let us consider g. One can verify that every element of the tree of g is Conway game-like.

The following propositions are true:

(25) For every Conway game chain f and for every non empty natural number n holds $f \restriction n$ is a Conway game chain.

(26) Let f_1, f_2 be Conway game chains. Given g such that $g = f_2(1)$ and $f_1(\operatorname{len} f_1) \in$ the options of g. Then $f_1 \frown f_2$ is a Conway game chain.

(27) $x \in$ the tree of g iff $x = g$ or there exists g_3 such that $g_3 \in$ the options of g and $x \in$ the tree of g_3.

(28) If $g_3 \in$ the tree of g, then $g_3 = g$ or ConwayRank $g_3 \in$ ConwayRank g.

(29) If $g_3 \in$ the tree of g, then ConwayRank $g_3 \subseteq$ ConwayRank g.

(30) For every set s such that $g \in s$ and for every g_1 such that $g_1 \in s$ holds the options of $g_1 \subseteq s$ holds the tree of $g \subseteq s$.

(31) If $g_1 \in$ the tree of g_2, then the tree of $g_1 \subseteq$ the tree of g_2.

(32) If $g_1 \in$ the tree of g_2, then the proper tree of $g_1 \subseteq$ the proper tree of g_2.

(33) The options of $g \subseteq$ the proper tree of g.

(34) The options of $g \subseteq$ the tree of g.

(35) If $g_1 \in$ the proper tree of g_2, then the tree of $g_1 \subseteq$ the proper tree of g_2.

(36) If $g_3 \in$ the options of g, then the tree of $g_3 \subseteq$ the proper tree of g.

(37) The tree of $\mathbf{0} = \{\mathbf{0}\}$.

(38) $\mathbf{0} \in$ the tree of g.

The scheme *ConwayGameMin2* concerns a unary predicate $\mathcal{P}$, and states that:

> There exists g such that $\mathcal{P}[g]$ and for every g_3 such that $g_3 \in$ the proper tree of g holds not $\mathcal{P}[g_3]$

provided the following condition is met:

- There exists g such that $\mathcal{P}[g]$.

5. Scheme about Definability of Functions by Recursion

Now we present two schemes. The scheme *Func1RecUniq* deals with a binary functor $\mathcal{F}$ yielding a set, and states that:

> Let given g and f_1, f_2 be functions. Suppose that
> (i) dom $f_1 =$ the tree of g,
> (ii) dom $f_2 =$ the tree of g,
> (iii) for every g_1 such that $g_1 \in \operatorname{dom} f_1$ holds $f_1(g_1) = \mathcal{F}(g_1, f_1 \restriction$the proper tree of $g_1)$, and
> (iv) for every g_1 such that $g_1 \in \operatorname{dom} f_2$ holds $f_2(g_1) = \mathcal{F}(g_1, f_2 \restriction$the proper tree of $g_1)$.
> Then $f_1 = f_2$

for all values of the parameter.

The scheme *Func1RecEx* deals with a binary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a function f such that dom $f =$ the tree of g and for every g_1 such that $g_1 \in \operatorname{dom} f$ holds $f(g_1) = \mathcal{F}(g_1, f \restriction$the proper tree of $g_1)$

for all values of the parameter.

6. The Negative and Signs

Let us consider g. The functor $-g$ is defined by the condition (Def. 14).

(Def. 14) There exists a function f such that

(i) $\operatorname{dom} f =$ the tree of g,

(ii) $-g = f(g)$, and

(iii) for every g_1 such that $g_1 \in \operatorname{dom} f$ holds $f(g_1) = \langle\{f(g_4); g_4$ ranges over elements of the right options of g_1: the right options of $g_1 \neq \emptyset\}, \{f(g_7); g_7$ ranges over elements of the left options of g_1: the left options of $g_1 \neq \emptyset\}\rangle$.

Let us consider g. One can check that $-g$ is Conway game-like.

We now state three propositions:

(39)(i) For every x holds $x \in$ the left options of $-g$ iff there exists g_4 such that $g_4 \in$ the right options of g and $x = -g_4$, and

(ii) for every x holds $x \in$ the right options of $-g$ iff there exists g_7 such that $g_7 \in$ the left options of g and $x = -g_7$.

(40) $--g = g$.

(41)(i) $g_3 \in$ the left options of $-g$ iff $-g_3 \in$ the right options of g,

(ii) $g_3 \in$ the left options of g iff $-g_3 \in$ the right options of $-g$,

(iii) $g_3 \in$ the right options of $-g$ iff $-g_3 \in$ the left options of g, and

(iv) $g_3 \in$ the right options of g iff $-g_3 \in$ the left options of $-g$.

Let us consider g. We say that g is non-negative if and only if the condition (Def. 15) is satisfied.

(Def. 15) There exists s such that

(i) $g \in s$, and

(ii) for every g_1 such that $g_1 \in s$ and for every g_4 such that $g_4 \in$ the right options of g_1 there exists g_8 such that $g_8 \in$ the left options of g_4 and $g_8 \in s$.

Let us consider g. We say that g is non-positive if and only if:

(Def. 16) $-g$ is non-negative.

Let us consider g. We say that g is zero if and only if:

(Def. 17) g is non-negative and non-positive.

We say that g is fuzzy if and only if:

(Def. 18) g is not non-negative and g is not non-positive.

Let us consider g. We say that g is positive if and only if:

(Def. 19) g is non-negative and g is not zero.

We say that g is negative if and only if:

(Def. 20) g is non-positive and g is not zero.

One can verify the following observations:

* every Conway game which is zero is also non-negative and non-positive,
* every Conway game which is non-positive and non-negative is also zero,

* every Conway game which is negative is also non-positive and non zero,
* every Conway game which is non-positive and non zero is also negative,
* every Conway game which is positive is also non-negative and non zero,
* every Conway game which is non-negative and non zero is also positive,
* every Conway game which is fuzzy is also non non-negative and non non-positive, and
* every Conway game which is non non-negative and non non-positive is also fuzzy.

One can prove the following propositions:

(42) g is zero, or positive, or negative, or fuzzy.

(43) g is non-negative if and only if for every g_4 such that $g_4 \in$ the right options of g there exists g_8 such that $g_8 \in$ the left options of g_4 and g_8 is non-negative.

(44) g is non-positive if and only if for every g_7 such that $g_7 \in$ the left options of g there exists g_6 such that $g_6 \in$ the right options of g_7 and g_6 is non-positive.

(45)(i) g is non-negative iff for every g_4 such that $g_4 \in$ the right options of g holds g_4 is fuzzy or positive, and

(ii) g is non-positive iff for every g_7 such that $g_7 \in$ the left options of g holds g_7 is fuzzy or negative.

(46) g is fuzzy if and only if the following conditions are satisfied:

(i) there exists g_7 such that $g_7 \in$ the left options of g and g_7 is non-negative, and

(ii) there exists g_4 such that $g_4 \in$ the right options of g and g_4 is non-positive.

(47) g is zero if and only if the following conditions are satisfied:

(i) for every g_7 such that $g_7 \in$ the left options of g holds g_7 is fuzzy or negative, and

(ii) for every g_4 such that $g_4 \in$ the right options of g holds g_4 is fuzzy or positive.

(48) g is positive if and only if the following conditions are satisfied:

(i) for every g_4 such that $g_4 \in$ the right options of g holds g_4 is fuzzy or positive, and

(ii) there exists g_7 such that $g_7 \in$ the left options of g and g_7 is non-negative.

(49) g is negative if and only if the following conditions are satisfied:

(i) for every g_7 such that $g_7 \in$ the left options of g holds g_7 is fuzzy or negative, and

(ii) there exists g_4 such that $g_4 \in$ the right options of g and g_4 is non-positive.

One can check that $\mathbf{0}$ is zero.

Let us observe that $\mathbf{1}$ is positive and $*$ is fuzzy.

One can verify the following observations:

* there exists a Conway game which is zero,
* there exists a Conway game which is positive, and
* there exists a Conway game which is fuzzy.

Let g be a non-positive Conway game. Note that $-g$ is non-negative.

Let g be a non-negative Conway game. Note that $-g$ is non-positive.

Let g be a positive Conway game. One can verify that $-g$ is negative.

Let us note that there exists a Conway game which is negative.

Let g be a negative Conway game. Note that $-g$ is positive.

Let g be a fuzzy Conway game. Note that $-g$ is fuzzy.

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Veblen Hierarchy

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Summary. The Veblen hierarchy is an extension of the construction of epsilon numbers (fixpoints of the exponential map: $\omega^\epsilon = \epsilon$). It is a collection φ_α of the Veblen Functions where $\varphi_0(\beta) = \omega^\beta$ and $\varphi_1(\beta) = \epsilon_\beta$. The sequence of fixpoints of φ_1 function form φ_2, etc. For a limit non empty ordinal λ the function φ_λ is the sequence of common fixpoints of all functions φ_α where $\alpha < \lambda$.

The Mizar formalization of the concept cannot be done directly as the Veblen functions are classes (not (small) sets). It is done with use of universal sets (Tarski classes). Namely, we define the Veblen functions in a given universal set and $\varphi_\alpha(\beta)$ as a value of Veblen function from the smallest universal set including α and β.

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The papers [16], [18], [2], [5], [14], [13], [9], [10], [15], [3], [4], [1], [8], [11], [19], [17], [6], [7], and [12] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following convention: α, β, γ, δ denote ordinal numbers, λ denotes a non empty limit ordinal ordinal number, A denotes a non empty ordinal number, e denotes an element of A, X, Y, x, y denote sets, and n denotes a natural number.

Next we state several propositions:

(1) Let φ be a function. Suppose φ is an isomorphism between $\subseteq_X$ and $\subseteq_Y$. Let given x, y. If x, $y \in X$, then $x \subseteq y$ iff $\varphi(x) \subseteq \varphi(y)$.

(2) Let X, Y be ordinal-membered sets and φ be a function. Suppose φ is an isomorphism between $\subseteq_X$ and $\subseteq_Y$. Let given x, y. If x, $y \in X$, then $x \in y$ iff $\varphi(x) \in \varphi(y)$.

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(3) If $\langle x, y \rangle \in \subseteq_X$, then $x \subseteq y$.

(4) For all transfinite sequences f_1, f_2 holds $f_1 \subseteq f_1 \frown f_2$.

(5) For all transfinite sequences f_1, f_2 holds $\operatorname{rng}(f_1 \frown f_2) = \operatorname{rng} f_1 \cup \operatorname{rng} f_2$.

(6) $\alpha \subseteq \beta$ iff $\varepsilon_\alpha \subseteq \varepsilon_\beta$.

(7) $\alpha \in \beta$ iff $\varepsilon_\alpha \in \varepsilon_\beta$.

Let X be an ordinal-membered set. Note that $\bigcup X$ is ordinal.

Let φ be an ordinal yielding function. Observe that $\operatorname{rng} \varphi$ is ordinal-membered.

Let us consider α. Note that id_α is transfinite sequence-like and ordinal yielding.

Let us consider α. Observe that id_α is increasing.

Let us consider α. Note that id_α is continuous.

Let us observe that there exists a sequence of ordinal numbers which is non empty, increasing, and continuous.

Let φ be a transfinite sequence. We say that φ is normal if and only if:

(Def. 1) φ is an increasing continuous sequence of ordinal numbers.

Let φ be a sequence of ordinal numbers. Let us observe that φ is normal if and only if:

(Def. 2) φ is increasing and continuous.

One can verify the following observations:

* every transfinite sequence which is normal is also ordinal yielding,
* every sequence of ordinal numbers which is normal is also increasing and continuous, and
* every sequence of ordinal numbers which is increasing and continuous is also normal.

Let us observe that there exists a transfinite sequence which is non empty and normal.

Next we state the proposition

(8) For every sequence φ of ordinal numbers such that φ is non-decreasing holds $\varphi \restriction \alpha$ is non-decreasing.

Let us consider X. The functor $\operatorname{ord-type} X$ yields an ordinal number and is defined by:

(Def. 3) $\operatorname{ord-type} X = \overline{\subseteq_{\mathrm{On}\, X}}$.

Let X be an ordinal-membered set. Then $\operatorname{ord-type} X$ can be characterized by the condition:

(Def. 4) $\operatorname{ord-type} X = \overline{\subseteq_X}$.

Let X be an ordinal-membered set. One can verify that $\subseteq_X$ is well-ordering.

Let E be an empty set. Observe that $\mathrm{On}\, E$ is empty.

Let E be an empty set. One can verify that $\overline{E}$ is empty.

Next we state four propositions:

(9) ord-type $\emptyset = 0$.

(10) ord-type$\{\alpha\} = 1$.

(11) If $\alpha \neq \beta$, then ord-type$\{\alpha, \beta\} = 2$.

(12) ord-type $\alpha = \alpha$.

Let us consider X. The functor numbering X yields a sequence of ordinal numbers and is defined as follows:

(Def. 5) numbering X = the canonical isomorphism between $\subseteq_{\text{ord-type}\,X}$ and $\subseteq_{\text{On}\,X}$.

Next we state four propositions:

(13) dom numbering X = ord-type X and rng numbering X = On X.

(14) For every ordinal-membered set X holds rng numbering $X = X$.

(15) Card dom numbering X = Card On X.

(16) numbering X is an isomorphism between $\subseteq_{\text{ord-type}\,X}$ and $\subseteq_{\text{On}\,X}$.

In the sequel φ denotes a sequence of ordinal numbers.

One can prove the following propositions:

(17) If φ = numbering X and x, $y \in \text{dom}\,\varphi$, then $x \subseteq y$ iff $\varphi(x) \subseteq \varphi(y)$.

(18) If φ = numbering X and x, $y \in \text{dom}\,\varphi$, then $x \in y$ iff $\varphi(x) \in \varphi(y)$.

Let us consider X. Note that numbering X is increasing.

Let X, Y be ordinal-membered sets. One can check that $X \cup Y$ is ordinal-membered.

Let X be an ordinal-membered set and let Y be a set. Observe that $X \setminus Y$ is ordinal-membered.

The following three propositions are true:

(19) Let X, Y be ordinal-membered sets. Suppose that for all x, y such that $x \in X$ and $y \in Y$ holds $x \in y$. Then (numbering X) $^\frown$ numbering Y is an isomorphism between $\subseteq_{\text{ord-type}\,X+\text{ord-type}\,Y}$ and $\subseteq_{X \cup Y}$.

(20) For all ordinal-membered sets X, Y such that for all x, y such that $x \in X$ and $y \in Y$ holds $x \in y$ holds numbering$(X \cup Y)$ = (numbering X) $^\frown$ numbering Y.

(21) For all ordinal-membered sets X, Y such that for all x, y such that $x \in X$ and $y \in Y$ holds $x \in y$ holds ord-type$(X \cup Y)$ = ord-type X + ord-type Y.

2. Fixpoints of a Normal Function

Next we state the proposition

(27) For every function φ such that x is a fixpoint of φ holds $x \in \text{rng}\,\varphi$.

Let φ be a sequence of ordinal numbers. The functor criticals φ yields a sequence of ordinal numbers and is defined by:

(Def. 7) $\text{criticals}\,\varphi = \text{numbering}\{\alpha \in \text{dom}\,\varphi : \alpha \text{ is a fixpoint of } \varphi\}$.

We now state three propositions:

(28) $\text{On}\{\alpha \in \text{dom}\,\varphi : \alpha \text{ is a fixpoint of } \varphi\} = \{\alpha \in \text{dom}\,\varphi : \alpha \text{ is a fixpoint of } \varphi\}$.

(29) If $x \in \text{dom}\,\text{criticals}\,\varphi$, then $(\text{criticals}\,\varphi)(x)$ is a fixpoint of φ.

(30) $\text{rng}\,\text{criticals}\,\varphi = \{\alpha \in \text{dom}\,\varphi : \alpha \text{ is a fixpoint of } \varphi\}$ and $\text{rng}\,\text{criticals}\,\varphi \subseteq \text{rng}\,\varphi$.

Let us consider φ. One can verify that $\text{criticals}\,\varphi$ is increasing.

We now state several propositions:

(31) If $x \in \text{dom}\,\text{criticals}\,\varphi$, then $x \subseteq (\text{criticals}\,\varphi)(x)$.

(32) $\text{dom}\,\text{criticals}\,\varphi \subseteq \text{dom}\,\varphi$.

(33) If β is a fixpoint of φ, then there exists α such that $\alpha \in \text{dom}\,\text{criticals}\,\varphi$ and $\beta = (\text{criticals}\,\varphi)(\alpha)$.

(34) If $\alpha \in \text{dom}\,\text{criticals}\,\varphi$ and β is a fixpoint of φ and $(\text{criticals}\,\varphi)(\alpha) \in \beta$, then $\text{succ}\,\alpha \in \text{dom}\,\text{criticals}\,\varphi$.

(35) If $\text{succ}\,\alpha \in \text{dom}\,\text{criticals}\,\varphi$ and β is a fixpoint of φ and $(\text{criticals}\,\varphi)(\alpha) \in \beta$, then $(\text{criticals}\,\varphi)(\text{succ}\,\alpha) \subseteq \beta$.

(36) Suppose φ is normal and $\bigcup X \in \text{dom}\,\varphi$ and X is non empty and for every x such that $x \in X$ there exists y such that $x \subseteq y$ and $y \in X$ and y is a fixpoint of φ. Then $\bigcup X = \varphi(\bigcup X)$.

(37) If φ is normal and $\bigcup X \in \text{dom}\,\varphi$ and X is non empty and for every x such that $x \in X$ holds x is a fixpoint of φ, then $\bigcup X = \varphi(\bigcup X)$.

(38) If $\lambda \subseteq \text{dom}\,\text{criticals}\,\varphi$ and α is a fixpoint of φ and for every x such that $x \in \lambda$ holds $(\text{criticals}\,\varphi)(x) \in \alpha$, then $\lambda \in \text{dom}\,\text{criticals}\,\varphi$.

(39) If φ is normal and $\lambda \in \text{dom}\,\text{criticals}\,\varphi$, then $(\text{criticals}\,\varphi)(\lambda) = \bigcup((\text{criticals}\,\varphi){\restriction}\lambda)$.

Let φ be a normal sequence of ordinal numbers. Observe that $\text{criticals}\,\varphi$ is continuous.

Next we state the proposition

(40) For all sequences f_1, f_2 of ordinal numbers such that $f_1 \subseteq f_2$ holds $\text{criticals}\,f_1 \subseteq \text{criticals}\,f_2$.

3. Fixpoints in a Universal Set

In the sequel U, W are universal classes.

Let us consider U. One can check that there exists a transfinite sequence of ordinals of U which is normal.

Let us consider U, α. An ordinal-sequence from α to U is a function from α into $\text{On}\,U$.

Let us consider U, α. Note that every ordinal-sequence from α to U is transfinite sequence-like and ordinal yielding.

Let us consider U, α, let φ be an ordinal-sequence from α to U, and let us consider x. Then $\varphi(x)$ is an ordinal of U.

The following two propositions are true:

(41) If $\alpha \in U$, then for every ordinal-sequence φ from α to U holds $\bigcup \varphi \in U$.

(42) If $\alpha \in U$, then for every ordinal-sequence φ from α to U holds $\sup \varphi \in U$.

In this article we present several logical schemes. The scheme *CriticalNumber2* deals with a universal class $\mathcal{A}$, an ordinal $\mathcal{B}$ of $\mathcal{A}$, an ordinal-sequence $\mathcal{C}$ from ω to $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding an ordinal number, and states that:

$\mathcal{B} \subseteq \bigcup \mathcal{C}$ and $\mathcal{F}(\bigcup \mathcal{C}) = \bigcup \mathcal{C}$ and for every β such that $\mathcal{B} \subseteq \beta$ and $\beta \in \mathcal{A}$ and $\mathcal{F}(\beta) = \beta$ holds $\bigcup \mathcal{C} \subseteq \beta$

provided the following conditions are met:

- $\omega \in \mathcal{A}$,
- For every α such that $\alpha \in \mathcal{A}$ holds $\mathcal{F}(\alpha) \in \mathcal{A}$,
- For all α, β such that $\alpha \in \beta$ and $\beta \in \mathcal{A}$ holds $\mathcal{F}(\alpha) \in \mathcal{F}(\beta)$,
- Let α be an ordinal of $\mathcal{A}$. Suppose α is non empty and limit ordinal. Let φ_1 be a sequence of ordinal numbers. If $\operatorname{dom} \varphi_1 = \alpha$ and for every β such that $\beta \in \alpha$ holds $\varphi_1(\beta) = \mathcal{F}(\beta)$, then $\mathcal{F}(\alpha)$ is the limit of φ_1,
- $\mathcal{C}(0) = \mathcal{B}$, and
- For every α such that $\alpha \in \omega$ holds $\mathcal{C}(\operatorname{succ} \alpha) = \mathcal{F}(\mathcal{C}(\alpha))$.

The scheme *CriticalNumber3* deals with a universal class $\mathcal{A}$, an ordinal $\mathcal{B}$ of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding an ordinal number, and states that:

There exists an ordinal α of $\mathcal{A}$ such that $\mathcal{B} \in \alpha$ and $\mathcal{F}(\alpha) = \alpha$

provided the following requirements are met:

- $\omega \in \mathcal{A}$,
- For every α such that $\alpha \in \mathcal{A}$ holds $\mathcal{F}(\alpha) \in \mathcal{A}$,
- For all α, β such that $\alpha \in \beta$ and $\beta \in \mathcal{A}$ holds $\mathcal{F}(\alpha) \in \mathcal{F}(\beta)$, and
- Let α be an ordinal of $\mathcal{A}$. Suppose α is non empty and limit ordinal. Let φ_1 be a sequence of ordinal numbers. If $\operatorname{dom} \varphi_1 = \alpha$ and for every β such that $\beta \in \alpha$ holds $\varphi_1(\beta) = \mathcal{F}(\beta)$, then $\mathcal{F}(\alpha)$ is the limit of φ_1.

In the sequel F, φ_1 denote normal transfinite sequences of ordinals of W.

One can prove the following propositions:

(43) If ω, $\beta \in W$, then there exists α such that $\beta \in \alpha$ and α is a fixpoint of φ_1.

(44) If $\omega \in W$, then criticals F is a transfinite sequence of ordinals of W.

(45) If φ is normal, then for every α such that $\alpha \in \operatorname{dom} \operatorname{criticals} \varphi$ holds $\varphi(\alpha) \subseteq (\operatorname{criticals} \varphi)(\alpha)$.

4. Sequences of Sequences of Ordinals

Let us consider U and let α, β be ordinals of U. Then α^{β} is an ordinal of U.

Let us consider U, α. Let us assume that $\alpha \in U$. The functor $U \exp \alpha$ yields a transfinite sequence of ordinals of U and is defined as follows:

(Def. 8) For every ordinal β of U holds $(U \exp \alpha)(\beta) = \alpha^{\beta}$.

Let us observe that ω is non trivial.

Let us consider U. Observe that there exists an ordinal of U which is non trivial and finite.

One can verify that there exists an ordinal number which is non trivial and finite.

Let us consider U and let α be a non trivial ordinal of U. Note that $U \exp \alpha$ is normal.

Let ψ be a function. We say that ψ is ordinal-sequence-valued if and only if:

(Def. 9) If $x \in \operatorname{rng} \psi$, then x is a sequence of ordinal numbers.

Let φ be a sequence of ordinal numbers. Observe that $\langle \varphi \rangle$ is ordinal-sequence-valued.

Let φ be a function. We say that φ has the same dom if and only if:

(Def. 10) $\operatorname{rng} \varphi$ has common domain.

Let φ be a function. Observe that $\{\varphi\}$ has common domain.

Let φ be a function. One can verify that $\langle \varphi \rangle$ has the same dom.

One can verify that there exists a transfinite sequence which is non empty and ordinal-sequence-valued and has the same dom.

Let ψ be an ordinal-sequence-valued function and let us consider x. Observe that $\psi(x)$ is relation-like and function-like.

Let ψ be an ordinal-sequence-valued function and let us consider x. Observe that $\psi(x)$ is transfinite sequence-like and ordinal yielding.

Let ψ be a transfinite sequence and let us consider α. Note that $\psi \restriction \alpha$ is transfinite sequence-like.

Let ψ be an ordinal-sequence-valued function and let us consider X. One can check that $\psi \restriction X$ is ordinal-sequence-valued.

Let us consider α, β. Observe that every function from α into β is ordinal yielding and transfinite sequence-like.

Let ψ be an ordinal-sequence-valued transfinite sequence. The functor criticals ψ yields a sequence of ordinal numbers and is defined as follows:

(Def. 11) $\operatorname{criticals} \psi = \operatorname{numbering}\{\alpha \in \operatorname{dom} \psi(0) : \alpha \in \operatorname{dom} \psi(0) \wedge \bigwedge_{\varphi} (\varphi \in \operatorname{rng} \psi \Rightarrow \alpha \text{ is a fixpoint of } \varphi)\}$.

In the sequel ψ is an ordinal-sequence-valued transfinite sequence.

One can prove the following propositions:

(46) Let given ψ. Then $\{\alpha \in \operatorname{dom} \psi(0) : \alpha \in \operatorname{dom} \psi(0) \wedge \bigwedge_{\varphi} (\varphi \in \operatorname{rng} \psi \Rightarrow \alpha \text{ is a fixpoint of } \varphi)\}$ is ordinal-membered.

(47) If $\alpha \in \operatorname{dom}\psi$ and $\beta \in \operatorname{dom}\operatorname{criticals}\psi$, then $(\operatorname{criticals}\psi)(\beta)$ is a fixpoint of $\psi(\alpha)$.

(48) If $x \in \operatorname{dom}\operatorname{criticals}\psi$, then $x \subseteq (\operatorname{criticals}\psi)(x)$.

(49) If $\varphi \in \operatorname{rng}\psi$, then $\operatorname{dom}\operatorname{criticals}\psi \subseteq \operatorname{dom}\varphi$.

(50) If $\operatorname{dom}\psi \neq \emptyset$ and for every γ such that $\gamma \in \operatorname{dom}\psi$ holds β is a fixpoint of $\psi(\gamma)$, then there exists α such that $\alpha \in \operatorname{dom}\operatorname{criticals}\psi$ and $\beta = (\operatorname{criticals}\psi)(\alpha)$.

(51) Suppose $\operatorname{dom}\psi \neq \emptyset$ and $\lambda \subseteq \operatorname{dom}\operatorname{criticals}\psi$ and for every φ such that $\varphi \in \operatorname{rng}\psi$ holds α is a fixpoint of φ and for every x such that $x \in \lambda$ holds $(\operatorname{criticals}\psi)(x) \in \alpha$. Then $\lambda \in \operatorname{dom}\operatorname{criticals}\psi$.

(52) For every ψ such that $\operatorname{dom}\psi \neq \emptyset$ and for every α such that $\alpha \in \operatorname{dom}\psi$ holds $\psi(\alpha)$ is normal holds if $\lambda \in \operatorname{dom}\operatorname{criticals}\psi$, then $(\operatorname{criticals}\psi)(\lambda) = \bigcup((\operatorname{criticals}\psi)\restriction\lambda)$.

(53) For every ψ such that $\operatorname{dom}\psi \neq \emptyset$ and for every α such that $\alpha \in \operatorname{dom}\psi$ holds $\psi(\alpha)$ is normal holds $\operatorname{criticals}\psi$ is continuous.

(54) Let given ψ. Suppose $\operatorname{dom}\psi \neq \emptyset$ and for every α such that $\alpha \in \operatorname{dom}\psi$ holds $\psi(\alpha)$ is normal. Let given α, φ. If $\alpha \in \operatorname{dom}\operatorname{criticals}\psi$ and $\varphi \in \operatorname{rng}\psi$, then $\varphi(\alpha) \subseteq (\operatorname{criticals}\psi)(\alpha)$.

(55) Let g_1, g_2 be ordinal-sequence-valued transfinite sequences. If $\operatorname{dom} g_1 = \operatorname{dom} g_2$ and $\operatorname{dom} g_1 \neq \emptyset$ and for every α such that $\alpha \in \operatorname{dom} g_1$ holds $g_1(\alpha) \subseteq g_2(\alpha)$, then $\operatorname{criticals} g_1 \subseteq \operatorname{criticals} g_2$.

Let ψ be an ordinal-sequence-valued transfinite sequence. The functor $\operatorname{lims}\psi$ yielding a sequence of ordinal numbers is defined by:

(Def. 12) $\operatorname{dom}\operatorname{lims}\psi = \operatorname{dom}\psi(0)$ and for every α such that $\alpha \in \operatorname{dom}\operatorname{lims}\psi$ holds $(\operatorname{lims}\psi)(\alpha) = \bigcup\{\psi(\beta)(\alpha); \beta \text{ ranges over elements of } \operatorname{dom}\psi : \beta \in \operatorname{dom}\psi\}$.

Next we state the proposition

(56) Let ψ be an ordinal-sequence-valued transfinite sequence. Suppose $\operatorname{dom}\psi \neq \emptyset$ and $\operatorname{dom}\psi \in U$ and for every α such that $\alpha \in \operatorname{dom}\psi$ holds $\psi(\alpha)$ is a transfinite sequence of ordinals of U. Then $\operatorname{lims}\psi$ is a transfinite sequence of ordinals of U.

5. Veblen Hierarchy

Let us consider x. The functor OSx yields a sequence of ordinal numbers and is defined by:

(Def. 13) $$OSx = \begin{cases} x, & \text{if } x \text{ is a sequence of ordinal numbers,} \\ \text{the sequence of ordinal numbers}, & \text{otherwise.} \end{cases}$$

The functor $OSVx$ yielding an ordinal-sequence-valued transfinite sequence is defined by:

(Def. 14) $OSV\,x = \begin{cases} x, \text{ if } x \text{ is an ordinal-sequence-valued transfinite sequence,} \\ \text{the ordinal-sequence-valued transfinite sequence, otherwise.} \end{cases}$

Let us consider U. The functor U-Veblen yields an ordinal-sequence-valued transfinite sequence and is defined by the conditions (Def. 15).

(Def. 15)(i) $\mathrm{dom}(U\text{-Veblen}) = \mathrm{On}\,U$,

(ii) $U\text{-Veblen}(0) = U \exp \omega$,

(iii) for every β such that $\mathrm{succ}\,\beta \in \mathrm{On}\,U$ holds $U\text{-Veblen}(\mathrm{succ}\,\beta) = \mathrm{criticals}\,U\text{-Veblen}(\beta)$, and

(iv) for every λ such that $\lambda \in \mathrm{On}\,U$ holds $U\text{-Veblen}(\lambda) = \mathrm{criticals}(U\text{-Veblen}\restriction\lambda)$.

Let us note that there exists a universal class which is uncountable.

One can prove the following propositions:

(57) For every universal class U holds U is uncountable iff $\omega \in U$.

(58) If $\alpha \in \beta$ and β, $\omega \in U$ and $\gamma \in \mathrm{dom}\,U\text{-Veblen}(\beta)$, then $U\text{-Veblen}(\beta)(\gamma)$ is a fixpoint of $U\text{-Veblen}(\alpha)$.

(59) If $\lambda \in U$ and for every γ such that $\gamma \in \lambda$ holds α is a fixpoint of $U\text{-Veblen}(\gamma)$, then α is a fixpoint of $\mathrm{lims}(U\text{-Veblen}\restriction\lambda)$.

(60) If $\alpha \subseteq \beta$ and β, $\omega \in U$ and $\gamma \in \mathrm{dom}\,U\text{-Veblen}(\beta)$ and for every γ such that $\gamma \in \beta$ holds $U\text{-Veblen}(\gamma)$ is normal, then $U\text{-Veblen}(\alpha)(\gamma) \subseteq U\text{-Veblen}(\beta)(\gamma)$.

(61) Suppose λ, $\alpha \in U$ and $\beta \in \lambda$ and for every γ such that $\gamma \in \lambda$ holds $U\text{-Veblen}(\gamma)$ is a normal transfinite sequence of ordinals of U. Then $(\mathrm{lims}(U\text{-Veblen}\restriction\lambda))(\alpha)$ is a fixpoint of $U\text{-Veblen}(\beta)$.

(62) If ω, $\alpha \in U$, then $U\text{-Veblen}(\alpha)$ is a normal transfinite sequence of ordinals of U.

(63) If $\omega \in U$ and $U \subseteq W$ and $\alpha \in U$, then $U\text{-Veblen}(\alpha) \subseteq W\text{-Veblen}(\alpha)$.

(64) If ω, α, $\beta \in U$ and ω, α, $\beta \in W$, then $U\text{-Veblen}(\beta)(\alpha) = W\text{-Veblen}(\beta)(\alpha)$.

(65) Suppose $\lambda \in U$ and for every α such that $\alpha \in \lambda$ holds $U\text{-Veblen}(\alpha)$ is a normal transfinite sequence of ordinals of U. Then $\mathrm{lims}(U\text{-Veblen}\restriction\lambda)$ is a non-decreasing continuous transfinite sequence of ordinals of U.

Let us consider α. Note that $\mathbf{T}(\alpha \cup \omega)$ is uncountable.

Let us consider α, β. The functor $\varphi_\alpha(\beta)$ yields an ordinal number and is defined by:

(Def. 16) $\varphi_\alpha(\beta) = \mathbf{T}(\alpha \cup \beta \cup \omega)\text{-Veblen}(\alpha)(\beta)$.

Let us consider n, β. Then $\varphi_n(\beta)$ is an ordinal number and it can be characterized by the condition:

(Def. 17) $\varphi_n(\beta) = \mathbf{T}(\beta \cup \omega)\text{-Veblen}(n)(\beta)$.

One can prove the following propositions:

(66) $\alpha \in \mathbf{T}(\alpha \cup \beta \cup \gamma)$.

(67) If ω, α, $\beta \in U$, then $\varphi_\beta(\alpha) = U$-Veblen$(\beta)(\alpha)$.

(68) $\varphi_0(\alpha) = \omega^\alpha$.

(69) $\varphi_\beta(\varphi_{\mathrm{succ}\,\beta}(\alpha)) = \varphi_{\mathrm{succ}\,\beta}(\alpha)$.

(70) If $\beta \in \gamma$, then $\varphi_\beta(\varphi_\gamma(\alpha)) = \varphi_\gamma(\alpha)$.

(71) $\alpha \subseteq \beta$ iff $\varphi_\gamma(\alpha) \subseteq \varphi_\gamma(\beta)$.

(72) $\alpha \in \beta$ iff $\varphi_\gamma(\alpha) \in \varphi_\gamma(\beta)$.

(73) $\varphi_\alpha(\beta) \in \varphi_\gamma(\delta)$ iff $\alpha = \gamma$ and $\beta \in \delta$ or $\alpha \in \gamma$ and $\beta \in \varphi_\gamma(\delta)$ or $\gamma \in \alpha$ and $\varphi_\alpha(\beta) \in \delta$.

6. Epsilon Numbers

In the sequel U is an uncountable universal class.

Next we state four propositions:

(74) U-Veblen$(1) =$ criticals$(U \exp \omega)$.

(75) $\varphi_1(\alpha)$ is epsilon.

(76) For every epsilon ordinal number e there exists α such that $e = \varphi_1(\alpha)$.

(77) $\varphi_1(\alpha) = \varepsilon_\alpha$.

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Sorting by Exchanging

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Summary. We show that exchanging of pairs in an array which are in incorrect order leads to sorted array. It justifies correctness of Bubble Sort, Insertion Sort, and Quicksort.

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The notation and terminology used here have been introduced in the following papers: [20], [6], [11], [1], [8], [16], [12], [13], [10], [9], [17], [18], [3], [4], [2], [7], [14], [21], [22], [19], [5], and [15].

1. Preliminaries

We adopt the following convention: α, β, γ, δ denote ordinal numbers, k denotes a natural number, and x, y, z, t, X, Y, Z denote sets.

The following propositions are true:

(1) $x \in (\alpha + \beta) \setminus \alpha$ iff there exists γ such that $x = \alpha + \gamma$ and $\gamma \in \beta$.

(2) Suppose $\alpha \in \beta$ and $\gamma \in \delta$. Then $\gamma \neq \alpha$ and $\gamma \neq \beta$ and $\delta \neq \alpha$ and $\delta \neq \beta$ or $\gamma \in \alpha$ and $\delta = \alpha$ or $\gamma \in \alpha$ and $\delta = \beta$ or $\gamma = \alpha$ and $\delta \in \beta$ or $\gamma = \alpha$ and $\delta = \beta$ or $\gamma = \alpha$ and $\beta \in \delta$ or $\alpha \in \gamma$ and $\delta = \beta$ or $\gamma = \beta$ and $\beta \in \delta$.

(3) If $x \notin y$, then $(y \cup \{x\}) \setminus y = \{x\}$.

(4) $\mathrm{succ}\, x \setminus x = \{x\}$.

(5) Let f be a function, r be a binary relation, and given x. Then $x \in f^{\circ} r$ if and only if there exist y, z such that $\langle y, z\rangle \in r$ and $\langle y, z\rangle \in \mathrm{dom}\, f$ and $f(y, z) = x$.

(6) If $\alpha \setminus \beta \neq \emptyset$, then $\inf(\alpha \setminus \beta) = \beta$ and $\sup(\alpha \setminus \beta) = \alpha$ and $\bigcup(\alpha \setminus \beta) = \bigcup \alpha$.

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(7) If $\alpha \setminus \beta$ is non empty and finite, then there exists a natural number n such that $\alpha = \beta + n$.

2. Arrays

Let f be a set. We say that f is segmental if and only if:

(Def. 1) There exist α, β such that $\pi_1(f) = \alpha \setminus \beta$.

In the sequel f, g denote functions.

The following two propositions are true:

(8) If $\mathrm{dom}\, f = \mathrm{dom}\, g$ and f is segmental, then g is segmental.

(9) If f is segmental, then for all α, β, γ such that $\alpha \subseteq \beta \subseteq \gamma$ and α, $\gamma \in \mathrm{dom}\, f$ holds $\beta \in \mathrm{dom}\, f$.

Let us observe that every function which is transfinite sequence-like is also segmental and every function which is finite sequence-like is also segmental.

Let us consider α and let s be a set. We say that s is α-based if and only if:

(Def. 2) If $\beta \in \pi_1(s)$, then $\alpha \in \pi_1(s)$ and $\alpha \subseteq \beta$.

We say that s is α-limited if and only if:

(Def. 3) $\alpha = \sup \pi_1(s)$.

Next we state two propositions:

(10) f is α-based and segmental iff there exists β such that $\mathrm{dom}\, f = \beta \setminus \alpha$ and $\alpha \subseteq \beta$.

(11) f is β-limited, non empty, and segmental iff there exists α such that $\mathrm{dom}\, f = \beta \setminus \alpha$ and $\alpha \in \beta$.

Let us observe that every function which is transfinite sequence-like is also 0-based and every function which is finite sequence-like is also 1-based.

The following three propositions are true:

(12) f is $\inf \mathrm{dom}\, f$-based.

(13) f is $\sup \mathrm{dom}\, f$-limited.

(14) If f is β-limited and $\alpha \in \mathrm{dom}\, f$, then $\alpha \in \beta$.

Let us consider f. The functor $\mathrm{base}\, f$ yielding an ordinal number is defined as follows:

(Def. 4)(i) f is $\mathrm{base}\, f$-based if there exists α such that $\alpha \in \mathrm{dom}\, f$,

(ii) $\mathrm{base}\, f = 0$, otherwise.

The functor $\mathrm{limit}\, f$ yields an ordinal number and is defined as follows:

(Def. 5)(i) f is $\mathrm{limit}\, f$-limited if there exists α such that $\alpha \in \mathrm{dom}\, f$,

(ii) $\mathrm{limit}\, f = 0$, otherwise.

Let us consider f. The functor $\mathrm{length}\, f$ yielding an ordinal number is defined as follows:

(Def. 6) length f = limit f − base f.

We now state four propositions:

(15) base $\emptyset = 0$ and limit $\emptyset = 0$ and length $\emptyset = 0$.

(16) limit f = sup dom f.

(17) f is limit f-limited.

(18) Every empty set is α-based.

Let us consider α, X, Y. Note that there exists a transfinite sequence which is Y-defined, X-valued, α-based, segmental, finite, and empty.

An array is a segmental function.

Let A be an array. Observe that dom A is ordinal-membered.

We now state the proposition

(19) For every array f holds f is 0-limited iff f is empty.

Let us mention that every array which is 0-based is also transfinite sequence-like.

Let us consider X. An array of X is an X-valued array.

Let X be a 1-sorted structure. An array of X is an array of the carrier of X.

Let us consider α, X. An array of α, X is an α-defined array of X.

In the sequel A, B, C denote arrays.

Next we state several propositions:

(20) base f = inf dom f.

(21) f is base f-based.

(22) dom A = limit $A \setminus$ base A.

(23) If dom $A = \alpha \setminus \beta$ and A is non empty, then base $A = \beta$ and limit $A = \alpha$.

(24) For every transfinite sequence f holds base $f = 0$ and limit f = dom f and length f = dom f.

Let us consider α, β, X. Note that there exists an array of α, X which is β-based, natural-valued, integer-valued, real-valued, complex-valued, and finite.

Let us consider α, x. Note that $\{\langle\alpha, x\rangle\}$ is segmental.

Let us consider α and let x be a natural number. Observe that $\{\langle\alpha, x\rangle\}$ is natural-valued.

Let us consider α and let x be a real number. One can verify that $\{\langle\alpha, x\rangle\}$ is real-valued.

Let us consider α, let X be a non empty set, and let x be an element of X. One can check that $\{\langle\alpha, x\rangle\}$ is X-valued.

Let us consider α, x. One can check that $\{\langle\alpha, x\rangle\}$ is α-based and succ α-limited.

Let us consider β. Note that there exists an array which is non empty, β-based, natural-valued, integer-valued, real-valued, complex-valued, and finite. Let X be a non empty set. Note that there exists an array which is non empty, β-based, finite, and X-valued.

Let s be a transfinite sequence. We introduce $s\,\mathrm{last}$ as a synonym of $\mathrm{last}\,s$.

Let A be an array. The functor $\mathrm{last}\,A$ is defined by:

(Def. 7) $\mathrm{last}\,A = A(\bigcup \mathrm{dom}\,A)$.

3. Descending Sequences

Let f be a function. We say that f is descending if and only if:

(Def. 8) For all α, β such that $\alpha, \beta \in \mathrm{dom}\,f$ and $\alpha \in \beta$ holds $f(\beta) \subset f(\alpha)$.

We now state four propositions:

(25) For every finite array f such that for every α such that $\alpha, \mathrm{succ}\,\alpha \in \mathrm{dom}\,f$ holds $f(\mathrm{succ}\,\alpha) \subset f(\alpha)$ holds f is descending.

(26) For every array f such that $\mathrm{length}\,f = \omega$ and for every α such that α, $\mathrm{succ}\,\alpha \in \mathrm{dom}\,f$ holds $f(\mathrm{succ}\,\alpha) \subset f(\alpha)$ holds f is descending.

(27) For every transfinite sequence f such that f is descending and $f(0)$ is finite holds f is finite.

(28) Let f be a transfinite sequence. Suppose f is descending and $f(0)$ is finite and for every α such that $f(\alpha) \neq \emptyset$ holds $\mathrm{succ}\,\alpha \in \mathrm{dom}\,f$. Then $\mathrm{last}\,f = \emptyset$.

The scheme A deals with a transfinite sequence $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\mathcal{A}$ is finite

provided the parameters meet the following requirements:

- $\mathcal{F}(\mathcal{A}(0))$ is finite, and
- For every α such that $\mathrm{succ}\,\alpha \in \mathrm{dom}\,\mathcal{A}$ and $\mathcal{F}(\mathcal{A}(\alpha))$ is finite holds $\mathcal{F}(\mathcal{A}(\mathrm{succ}\,\alpha)) \subset \mathcal{F}(\mathcal{A}(\alpha))$.

4. Swap

Let us consider X, let f be an X-defined function, and let α, β be sets. Note that $\mathrm{Swap}(f, \alpha, \beta)$ is X-defined.

Let X be a set, let f be an X-valued function, and let x, y be sets. Note that $\mathrm{Swap}(f, x, y)$ is X-valued.

The following propositions are true:

(29) If $x, y \in \mathrm{dom}\,f$, then $(\mathrm{Swap}(f, x, y))(x) = f(y)$.

(30) For every array f of X such that $x, y \in \mathrm{dom}\,f$ holds $(\mathrm{Swap}(f, x, y))_x = f_y$.

(31) If $x, y \in \mathrm{dom}\,f$, then $(\mathrm{Swap}(f, x, y))(y) = f(x)$.

(32) For every array f of X such that $x, y \in \mathrm{dom}\,f$ holds $(\mathrm{Swap}(f, x, y))_y = f_x$.

(33) If $z \neq x$ and $z \neq y$, then $(\mathrm{Swap}(f, x, y))(z) = f(z)$.

(34) For every array f of X such that $z \in \mathrm{dom}\, f$ and $z \neq x$ and $z \neq y$ holds $(\mathrm{Swap}(f, x, y))_z = f_z$.

(35) If x, $y \in \mathrm{dom}\, f$, then $\mathrm{Swap}(f, x, y) = \mathrm{Swap}(f, y, x)$.

Let X be a non empty set. Observe that there exists an X-valued non empty function which is onto.

Let X be a non empty set, let f be an onto X-valued non empty function, and let x, y be elements of $\mathrm{dom}\, f$. Note that $\mathrm{Swap}(f, x, y)$ is onto.

Let us consider A and let us consider x, y. Note that $\mathrm{Swap}(A, x, y)$ is segmental.

We now state the proposition

(36) If x, $y \in \mathrm{dom}\, A$, then $\mathrm{Swap}(\mathrm{Swap}(A, x, y), x, y) = A$.

Let A be a real-valued array and let us consider x, y. One can verify that $\mathrm{Swap}(A, x, y)$ is real-valued.

5. Permutations

Let A be an array. An array is called a permutation of A if:

(Def. 9) There exists a permutation f of $\mathrm{dom}\, A$ such that it $= A \cdot f$.

We now state several propositions:

(37) For every permutation B of A holds $\mathrm{dom}\, B = \mathrm{dom}\, A$ and $\mathrm{rng}\, B = \mathrm{rng}\, A$.

(38) A is a permutation of A.

(39) If A is a permutation of B, then B is a permutation of A.

(40) If A is a permutation of B and B is a permutation of C, then A is a permutation of C.

(41) $\mathrm{Swap}(\mathrm{id}_X, x, y)$ is one-to-one.

Let X be a non empty set and let x, y be elements of X.

Note that $\mathrm{Swap}(\mathrm{id}_X, x, y)$ is one-to-one, X-valued, and X-defined.

Let X be a non empty set and let x, y be elements of X.

Note that $\mathrm{Swap}(\mathrm{id}_X, x, y)$ is onto and total.

Let X, Y be non empty sets, let f be a function from X into Y, and let x, y be elements of X. Then $\mathrm{Swap}(f, x, y)$ is a function from X into Y.

Next we state three propositions:

(42) If x, $y \in X$ and $f = \mathrm{Swap}(\mathrm{id}_X, x, y)$ and $X = \mathrm{dom}\, A$, then $\mathrm{Swap}(A, x, y) = A \cdot f$.

(43) If x, $y \in \mathrm{dom}\, A$, then $\mathrm{Swap}(A, x, y)$ is a permutation of A and A is a permutation of $\mathrm{Swap}(A, x, y)$.

(44) Suppose x, $y \in \mathrm{dom}\, A$ and A is a permutation of B. Then $\mathrm{Swap}(A, x, y)$ is a permutation of B and A is a permutation of $\mathrm{Swap}(B, x, y)$.

6. Exchanging

Let O be a relational structure and let A be an array of O. We say that A is ascending if and only if:

(Def. 10) For all α, β such that α, $\beta \in \operatorname{dom} A$ and $\alpha \in \beta$ holds $A_\alpha \leq A_\beta$.

The functor inversions A is defined by:

(Def. 11) inversions $A = \{\langle \alpha, \beta \rangle; \alpha$ ranges over elements of $\operatorname{dom} A, \beta$ ranges over elements of $\operatorname{dom} A : \alpha \in \beta \ \wedge \ A_\alpha \not\leq A_\beta\}$.

Let O be a relational structure. One can verify that every empty array of O is ascending. Let A be an empty array of O. One can verify that inversions A is empty.

In the sequel O is a connected non empty poset and R, Q are arrays of O.

We now state the proposition

(45) For every O and for all elements x, y of O holds $x > y$ iff $x \not\leq y$.

Let us consider O, R. Then inversions R can be characterized by the condition:

(Def. 12) inversions $R = \{\langle \alpha, \beta \rangle; \alpha$ ranges over elements of $\operatorname{dom} R, \beta$ ranges over elements of $\operatorname{dom} R : \alpha \in \beta \ \wedge \ R_\alpha > R_\beta\}$.

Next we state two propositions:

(46) $\langle x, y \rangle \in$ inversions R iff x, $y \in \operatorname{dom} R$ and $x \in y$ and $R_x > R_y$.

(47) inversions $R \subseteq \operatorname{dom} R \times \operatorname{dom} R$.

Let us consider O and let R be a finite array of O. Observe that inversions R is finite.

We now state three propositions:

(48) R is ascending iff inversions $R = \emptyset$.

(49) If $\langle x, y \rangle \in$ inversions R, then $\langle y, x \rangle \notin$ inversions R.

(50) If $\langle x, y \rangle$, $\langle y, z \rangle \in$ inversions R, then $\langle x, z \rangle \in$ inversions R.

Let us consider O, R. Note that inversions R is relation-like.

Let us consider O, R. Note that inversions R is asymmetric and transitive.

Let us consider O and let α, β be elements of O. Let us note that the predicate $\alpha < \beta$ is antisymmetric.

Next we state several propositions:

(51) If $\langle x, y \rangle \in$ inversions R, then $\langle x, y \rangle \notin$ inversions $\mathrm{Swap}(R, x, y)$.

(52) If x, $y \in \operatorname{dom} R$ and $z \neq x$ and $z \neq y$ and $t \neq x$ and $t \neq y$, then $\langle z, t \rangle \in$ inversions R iff $\langle z, t \rangle \in$ inversions $\mathrm{Swap}(R, x, y)$.

(53) If $\langle x, y \rangle \in$ inversions R, then $\langle z, y \rangle \in$ inversions R and $z \in x$ iff $\langle z, x \rangle \in$ inversions $\mathrm{Swap}(R, x, y)$.

(54) If $\langle x, y \rangle \in$ inversions R, then $\langle z, x \rangle \in$ inversions R iff $z \in x$ and $\langle z, y \rangle \in$ inversions $\mathrm{Swap}(R, x, y)$.

(55) If $\langle x, y\rangle \in \text{inversions}\, R$ and $z \in y$ and $\langle x, z\rangle \in \text{inversions}\,\text{Swap}(R, x, y)$, then $\langle x, z\rangle \in \text{inversions}\, R$.

(56) If $\langle x, y\rangle \in \text{inversions}\, R$ and $x \in z$ and $\langle z, y\rangle \in \text{inversions}\,\text{Swap}(R, x, y)$, then $\langle z, y\rangle \in \text{inversions}\, R$.

(57) If $\langle x, y\rangle \in \text{inversions}\, R$ and $y \in z$ and $\langle x, z\rangle \in \text{inversions}\,\text{Swap}(R, x, y)$, then $\langle y, z\rangle \in \text{inversions}\, R$.

(58) If $\langle x, y\rangle \in \text{inversions}\, R$, then $y \in z$ and $\langle x, z\rangle \in \text{inversions}\, R$ iff $\langle y, z\rangle \in \text{inversions}\,\text{Swap}(R, x, y)$.

Let us consider O, R, x, y. The functor $\subseteq^R_{x,y}$ yields a function and is defined by:

(Def. 13) $\subseteq^R_{x,y} = \text{Swap}(\text{id}_{\text{dom}\, R}, x, y) \times \text{Swap}(\text{id}_{\text{dom}\, R}, x, y) {+\cdot}\, \text{id}_{\{x\}\times(\text{succ}\, y \setminus x) \cup (\text{succ}\, y \setminus x)\times\{y\}}$.

Next we state the proposition

(59) $\gamma \in \text{succ}\, \beta \setminus \alpha$ iff $\alpha \subseteq \gamma \subseteq \beta$.

We adopt the following convention: T is a non empty array of O and p, q, r, s are elements of $\text{dom}\, T$.

The following propositions are true:

(60) $\text{succ}\, q \setminus p \subseteq \text{dom}\, T$.

(61) $\text{dom} \subseteq^T_{p,q} = \text{dom}\, T \times \text{dom}\, T$ and $\text{rng} \subseteq^T_{p,q} \subseteq \text{dom}\, T \times \text{dom}\, T$.

(62) If $p \subseteq r \subseteq q$, then $(\subseteq^T_{p,q})(p, r) = \langle p, r\rangle$ and $(\subseteq^T_{p,q})(r, q) = \langle r, q\rangle$.

(63) If $r \neq p$ and $s \neq q$ and $f = \text{Swap}(\text{id}_{\text{dom}\, T}, p, q)$, then $(\subseteq^T_{p,q})(r, s) = \langle f(r), f(s)\rangle$.

(64) If $r \in p$ and $f = \text{Swap}(\text{id}_{\text{dom}\, T}, p, q)$, then $(\subseteq^T_{p,q})(r, q) = \langle f(r), f(q)\rangle$ and $(\subseteq^T_{p,q})(r, p) = \langle f(r), f(p)\rangle$.

(65) If $q \in r$ and $f = \text{Swap}(\text{id}_{\text{dom}\, T}, p, q)$, then $(\subseteq^T_{p,q})(p, r) = \langle f(p), f(r)\rangle$ and $(\subseteq^T_{p,q})(q, r) = \langle f(q), f(r)\rangle$.

(66) If $p \in q$, then $(\subseteq^T_{p,q})(p, q) = \langle p, q\rangle$.

(67) If $p \in q$ and $r \neq p$ and $r \neq q$ and $s \neq p$ and $s \neq q$, then $(\subseteq^T_{p,q})(r, s) = \langle r, s\rangle$.

(68) If $r \in p$ and $p \in q$, then $(\subseteq^T_{p,q})(r, p) = \langle r, q\rangle$ and $(\subseteq^T_{p,q})(r, q) = \langle r, p\rangle$.

(69) If $p \in s$ and $s \in q$, then $(\subseteq^T_{p,q})(p, s) = \langle p, s\rangle$ and $(\subseteq^T_{p,q})(s, q) = \langle s, q\rangle$.

(70) If $p \in q$ and $q \in s$, then $(\subseteq^T_{p,q})(p, s) = \langle q, s\rangle$ and $(\subseteq^T_{p,q})(q, s) = \langle p, s\rangle$.

(71) If $p \in q$, then $\subseteq^T_{p,q} \restriction (\text{inversions}\,\text{Swap}(T, p, q)$ **qua** set) is one-to-one.

Let us consider O, R, x, y, z. Note that $(\subseteq^R_{x,y})^\circ z$ is relation-like.

7. Correctness of Sorting by Exchanging

The following proposition is true

(72) If $\langle x,\, y\rangle \in \operatorname{inversions} R$, then $(\subseteq^{R}_{x,y})^{\circ} \operatorname{inversions} \operatorname{Swap}(R,x,y) \subset \operatorname{inversions} R$.

Let R be a finite function and let us consider x, y. One can check that $\operatorname{Swap}(R,x,y)$ is finite.

Next we state two propositions:

(73) For every array R of O such that $\langle x,\, y\rangle \in \operatorname{inversions} R$ and $\operatorname{inversions} R$ is finite holds $\overline{\overline{\operatorname{inversions} \operatorname{Swap}(R,x,y)}} \in \overline{\overline{\operatorname{inversions} R}}$.

(74) For every finite array R of O such that $\langle x,\, y\rangle \in \operatorname{inversions} R$ holds $\overline{\overline{\operatorname{inversions} \operatorname{Swap}(R,x,y)}} < \overline{\overline{\operatorname{inversions} R}}$.

Let us consider O, R. A non empty array is called a computation of R if it satisfies the conditions (Def. 14).

(Def. 14)(i) $\operatorname{It}(\operatorname{base} \operatorname{it}) = R$,

(ii) for every α such that $\alpha \in \operatorname{dom} \operatorname{it}$ holds $\operatorname{it}(\alpha)$ is an array of O, and

(iii) for every α such that α, $\operatorname{succ} \alpha \in \operatorname{dom} \operatorname{it}$ there exist R, x, y such that $\langle x,\, y\rangle \in \operatorname{inversions} R$ and $\operatorname{it}(\alpha) = R$ and $\operatorname{it}(\operatorname{succ} \alpha) = \operatorname{Swap}(R,x,y)$.

We now state the proposition

(75) $\{\langle \alpha,\, R\rangle\}$ is a computation of R.

Let us consider O, R, α. One can check that there exists a computation of R which is α-based and finite.

Let us consider O, R, let C be a computation of R, and let us consider x. One can check that $C(x)$ is segmental, function-like, and relation-like.

Let us consider O, R, let C be a computation of R, and let us consider x. Observe that $C(x)$ is the carrier of O-valued.

Let us consider O, R and let C be a computation of R. Observe that $\operatorname{last} C$ is segmental, relation-like, and function-like.

Let us consider O, R and let C be a computation of R. Observe that $\operatorname{last} C$ is the carrier of O-valued.

Let us consider O, R and let C be a computation of R. We say that C is complete if and only if:

(Def. 15) $\operatorname{last} C$ is ascending.

One can prove the following three propositions:

(76) For every 0-based computation C of R such that R is a finite array of O holds C is finite.

(77) Let C be a 0-based computation of R. Suppose R is a finite array of O and for every α such that $\operatorname{inversions} C(\alpha) \neq \emptyset$ holds $\operatorname{succ} \alpha \in \operatorname{dom} C$. Then C is complete.

(78) Let C be a finite computation of R. Then $\operatorname{last} C$ is a permutation of R and for every α such that $\alpha \in \operatorname{dom} C$ holds $C(\alpha)$ is a permutation of R.

8. Existence of Complete Computations

Next we state three propositions:

(79) For every 0-based finite array A of X such that $A \neq \emptyset$ holds $\operatorname{last} A \in X$.

(80) $\operatorname{last}\langle x \rangle = x$.

(81) For every 0-based finite array A holds $\operatorname{last}(A \frown \langle x \rangle) = x$.

Let X be a set. Observe that every element of X^{ω} is X-valued.

The scheme A deals with a unary functor $\mathcal{F}$ yielding a set, a non empty set $\mathcal{A}$, a set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a finite 0-based non empty array f and there exists an element k of $\mathcal{A}$ such that

(i) $k = \operatorname{last} f$,

(ii) $\mathcal{F}(k) = \emptyset$,

(iii) $f(0) = \mathcal{B}$, and

(iv) for every α such that $\operatorname{succ} \alpha \in \operatorname{dom} f$ there exist elements x, y of $\mathcal{A}$ such that $x = f(\alpha)$ and $y = f(\operatorname{succ} \alpha)$ and $\mathcal{P}[x, y]$

provided the following requirements are met:

- $\mathcal{B} \in \mathcal{A}$,
- $\mathcal{F}(\mathcal{B})$ is finite, and
- For every element x of $\mathcal{A}$ such that $\mathcal{F}(x) \neq \emptyset$ there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{F}(y) \subset \mathcal{F}(x)$.

In the sequel A is an array and B is a permutation of A.

We now state the proposition

(82) $B \in (\operatorname{rng} A)^{\operatorname{dom} A}$.

Let A be a real-valued array. One can verify that every permutation of A is real-valued.

Let us consider α and let X be a non empty set. Observe that every element of X^{α} is transfinite sequence-like.

Let us consider X and let Y be a real-membered non empty set. One can check that every element of Y^X is real-valued.

Let us consider X and let A be an array of X. One can check that every permutation of A is X-valued.

Let X be a set, let Z be a set, and let Y be a subset of Z. Note that every element of Y^X is Z-valued.

One can prove the following propositions:

(83) Every X-defined Y-valued binary relation is a relation between X and Y.

(84) For every finite ordinal number α and for every x such that $x \in \alpha$ holds $x = 0$ or there exists β such that $x = \operatorname{succ} \beta$.

(85) For every 0-based finite non empty array A of O holds there exists a 0-based computation of A which is complete.

(86) For every 0-based finite non empty array A of O holds there exists a permutation of A which is ascending.

Let us consider O and let A be a 0-based finite array of O. Observe that there exists a permutation of A which is ascending.

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Linear Transformations of Euclidean Topological Spaces

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Summary. We introduce linear transformations of Euclidean topological spaces given by a transformation matrix. Next, we prove selected properties and basic arithmetic operations on these linear transformations. Finally, we show that a linear transformation given by an invertible matrix is a homeomorphism.

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The papers [2], [12], [6], [26], [7], [8], [30], [21], [22], [23], [15], [31], [29], [19], [24], [3], [4], [9], [16], [5], [20], [18], [1], [14], [28], [13], [10], [25], [27], [11], and [17] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: X, Y denote sets, n, m, k, i denote natural numbers, r denotes a real number, R denotes an element of $\mathbb{R}_{\rm F}$, K denotes a field, f, f_1, f_2, g_1, g_2 denote finite sequences, r_1, r_2, r_3 denote real-valued finite sequences, c_1, c_2 denote complex-valued finite sequences, and F denotes a function.

Let us consider X, Y and let F be a positive yielding partial function from X to $\mathbb{R}$. One can check that $F{\restriction}Y$ is positive yielding.

Let us consider X, Y and let F be a negative yielding partial function from X to $\mathbb{R}$. One can verify that $F{\restriction}Y$ is negative yielding.

Let us consider X, Y and let F be a non-positive yielding partial function from X to $\mathbb{R}$. Note that $F{\restriction}Y$ is non-positive yielding.

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Let us consider X, Y and let F be a non-negative yielding partial function from X to $\mathbb{R}$. Note that $F \restriction Y$ is non-negative yielding.

Let us consider r_1. One can check that $\sqrt{r_1}$ is finite sequence-like.

Let us consider r_1. The functor ${}^{@}r_1$ yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by:

(Def. 1) ${}^{@}r_1 = r_1$.

Let p be a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$. The functor ${}^{@}p$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 2) ${}^{@}p = p$.

We now state several propositions:

(1) $({}^{@}r_2) + {}^{@}r_3 = r_2 + r_3$.

(2) $\sqrt{r_2 \frown r_3} = \sqrt{r_2} \frown \sqrt{r_3}$.

(3) $\sqrt{\langle r \rangle} = \langle \sqrt{r} \rangle$.

(4) $\sqrt{r_1{}^{\mathbf{2}}} = |r_1|$.

(5) If r_1 is non-negative yielding, then $\sqrt{\sum r_1} \leq \sum \sqrt{r_1}$.

(6) There exists X such that $X \subseteq \operatorname{dom} F$ and $\operatorname{rng} F = \operatorname{rng}(F \restriction X)$ and $F \restriction X$ is one-to-one.

Let us consider c_1, X. Observe that $c_1 - X$ is complex-valued.

Let us consider r_1, X. Observe that $r_1 - X$ is real-valued.

Let c_1 be a complex-valued finite subsequence. Note that $\operatorname{Seq} c_1$ is complex-valued.

Let r_1 be a real-valued finite subsequence. Observe that $\operatorname{Seq} r_1$ is real-valued.

One can prove the following propositions:

(7) For every permutation P of $\operatorname{dom} f$ such that $f_1 = f \cdot P$ there exists a permutation Q of $\operatorname{dom}(f - X)$ such that $f_1 - X = (f - X) \cdot Q$.

(8) For every permutation P of $\operatorname{dom} c_1$ such that $c_2 = c_1 \cdot P$ holds $\sum(c_2 - X) = \sum(c_1 - X)$.

(9) Let f, f_1 be finite subsequences and P be a permutation of $\operatorname{dom} f$. If $f_1 = f \cdot P$, then there exists a permutation Q of $\operatorname{dom} \operatorname{Seq}(f_1 \restriction P^{-1}(X))$ such that $\operatorname{Seq}(f \restriction X) = \operatorname{Seq}(f_1 \restriction P^{-1}(X)) \cdot Q$.

(10) Let c_1, c_2 be complex-valued finite subsequences and P be a permutation of $\operatorname{dom} c_1$. If $c_2 = c_1 \cdot P$, then $\sum \operatorname{Seq}(c_1 \restriction X) = \sum \operatorname{Seq}(c_2 \restriction P^{-1}(X))$.

(11) Let f be a finite subsequence and n be an element of $\mathbb{N}$. If for every i holds $i + n \in X$ iff $i \in Y$, then $\operatorname{Shift}^n f \restriction X = \operatorname{Shift}^n (f \restriction Y)$.

(12) There exists a subset Y of $\mathbb{N}$ such that $\operatorname{Seq}((f_1 \frown f_2) \restriction X) = (\operatorname{Seq}(f_1 \restriction X)) \frown \operatorname{Seq}(f_2 \restriction Y)$ and for every n such that $n > 0$ holds $n \in Y$ iff $n + \operatorname{len} f_1 \in X \cap \operatorname{dom}(f_1 \frown f_2)$.

(13) If $\operatorname{len} g_1 = \operatorname{len} f_1$ and $\operatorname{len} g_2 \leq \operatorname{len} f_2$, then $\operatorname{Seq}((f_1 \frown f_2) \restriction (g_1 \frown g_2)^{-1}(X)) = (\operatorname{Seq}(f_1 \restriction g_1{}^{-1}(X))) \frown \operatorname{Seq}(f_2 \restriction g_2{}^{-1}(X))$.

(14) Let D be a non empty set and M be a matrix over D of dimension $n \times m$. Then $M - X$ is a matrix over D of dimension $n -' \overline{\overline{M^{-1}(X)}} \times m$.

(15) Let F be a function from $\mathrm{Seg}\, n$ into $\mathrm{Seg}\, n$, D be a non empty set, M be a matrix over D of dimension $n \times m$, and given i. If $i \in \mathrm{Seg}\,\mathrm{width}\, M$, then $(M \cdot F)_{\square,i} = M_{\square,i} \cdot F$.

(16) Let A be a matrix over K of dimension $n \times m$. Suppose $\mathrm{rk}(A) = n$. Then there exists a matrix B over K of dimension $m -' n \times m$ such that $\mathrm{rk}(A \frown B) = m$.

(17) Let A be a matrix over K of dimension $n \times m$. Suppose $\mathrm{rk}(A) = m$. Then there exists a matrix B over K of dimension $n \times n -' m$ such that $\mathrm{rk}(A \frown B) = n$.

For simplicity, we adopt the following convention: f, f_1, f_2 denote n-element real-valued finite sequences, p, p_1, p_2 denote points of $\mathcal{E}^n_{\mathrm{T}}$, M, M_1, M_2 denote matrices over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$, and A, B denote square matrices over $\mathbb{R}_{\mathrm{F}}$ of dimension n.

2. Linear Transformations of Euclidean Topological Spaces Given by a Transformation Matrix

Let us consider n, m, M. The functor $\mathrm{Mx2Tran}\, M$ yielding a function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^m_{\mathrm{T}}$ is defined by:

(Def. 3)(i) $(\mathrm{Mx2Tran}\, M)(f) = \mathrm{Line}(\mathrm{LineVec2Mx}({}^{@}f) \cdot M, 1)$ if $n \neq 0$,

(ii) $(\mathrm{Mx2Tran}\, M)(f) = 0_{\mathcal{E}^m_{\mathrm{T}}}$, otherwise.

Let us consider n, m, M and let x be a set. One can check that $(\mathrm{Mx2Tran}\, M)(x)$ is function-like and relation-like and $(\mathrm{Mx2Tran}\, M)(x)$ is real-valued and finite sequence-like.

Let us consider n, m, M, f. One can check that $(\mathrm{Mx2Tran}\, M)(f)$ is m-element.

One can prove the following propositions:

(18) If $1 \leq i \leq m$ and $n \neq 0$, then $(\mathrm{Mx2Tran}\, M)(f)(i) = ({}^{@}f) \cdot M_{\square,i}$.

(19) $\mathrm{len}\,\mathrm{MX2FinS}(I^{n\times n}_K) = n$.

(20) Let B_1 be an ordered basis of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ and B_2 be an ordered basis of the m-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. Suppose $B_1 = \mathrm{MX2FinS}(I^{n\times n}_{\mathbb{R}_{\mathrm{F}}})$ and $B_2 = \mathrm{MX2FinS}(I^{m\times m}_{\mathbb{R}_{\mathrm{F}}})$. Let M_1 be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $\mathrm{len}\, B_1 \times \mathrm{len}\, B_2$. If $M_1 = M$, then $\mathrm{Mx2Tran}\, M = \mathrm{Mx2Tran}(M_1, B_1, B_2)$.

(21) For every permutation P of $\mathrm{Seg}\, n$ holds $(\mathrm{Mx2Tran}\, M)(f) = (\mathrm{Mx2Tran}(M \cdot P))(f \cdot P)$ and $f \cdot P$ is an n-element finite sequence of elements of $\mathbb{R}$.

(22) $(\mathrm{Mx2Tran}\, M)(f_1 + f_2) = (\mathrm{Mx2Tran}\, M)(f_1) + (\mathrm{Mx2Tran}\, M)(f_2)$.

(23) $(\mathrm{Mx2Tran}\, M)(r \cdot f) = r \cdot (\mathrm{Mx2Tran}\, M)(f)$.

(24) $(\mathrm{Mx2Tran}\, M)(f_1 - f_2) = (\mathrm{Mx2Tran}\, M)(f_1) - (\mathrm{Mx2Tran}\, M)(f_2)$.

(25) $(\mathrm{Mx2Tran}(M_1 + M_2))(f) = (\mathrm{Mx2Tran}\, M_1)(f) + (\mathrm{Mx2Tran}\, M_2)(f)$.

(26) $(R) \cdot (\mathrm{Mx2Tran}\, M)(f) = (\mathrm{Mx2Tran}(R \cdot M))(f)$.

(27) $(\mathrm{Mx2Tran}\, M)(p_1 + p_2) = (\mathrm{Mx2Tran}\, M)(p_1) + (\mathrm{Mx2Tran}\, M)(p_2)$.

(28) $(\mathrm{Mx2Tran}\, M)(p_1 - p_2) = (\mathrm{Mx2Tran}\, M)(p_1) - (\mathrm{Mx2Tran}\, M)(p_2)$.

(29) $(\mathrm{Mx2Tran}\, M)(0_{\mathcal{E}^n_{\mathrm{T}}}) = 0_{\mathcal{E}^m_{\mathrm{T}}}$.

(30) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ holds $(\mathrm{Mx2Tran}\, M)^\circ(p + A) = (\mathrm{Mx2Tran}\, M)(p) + (\mathrm{Mx2Tran}\, M)^\circ A$.

(31) For every subset A of $\mathcal{E}^m_{\mathrm{T}}$ holds $(\mathrm{Mx2Tran}\, M)^{-1}((\mathrm{Mx2Tran}\, M)(p)+A) = p + (\mathrm{Mx2Tran}\, M)^{-1}(A)$.

(32) Let A be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$ and B be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $\mathrm{width}\, A \times k$. If if $n = 0$, then $m = 0$ and if $m = 0$, then $k = 0$, then $\mathrm{Mx2Tran}\, B \cdot \mathrm{Mx2Tran}\, A = \mathrm{Mx2Tran}(A \cdot B)$.

(33) $\mathrm{Mx2Tran}(I^{n \times n}_{\mathbb{R}_{\mathrm{F}}}) = \mathrm{id}_{\mathcal{E}^n_{\mathrm{T}}}$.

(34) If $\mathrm{Mx2Tran}\, M_1 = \mathrm{Mx2Tran}\, M_2$, then $M_1 = M_2$.

(35) Let A be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$ and B be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $k \times m$. Then $(\mathrm{Mx2Tran}(A \frown B))(f \frown (k \mapsto 0)) = (\mathrm{Mx2Tran}\, A)(f)$ and $(\mathrm{Mx2Tran}(B \frown A))((k \mapsto 0) \frown f) = (\mathrm{Mx2Tran}\, A)(f)$.

(36) Let A be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$, B be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $k \times m$, and g be a k-element real-valued finite sequence. Then $(\mathrm{Mx2Tran}(A \frown B))(f \frown g) = (\mathrm{Mx2Tran}\, A)(f) + (\mathrm{Mx2Tran}\, B)(g)$.

(37) Let A be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times k$ and B be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$ such that if $n = 0$, then $k + m = 0$. Then $(\mathrm{Mx2Tran}(A \frown B))(f) = (\mathrm{Mx2Tran}\, A)(f) \frown (\mathrm{Mx2Tran}\, B)(f)$.

(38) $(\mathrm{Mx2Tran}(I^{m \times m}_{\mathbb{R}_{\mathrm{F}}} \restriction n))(f) \restriction n = f$.

3. Selected Properties of the Mx2Tran Operator

Next we state several propositions:

(39) $\mathrm{Mx2Tran}\, M$ is one-to-one iff $\mathrm{rk}(M) = n$.

(40) $\mathrm{Mx2Tran}\, A$ is one-to-one iff $\mathrm{Det}\, A \neq 0_{\mathbb{R}_{\mathrm{F}}}$.

(41) $\mathrm{Mx2Tran}\, M$ is onto iff $\mathrm{rk}(M) = m$.

(42) $\mathrm{Mx2Tran}\, A$ is onto iff $\mathrm{Det}\, A \neq 0_{\mathbb{R}_{\mathrm{F}}}$.

(43) For all A, B such that $\mathrm{Det}\, A \neq 0_{\mathbb{R}_{\mathrm{F}}}$ holds $(\mathrm{Mx2Tran}\, A)^{-1} = \mathrm{Mx2Tran}\, B$ iff $A^\smile = B$.

(44) There exists an m-element finite sequence L of elements of $\mathbb{R}$ such that for every i such that $i \in \mathrm{dom}\, L$ holds $L(i) = |^{@}(M_{\square,i})|$ and for every f holds $|(\mathrm{Mx2Tran}\, M)(f)| \leq \sum L \cdot |f|$.

(45) There exists a real number L such that $L > 0$ and for every f holds $|(\mathrm{Mx2Tran}\, M)(f)| \leq L \cdot |f|$.

(46) If $\mathrm{rk}(M) = n$, then there exists a real number L such that $L > 0$ and for every f holds $|f| \leq L \cdot |(\mathrm{Mx2Tran}\, M)(f)|$.

(47) $\mathrm{Mx2Tran}\, M$ is continuous.

Let us consider n, K. One can check that there exists a square matrix over K of dimension n which is invertible.

Let us consider n and let A be an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension n. Note that $\mathrm{Mx2Tran}\, A$ is homeomorphism.

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Linear Transformations of Euclidean Topological Spaces. Part II

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Summary. We prove a number of theorems concerning various notions used in the theory of continuity of barycentric coordinates.

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The papers [2], [9], [4], [5], [6], [14], [10], [25], [13], [16], [3], [7], [12], [1], [24], [15], [21], [23], [19], [17], [8], [11], [22], [20], and [18] provide the terminology and notation for this paper.

1. Correspondence Between Euclidean Topological Space and Vector Space over $\mathbb{R}_{\rm F}$

For simplicity, we follow the rules: X denotes a set, n, m, k denote natural numbers, K denotes a field, f denotes an n-element real-valued finite sequence, and M denotes a matrix over $\mathbb{R}_{\rm F}$ of dimension $n \times m$.

One can prove the following propositions:

(1) X is a linear combination of the n-dimension vector space over $\mathbb{R}_{\rm F}$ if and only if X is a linear combination of $\mathcal{E}^n_{\rm T}$.

(2) Let L_2 be a linear combination of the n-dimension vector space over $\mathbb{R}_{\rm F}$ and L_1 be a linear combination of $\mathcal{E}^n_{\rm T}$. If $L_1 = L_2$, then the support of $L_1 =$ the support of L_2.

(3) Let F be a finite sequence of elements of $\mathcal{E}^n_{\rm T}$, f_1 be a function from $\mathcal{E}^n_{\rm T}$ into $\mathbb{R}$, F_1 be a finite sequence of elements of the n-dimension vector space over $\mathbb{R}_{\rm F}$, and f_2 be a function from the n-dimension vector space over $\mathbb{R}_{\rm F}$ into $\mathbb{R}_{\rm F}$. If $f_1 = f_2$ and $F = F_1$, then $f_1\, F = f_2\, F_1$.

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(4) Let F be a finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$ and F_1 be a finite sequence of elements of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. If $F_1 = F$, then $\sum F = \sum F_1$.

(5) Let L_2 be a linear combination of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ and L_1 be a linear combination of $\mathcal{E}^n_{\mathrm{T}}$. If $L_1 = L_2$, then $\sum L_1 = \sum L_2$.

(6) Let A_2 be a subset of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ and A_1 be a subset of $\mathcal{E}^n_{\mathrm{T}}$. If $A_2 = A_1$, then $\Omega_{\mathrm{Lin}(A_1)} = \Omega_{\mathrm{Lin}(A_2)}$.

(7) Let A_2 be a subset of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ and A_1 be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $A_2 = A_1$. Then A_2 is linearly independent if and only if A_1 is linearly independent.

(8) Let V be a vector space over K, W be a subspace of V, and L be a linear combination of V. Then $L{\restriction}$the carrier of W is a linear combination of W.

(9) Let V be a vector space over K, A be a linearly independent subset of V, and L_3, L_4 be linear combinations of V. Suppose the support of $L_3 \subseteq A$ and the support of $L_4 \subseteq A$ and $\sum L_3 = \sum L_4$. Then $L_3 = L_4$.

(10) Let V be a real linear space, W be a subspace of V, and L be a linear combination of V. Then $L{\restriction}$the carrier of W is a linear combination of W.

(11) Let U be a subspace of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ and W be a subspace of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $\Omega_U = \Omega_W$. Then X is a linear combination of U if and only if X is a linear combination of W.

(12) Let U be a subspace of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$, W be a subspace of $\mathcal{E}^n_{\mathrm{T}}$, L_5 be a linear combination of U, and L_6 be a linear combination of W. If $L_5 = L_6$, then the support of $L_5 =$ the support of L_6 and $\sum L_5 = \sum L_6$.

Let us consider m, K and let A be a subset of the m-dimension vector space over K. Note that $\mathrm{Lin}(A)$ is finite dimensional.

2. Correspondence Between the Mx2Tran Operator and Decomposition of a Vector in Basis

The following propositions are true:

(13) If $\mathrm{rk}(M) = n$, then M is an ordered basis of $\mathrm{Lin}(\mathrm{lines}(M))$.

(14) Let V, W be vector spaces over K, T be a linear transformation from V to W, A be a subset of V, and L be a linear combination of A. If $T{\restriction}A$ is one-to-one, then $T(\sum L) = \sum(T^{@}L)$.

(15) Let S be a subset of $\mathrm{Seg}\, n$. Suppose $M{\restriction}S$ is one-to-one and $\mathrm{rng}(M{\restriction}S) = \mathrm{lines}(M)$. Then there exists a linear combination L of $\mathrm{lines}(M)$ such that $\sum L = (\mathrm{Mx2Tran}\, M)(f)$ and for every k such that $k \in S$ holds $L(\mathrm{Line}(M,k)) = \sum \mathrm{Seq}(f{\restriction}M^{-1}(\{\mathrm{Line}(M,k)\}))$.

(16) Suppose M is without repeated line. Then there exists a linear combination L of lines(M) such that $\sum L = (\mathrm{Mx2Tran}\, M)(f)$ and for every k such that $k \in \mathrm{dom}\, f$ holds $L(\mathrm{Line}(M, k)) = f(k)$.

(17) For every ordered basis B of Lin(lines(M)) such that $B = M$ and for every element M_1 of Lin(lines(M)) such that $M_1 = (\mathrm{Mx2Tran}\, M)(f)$ holds $M_1 \rightarrow B = f$.

(18) $\mathrm{rng}\, \mathrm{Mx2Tran}\, M = \Omega_{\mathrm{Lin}(\mathrm{lines}(M))}$.

(19) Let F be a one-to-one finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $\mathrm{rng}\, F$ is linearly independent. Then there exists a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension n such that M is invertible and $M \restriction \mathrm{len}\, F = F$.

(20) Let B be an ordered basis of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. If $B = \mathrm{MX2FinS}(I^{n \times n}_{\mathbb{R}_{\mathrm{F}}})$, then $f \in \mathrm{Lin}(\mathrm{rng}(B \restriction k))$ iff $f = (f \restriction k) \frown ((n -' k) \mapsto 0)$.

(21) Let F be a one-to-one finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $\mathrm{rng}\, F$ is linearly independent. Let B be an ordered basis of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. Suppose $B = \mathrm{MX2FinS}(I^{n \times n}_{\mathbb{R}_{\mathrm{F}}})$. Let M be a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension n. If M is invertible and $M \restriction \mathrm{len}\, F = F$, then $(\mathrm{Mx2Tran}\, M)^\circ(\Omega_{\mathrm{Lin}(\mathrm{rng}(B \restriction \mathrm{len}\, F))}) = \Omega_{\mathrm{Lin}(\mathrm{rng}\, F)}$.

(22) Let A, B be linearly independent subsets of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $\overline{\overline{A}} = \overline{\overline{B}}$. Then there exists a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension n such that M is invertible and $(\mathrm{Mx2Tran}\, M)^\circ(\Omega_{\mathrm{Lin}(A)}) = \Omega_{\mathrm{Lin}(B)}$.

3. Preservation of Linear and Affine Independence of Vectors by the Mx2Tran Operator

The following propositions are true:

(23) For every linearly independent subset A of $\mathcal{E}^n_{\mathrm{T}}$ such that $\mathrm{rk}(M) = n$ holds $(\mathrm{Mx2Tran}\, M)^\circ A$ is linearly independent.

(24) For every affinely independent subset A of $\mathcal{E}^n_{\mathrm{T}}$ such that $\mathrm{rk}(M) = n$ holds $(\mathrm{Mx2Tran}\, M)^\circ A$ is affinely independent.

(25) Let A be an affinely independent subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $\mathrm{rk}(M) = n$. Let v be an element of $\mathcal{E}^n_{\mathrm{T}}$. If $v \in \mathrm{Affin}\, A$, then $(\mathrm{Mx2Tran}\, M)(v) \in \mathrm{Affin}((\mathrm{Mx2Tran}\, M)^\circ A)$ and for every f holds $(v \rightarrow A)(f) = ((\mathrm{Mx2Tran}\, M)(v) \rightarrow (\mathrm{Mx2Tran}\, M)^\circ A)((\mathrm{Mx2Tran}\, M)(f))$.

(26) For every linearly independent subset A of $\mathcal{E}^m_{\mathrm{T}}$ such that $\mathrm{rk}(M) = n$ holds $(\mathrm{Mx2Tran}\, M)^{-1}(A)$ is linearly independent.

(27) For every affinely independent subset A of $\mathcal{E}^m_{\mathrm{T}}$ such that $\mathrm{rk}(M) = n$ holds $(\mathrm{Mx2Tran}\, M)^{-1}(A)$ is affinely independent.

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The Axiomatization of Propositional Linear Time Temporal Logic

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Summary. The article introduces propositional linear time temporal logic as a formal system. Axioms and rules of derivation are defined. Soundness Theorem and Deduction Theorem are proved [9].

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The terminology and notation used in this paper have been introduced in the following papers: [10], [3], [4], [5], [8], [11], [13], [1], [2], [6], [12], and [7].

1. Preliminaries

In this paper a, b, c denote boolean numbers.

Next we state three propositions:

(1) $(a \Rightarrow b \wedge c) \Rightarrow (a \Rightarrow b) = 1$.

(2) $(a \Rightarrow (b \Rightarrow c)) \Rightarrow (a \wedge b \Rightarrow c) = 1$.

(3) $(a \wedge b \Rightarrow c) \Rightarrow (a \Rightarrow (b \Rightarrow c)) = 1$.

2. The Language. Basic Operators. Further Operators as Abbreviations

We introduce the LTLB-WFF as a synonym of HP-WFF.

For simplicity, we adopt the following rules: p, q, r, s, A, B, C are elements of the LTLB-WFF, G is a subset of the LTLB-WFF, i, j, n are elements of $\mathbb{N}$, and f_1, f_2 are finite sequences of elements of the LTLB-WFF.

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We introduce $\perp_t$ as a synonym of VERUM.

Let us consider p, q. We introduce $p \mathcal{U}_s q$ as a synonym of $p \wedge q$.

We now state the proposition

(4) For every A holds $A = \perp_t$ or there exists n such that $A = \mathrm{prop}\, n$ or there exist p, q such that $A = p \Rightarrow q$ or there exist p, q such that $A = p \mathcal{U}_s q$.

Let us consider p. The functor $\neg p$ yields an element of the LTLB-WFF and is defined as follows:

(Def. 1) $\neg p = p \Rightarrow \perp_t$.

The functor $\mathcal{X}\, p$ yielding an element of the LTLB-WFF is defined as follows:

(Def. 2) $\mathcal{X}\, p = \perp_t \mathcal{U}_s\, p$.

The element $\top_t$ of the LTLB-WFF is defined by:

(Def. 3) $\top_t = \neg \perp_t$.

Let us consider p, q. The functor $p \,\&\&\, q$ yields an element of the LTLB-WFF and is defined as follows:

(Def. 4) $p \,\&\&\, q = (p \Rightarrow (q \Rightarrow \perp_t)) \Rightarrow \perp_t$.

Let us consider p, q. The functor $p \,||\, q$ yielding an element of the LTLB-WFF is defined as follows:

(Def. 5) $p \,||\, q = \neg(\neg p \,\&\&\, \neg q)$.

Let us consider p. The functor $\mathcal{G}\, p$ yielding an element of the LTLB-WFF is defined as follows:

(Def. 6) $\mathcal{G}\, p = \neg(\neg p \,||(\top_t \,\&\&(\top_t \mathcal{U}_s \neg p)))$.

Let us consider p. The functor $\mathcal{F}\, p$ yields an element of the LTLB-WFF and is defined as follows:

(Def. 7) $\mathcal{F}\, p = \neg \mathcal{G} \neg p$.

Let us consider p, q. The functor $p \mathcal{U}\, q$ yields an element of the LTLB-WFF and is defined as follows:

(Def. 8) $p \mathcal{U}\, q = q \,||(p \,\&\&(p \mathcal{U}_s\, q))$.

Let us consider p, q. The functor $p \mathcal{R} q$ yielding an element of the LTLB-WFF is defined as follows:

(Def. 9) $p \,\mathcal{R}\, q = \neg(\neg p \,\mathcal{U} \neg q)$.

3. The Semantics

The subset AP of the LTLB-WFF is defined by:

(Def. 10) For every set x holds $x \in AP$ iff there exists an element n of $\mathbb{N}$ such that $x = \mathrm{prop}\, n$.

A LTL Model is a sequence of 2^{AP}.

In the sequel M denotes a LTL Model.

Let M be a LTL Model. The functor SAT_M yielding a function from $\mathbb{N} \times$ the LTLB-WFF into *Boolean* is defined by the condition (Def. 11).

(Def. 11) Let given n. Then

(i) $\mathrm{SAT}_M(\langle n, \perp_t\rangle) = 0$,

(ii) for every k holds $\mathrm{SAT}_M(\langle n, \mathrm{prop}\, k\rangle) = 1$ iff $\mathrm{prop}\, k \in M(n)$, and

(iii) for all p, q holds $\mathrm{SAT}_M(\langle n, p \Rightarrow q\rangle) = \mathrm{SAT}_M(\langle n, p\rangle) \Rightarrow \mathrm{SAT}_M(\langle n, q\rangle)$ and $\mathrm{SAT}_M(\langle n, p\,\mathcal{U}_s\, q\rangle) = 1$ iff there exists i such that $0 < i$ and $\mathrm{SAT}_M(\langle n+i, q\rangle) = 1$ and for every j such that $1 \leq j < i$ holds $\mathrm{SAT}_M(\langle n+j, p\rangle) = 1$.

One can prove the following propositions:

(5) $\mathrm{SAT}_M(\langle n, \neg A\rangle) = 1$ iff $\mathrm{SAT}_M(\langle n, A\rangle) = 0$.

(6) $\mathrm{SAT}_M(\langle n, \top_t\rangle) = 1$.

(7) $\mathrm{SAT}_M(\langle n, A \mathbin{\&\&} B\rangle) = 1$ iff $\mathrm{SAT}_M(\langle n, A\rangle) = 1$ and $\mathrm{SAT}_M(\langle n, B\rangle) = 1$.

(8) $\mathrm{SAT}_M(\langle n, A \,||\, B\rangle) = 1$ iff $\mathrm{SAT}_M(\langle n, A\rangle) = 1$ or $\mathrm{SAT}_M(\langle n, B\rangle) = 1$.

(9) $\mathrm{SAT}_M(\langle n, \mathcal{X}\, A\rangle) = \mathrm{SAT}_M(\langle n+1, A\rangle)$.

(10) $\mathrm{SAT}_M(\langle n, \mathcal{G}\, A\rangle) = 1$ iff for every i holds $\mathrm{SAT}_M(\langle n+i, A\rangle) = 1$.

(11) $\mathrm{SAT}_M(\langle n, \mathcal{F}\, A\rangle) = 1$ iff there exists i such that $\mathrm{SAT}_M(\langle n+i, A\rangle) = 1$.

(12) $\mathrm{SAT}_M(\langle n, p\,\mathcal{U}\, q\rangle) = 1$ iff there exists i such that $\mathrm{SAT}_M(\langle n+i, q\rangle) = 1$ and for every j such that $j < i$ holds $\mathrm{SAT}_M(\langle n+j, p\rangle) = 1$.

(13) $\mathrm{SAT}_M(\langle n, p\,\mathcal{R}\, q\rangle) = 1$ if and only if one of the following conditions is satisfied:

(i) there exists i such that $\mathrm{SAT}_M(\langle n+i, p\rangle) = 1$ and for every j such that $j \leq i$ holds $\mathrm{SAT}_M(\langle n+j, q\rangle) = 1$, or

(ii) for every i holds $\mathrm{SAT}_M(\langle n+i, q\rangle) = 1$.

(14) $\mathrm{SAT}_M(\langle n, \neg\,\mathcal{X}\, B\rangle) = \mathrm{SAT}_M(\langle n, \mathcal{X}\,\neg B\rangle)$.

(15) $\mathrm{SAT}_M(\langle n, \neg\,\mathcal{X}\, B \Rightarrow \mathcal{X}\,\neg B\rangle) = 1$.

(16) $\mathrm{SAT}_M(\langle n, \mathcal{X}\,\neg B \Rightarrow \neg\,\mathcal{X}\, B\rangle) = 1$.

(17) $\mathrm{SAT}_M(\langle n, \mathcal{X}(B \Rightarrow C) \Rightarrow (\mathcal{X}\, B \Rightarrow \mathcal{X}\, C)\rangle) = 1$.

(18) $\mathrm{SAT}_M(\langle n, \mathcal{G}\, B \Rightarrow B \mathbin{\&\&} \mathcal{X}\,\mathcal{G}\, B\rangle) = 1$.

(19) $\mathrm{SAT}_M(\langle n, B\,\mathcal{U}_s\, C \Rightarrow \mathcal{X}\, C \,||\, \mathcal{X}(B \mathbin{\&\&} (B\,\mathcal{U}_s\, C))\rangle) = 1$.

(20) $\mathrm{SAT}_M(\langle n, \mathcal{X}\, C \,||\, \mathcal{X}(B \mathbin{\&\&} (B\,\mathcal{U}_s\, C)) \Rightarrow B\,\mathcal{U}_s\, C\rangle) = 1$.

(21) $\mathrm{SAT}_M(\langle n, B\,\mathcal{U}_s\, C \Rightarrow \mathcal{X}\,\mathcal{F}\, C\rangle) = 1$.

4. Validity. Consequence. Some Facts about the Semantical Notions

Let us consider M, p. The predicate $M \models p$ is defined as follows:

(Def. 12) For every element n of $\mathbb{N}$ holds $\mathrm{SAT}_M(\langle n, p\rangle) = 1$.

Let us consider M, F. The predicate $M \models F$ is defined by:

(Def. 13) For every p such that $p \in F$ holds $M \models p$.

Let us consider F, p. The predicate $F \models p$ is defined as follows:

(Def. 14) For every M such that $M \models F$ holds $M \models p$.

One can prove the following propositions:

(22) $M \models F$ and $M \models G$ iff $M \models F \cup G$.

(23) $M \models A$ iff $M \models \{A\}$.

(24) If $F \models A$ and $F \models A \Rightarrow B$, then $F \models B$.

(25) If $F \models A$, then $F \models \mathcal{X}\, A$.

(26) If $F \models A$, then $F \models \mathcal{G}\, A$.

(27) If $F \models A \Rightarrow B$ and $F \models A \Rightarrow \mathcal{X}\, A$, then $F \models A \Rightarrow \mathcal{G}\, B$.

(28) $\mathrm{SAT}_{(M \uparrow i)}(\langle j, A\rangle) = \mathrm{SAT}_M(\langle i + j, A\rangle)$.

(29) If $M \models F$, then $M \uparrow i \models F$.

(30) $F \cup \{A\} \models B$ iff $F \models \mathcal{G}\, A \Rightarrow B$.

Let f be a function from the LTLB-WFF into *Boolean*. The functor $\mathrm{VAL}\, f$ yielding a function from the LTLB-WFF into *Boolean* is defined as follows:

(Def. 15) $(\mathrm{VAL}\, f)(\bot_t) = 0$ and $(\mathrm{VAL}\, f)(\mathrm{prop}\, n) = f(\mathrm{prop}\, n)$ and $(\mathrm{VAL}\, f)(A \Rightarrow B) = (\mathrm{VAL}\, f)(A) \Rightarrow (\mathrm{VAL}\, f)(B)$ and $(\mathrm{VAL}\, f)(A\, \mathcal{U}_s\, B) = f(A\, \mathcal{U}_s\, B)$.

The following propositions are true:

(31) For every function f from the LTLB-WFF into *Boolean* and for all p, q holds $(\mathrm{VAL}\, f)(p \,\&\&\, q) = (\mathrm{VAL}\, f)(p) \wedge (\mathrm{VAL}\, f)(q)$.

(32) Let f be a function from the LTLB-WFF into *Boolean*. Suppose that for every set B such that $B \in$ the LTLB-WFF holds $f(B) = \mathrm{SAT}_M(\langle n, B\rangle)$. Then $(\mathrm{VAL}\, f)(A) = \mathrm{SAT}_M(\langle n, A\rangle)$.

Let us consider p. We say that p is tautologically valid if and only if:

(Def. 16) For every function f from the LTLB-WFF into *Boolean* holds $(\mathrm{VAL}\, f)(p) = 1$.

One can prove the following proposition

(33) If A is tautologically valid, then $F \models A$.

5. Axioms. Derivation Rules. Derivability. Soundness Theorem for LTL

Let D be a set. We say that D has LTL axioms if and only if the condition (Def. 17) is satisfied.

(Def. 17) Let given p, q. Then if p is tautologically valid, then $p \in D$,
$\neg\, \mathcal{X}\, p \Rightarrow \mathcal{X}\, \neg p \in D$,
$\mathcal{X}\, \neg p \Rightarrow \neg\, \mathcal{X}\, p \in D$,

$\mathcal{X}(p \Rightarrow q) \Rightarrow (\mathcal{X}\, p \Rightarrow \mathcal{X}\, q) \in D$,
$\mathcal{G}\, p \Rightarrow p \,\&\&\, \mathcal{X}\, \mathcal{G}\, p \in D$,
$p\, \mathcal{U}_s\, q \Rightarrow \mathcal{X}\, q \,||\, \mathcal{X}(p \,\&\&(p\, \mathcal{U}_s\, q)) \in D$,
$\mathcal{X}\, q \,||\, \mathcal{X}(p \,\&\&(p\, \mathcal{U}_s\, q)) \Rightarrow p\, \mathcal{U}_s\, q \in D$,
$p\, \mathcal{U}_s\, q \Rightarrow \mathcal{X}\, \mathcal{F}\, q \in D$.

The subset $AX_{\rm LTL}$ of the LTLB-WFF is defined as follows:

(Def. 18) $AX_{\rm LTL}$ has LTL axioms and for every subset D of the LTLB-WFF such that D has LTL axioms holds $AX_{\rm LTL} \subseteq D$.

Let us mention that $AX_{\rm LTL}$ has LTL axioms.

Next we state two propositions:

(34) $p \Rightarrow (q \Rightarrow p) \in AX_{\rm LTL}$.

(35) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \in AX_{\rm LTL}$.

Let us consider p, q. The predicate $\mathrm{NEX}(p, q)$ is defined as follows:

(Def. 19) $q = \mathcal{X}\, p$.

Let us consider r. The predicate $\mathrm{MP}(p, q, r)$ is defined as follows:

(Def. 20) $q = p \Rightarrow r$.

The predicate $\mathrm{IND}(p, q, r)$ is defined as follows:

(Def. 21) There exist A, B such that $p = A \Rightarrow B$ and $q = A \Rightarrow \mathcal{X}\, A$ and $r = A \Rightarrow \mathcal{G}\, B$.

Let us observe that $AX_{\rm LTL}$ is non empty.

Let us consider A. We say that A is LTL axiom 1 if and only if:

(Def. 22) There exists B such that $A = \neg\, \mathcal{X}\, B \Rightarrow \mathcal{X}\, \neg B$.

We say that A is LTL axiom 1a if and only if:

(Def. 23) There exists B such that $A = \mathcal{X}\, \neg B \Rightarrow \neg\, \mathcal{X}\, B$.

We say that A is LTL axiom 2 if and only if:

(Def. 24) There exist B, C such that $A = \mathcal{X}(B \Rightarrow C) \Rightarrow (\mathcal{X}\, B \Rightarrow \mathcal{X}\, C)$.

We say that A is LTL axiom 3 if and only if:

(Def. 25) There exists B such that $A = \mathcal{G}\, B \Rightarrow B \,\&\&\, \mathcal{X}\, \mathcal{G}\, B$.

We say that A is LTL axiom 4 if and only if:

(Def. 26) There exist B, C such that $A = B\, \mathcal{U}_s\, C \Rightarrow \mathcal{X}\, C \,||\, \mathcal{X}(B \,\&\&(B\, \mathcal{U}_s\, C))$.

We say that A is LTL axiom 5 if and only if:

(Def. 27) There exist B, C such that $A = \mathcal{X}\, C \,||\, \mathcal{X}(B \,\&\&(B\, \mathcal{U}_s\, C)) \Rightarrow B\, \mathcal{U}_s\, C$.

We say that A is LTL axiom 6 if and only if:

(Def. 28) There exist B, C such that $A = B\, \mathcal{U}_s\, C \Rightarrow \mathcal{X}\, \mathcal{F}\, C$.

Next we state two propositions:

(36) Every element of $AX_{\rm LTL}$ is tautologically valid, or LTL axiom 1, or LTL axiom 1a, or LTL axiom 2, or LTL axiom 3, or LTL axiom 4, or LTL axiom 5, or LTL axiom 6.

(37) Suppose that A is LTL axiom 1, or LTL axiom 1a, or LTL axiom 2, or LTL axiom 3, or LTL axiom 4, or LTL axiom 5, or LTL axiom 6. Then $F \models A$.

Let i be a natural number and let us consider f, X. The predicate $\mathrm{prc}(f, X, i)$ is defined by the conditions (Def. 29).

(Def. 29)(i) $f(i) \in AX_{\mathrm{LTL}}$, or

(ii) $f(i) \in X$, or

(iii) there exist natural numbers j, k such that $1 \leq j < i$ and $1 \leq k < i$ and $\mathrm{MP}(f_j, f_k, f_i)$ or $\mathrm{IND}(f_j, f_k, f_i)$, or

(iv) there exists a natural number j such that $1 \leq j < i$ and $\mathrm{NEX}(f_j, f_i)$.

Let us consider X, p. The predicate $X \vdash p$ is defined as follows:

(Def. 30) There exists f such that $f(\mathrm{len}\, f) = p$ and $1 \leq \mathrm{len}\, f$ and for every natural number i such that $1 \leq i \leq \mathrm{len}\, f$ holds $\mathrm{prc}(f, X, i)$.

We now state four propositions:

(38) Let i, n be natural numbers. Suppose $n + \mathrm{len}\, f \leq \mathrm{len}\, f_2$ and for every natural number k such that $1 \leq k \leq \mathrm{len}\, f$ holds $f(k) = f_2(k + n)$ and $1 \leq i \leq \mathrm{len}\, f$. If $\mathrm{prc}(f, X, i)$, then $\mathrm{prc}(f_2, X, i + n)$.

(39) Suppose that

(i) $f_2 = f \frown f_1$,

(ii) $1 \leq \mathrm{len}\, f$,

(iii) $1 \leq \mathrm{len}\, f_1$,

(iv) for every natural number i such that $1 \leq i \leq \mathrm{len}\, f$ holds $\mathrm{prc}(f, X, i)$, and

(v) for every natural number i such that $1 \leq i \leq \mathrm{len}\, f_1$ holds $\mathrm{prc}(f_1, X, i)$.

Let i be a natural number. If $1 \leq i \leq \mathrm{len}\, f_2$, then $\mathrm{prc}(f_2, X, i)$.

(40) Suppose $f = f_1 \frown \langle p \rangle$ and $1 \leq \mathrm{len}\, f_1$ and for every natural number i such that $1 \leq i \leq \mathrm{len}\, f_1$ holds $\mathrm{prc}(f_1, X, i)$ and $\mathrm{prc}(f, X, \mathrm{len}\, f)$. Then for every natural number i such that $1 \leq i \leq \mathrm{len}\, f$ holds $\mathrm{prc}(f, X, i)$ and $X \vdash p$.

(41)[1] If $F \vdash A$, then $F \models A$.

6. Derivation of Some Formulas. Deduction Theorem of LTL

We now state a number of propositions:

(42) If $p \in AX_{\mathrm{LTL}}$ or $p \in X$, then $X \vdash p$.

(43) If $X \vdash p$ and $X \vdash p \Rightarrow q$, then $X \vdash q$.

(44) If $X \vdash p$, then $X \vdash \mathcal{X}\, p$.

(45) If $X \vdash p \Rightarrow q$ and $X \vdash p \Rightarrow \mathcal{X}\, p$, then $X \vdash p \Rightarrow \mathcal{G}\, q$.

(46) If $X \vdash r \Rightarrow p \,\&\&\, q$, then $X \vdash r \Rightarrow p$ and $X \vdash r \Rightarrow q$.

[1]Soundness Theorem for LTL

(47) If $X \vdash p \Rightarrow q$ and $X \vdash q \Rightarrow r$, then $X \vdash p \Rightarrow r$.

(48) If $X \vdash p \Rightarrow (q \Rightarrow r)$, then $X \vdash p \,\&\&\, q \Rightarrow r$.

(49) If $X \vdash p \,\&\&\, q \Rightarrow r$, then $X \vdash p \Rightarrow (q \Rightarrow r)$.

(50) If $X \vdash p \,\&\&\, q \Rightarrow r$ and $X \vdash p \Rightarrow s$, then $X \vdash p \,\&\&\, q \Rightarrow s \,\&\&\, r$.

(51) If $X \vdash p \Rightarrow (q \Rightarrow r)$ and $X \vdash r \Rightarrow s$, then $X \vdash p \Rightarrow (q \Rightarrow s)$.

(52) If $X \vdash p \Rightarrow q$, then $X \vdash \neg q \Rightarrow \neg p$.

(53) $X \vdash \mathcal{X}\, p \,\&\&\, \mathcal{X}\, q \Rightarrow \mathcal{X}(p \,\&\&\, q)$.

(54) If $F \vdash p$, then $F \vdash \mathcal{G}\, p$.

(55) If $p \Rightarrow q \in F$, then $F \cup \{p\} \vdash \mathcal{G}\, q$.

(56) If $F \vdash q$, then $F \cup \{p\} \vdash q$.

(57)[2] If $F \cup \{p\} \vdash q$, then $F \vdash \mathcal{G}\, p \Rightarrow q$.

(58) If $F \vdash p \Rightarrow q$, then $F \cup \{p\} \vdash q$.

(59) If $F \vdash \mathcal{G}\, p \Rightarrow q$, then $F \cup \{p\} \vdash q$.

(60) $F \vdash \mathcal{G}(p \Rightarrow q) \Rightarrow (\mathcal{G}\, p \Rightarrow \mathcal{G}\, q)$.

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[2] Deduction Theorem of LTL

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Banach Algebra of Bounded Complex-Valued Functionals

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Summary. In this article, we describe some basic properties of the Banach algebra which is constructed from all bounded complex-valued functionals.

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The notation and terminology used in this paper are introduced in the following articles: [2], [16], [9], [14], [7], [8], [3], [18], [17], [4], [19], [5], [15], [1], [20], [12], [11], [10], [21], [13], and [6].

Let V be a complex algebra. A complex algebra is called a complex subalgebra of V if it satisfies the conditions (Def. 1).

(Def. 1)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the addition of it = (the addition of V) $\restriction$ (the carrier of it),

(iii) the multiplication of it = (the multiplication of V) $\restriction$ (the carrier of it),

(iv) the external multiplication of it = (the external multiplication of V)$\restriction$($\mathbb{C}$ $\times$ the carrier of it),

(v) $1_{\rm it} = 1_V$, and

(vi) $0_{\rm it} = 0_V$.

We now state the proposition

(1) Let X be a non empty set, V be a complex algebra, V_1 be a non empty subset of V, d_1, d_2 be elements of X, A be a binary operation on X, M be a function from $X \times X$ into X, and M_1 be a function from $\mathbb{C} \times X$ into X. Suppose that $V_1 = X$ and $d_1 = 0_V$ and $d_2 = 1_V$ and $A =$ (the addition of V) $\restriction$ (V_1) and $M =$ (the multiplication of V) $\restriction$ (V_1) and $M_1 =$ (the external multiplication of V)$\restriction$($\mathbb{C} \times V_1$) and V_1 has inverse. Then $\langle X, M, A, M_1, d_2, d_1\rangle$ is a complex subalgebra of V.

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Let V be a complex algebra. One can check that there exists a complex subalgebra of V which is strict.

Let V be a complex algebra and let V_1 be a subset of V. We say that V_1 is $\mathbb{C}$-additively-linearly-closed if and only if:

(Def. 2) V_1 is add closed and has inverse and for every complex number a and for every element v of V such that $v \in V_1$ holds $a \cdot v \in V_1$.

Let V be a complex algebra and let V_1 be a subset of V. Let us assume that V_1 is $\mathbb{C}$-additively-linearly-closed and non empty. The functor $\mathrm{Mult}(V_1, V)$ yielding a function from $\mathbb{C} \times V_1$ into V_1 is defined as follows:

(Def. 3) $\mathrm{Mult}(V_1, V) = (\text{the external multiplication of } V) \restriction (\mathbb{C} \times V_1)$.

Let X be a non empty set. The functor $\mathbb{C}\text{-BoundedFunctions}\, X$ yielding a non empty subset of $\mathrm{CAlgebra}(X)$ is defined by:

(Def. 4) $\mathbb{C}\text{-BoundedFunctions}\, X = \{f : X \to \mathbb{C}\colon f \restriction X \text{ is bounded}\}$.

Let X be a non empty set. Note that $\mathrm{CAlgebra}(X)$ is scalar unital.

Let X be a non empty set. One can verify that $\mathbb{C}\text{-BoundedFunctions}\, X$ is $\mathbb{C}$-additively-linearly-closed and multiplicatively-closed.

Let V be a complex algebra. Observe that there exists a non empty subset of V which is $\mathbb{C}$-additively-linearly-closed and multiplicatively-closed.

Let V be a non empty CLS structure. We say that V is scalar-multiplcation-cancelable if and only if:

(Def. 5) For every complex number a and for every element v of V such that $a \cdot v = 0_V$ holds $a = 0$ or $v = 0_V$.

One can prove the following two propositions:

(2) Let V be a complex algebra and V_1 be a $\mathbb{C}$-additively-linearly-closed multiplicatively-closed non empty subset of V. Then $\langle V_1, \mathrm{mult}(V_1, V), \mathrm{Add}(V_1, V), \mathrm{Mult}(V_1, V), \mathrm{One}(V_1, V), \mathrm{Zero}(V_1, V)\rangle$ is a complex subalgebra of V.

(3) Let V be a complex algebra and V_1 be a complex subalgebra of V. Then

(i) for all elements v_1, w_1 of V_1 and for all elements v, w of V such that $v_1 = v$ and $w_1 = w$ holds $v_1 + w_1 = v + w$,

(ii) for all elements v_1, w_1 of V_1 and for all elements v, w of V such that $v_1 = v$ and $w_1 = w$ holds $v_1 \cdot w_1 = v \cdot w$,

(iii) for every element v_1 of V_1 and for every element v of V and for every complex number a such that $v_1 = v$ holds $a \cdot v_1 = a \cdot v$,

(iv) $\mathbf{1}_{(V_1)} = \mathbf{1}_V$, and

(v) $0_{(V_1)} = 0_V$.

Let X be a non empty set. The $\mathbb{C}$-algebra of bounded functions of X yielding a complex algebra is defined by:

(Def. 6) The $\mathbb{C}$-algebra of bounded functions of $X =$
$\langle \mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{mult}(\mathbb{C}\text{-BoundedFunctions}\, X, \mathrm{CAlgebra}(X)),$

Add($\mathbb{C}$-BoundedFunctions X, CAlgebra(X)),
Mult($\mathbb{C}$-BoundedFunctions X, CAlgebra(X)),
One($\mathbb{C}$-BoundedFunctions X, CAlgebra(X)),
Zero($\mathbb{C}$-BoundedFunctions X, CAlgebra(X))$\rangle$.

One can prove the following proposition

(4) For every non empty set X holds the $\mathbb{C}$-algebra of bounded functions of X is a complex subalgebra of CAlgebra(X).

Let X be a non empty set. Note that the $\mathbb{C}$-algebra of bounded functions of X is vector distributive and scalar unital.

Next we state several propositions:

(5) Let X be a non empty set, F, G, H be vectors of the $\mathbb{C}$-algebra of bounded functions of X, and f, g, h be functions from X into $\mathbb{C}$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element x of X holds $h(x) = f(x) + g(x)$.

(6) Let X be a non empty set, a be a complex number, F, G be vectors of the $\mathbb{C}$-algebra of bounded functions of X, and f, g be functions from X into $\mathbb{C}$. Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element x of X holds $g(x) = a \cdot f(x)$.

(7) Let X be a non empty set, F, G, H be vectors of the $\mathbb{C}$-algebra of bounded functions of X, and f, g, h be functions from X into $\mathbb{C}$. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element x of X holds $h(x) = f(x) \cdot g(x)$.

(8) For every non empty set X holds $0_{\text{the } \mathbb{C}\text{-algebra of bounded functions of } X} = X \longmapsto 0$.

(9) For every non empty set X holds $\mathbf{1}_{\text{the } \mathbb{C}\text{-algebra of bounded functions of } X} = X \longmapsto 1_{\mathbb{C}}$.

Let X be a non empty set and let F be a set. Let us assume that $F \in$ $\mathbb{C}$-BoundedFunctions X. The functor modetrans(F, X) yields a function from X into $\mathbb{C}$ and is defined by:

(Def. 7) modetrans(F, X) $= F$ and modetrans(F, X)$\restriction X$ is bounded.

Let X be a non empty set and let f be a function from X into $\mathbb{C}$. The functor PreNorms(f) yields a non empty subset of $\mathbb{R}$ and is defined by:

(Def. 8) PreNorms(f) $= \{|f(x)| : x$ ranges over elements of $X\}$.

We now state two propositions:

(10) For every non empty set X and for every function f from X into $\mathbb{C}$ such that $f\restriction X$ is bounded holds PreNorms(f) is upper bounded.

(11) Let X be a non empty set and f be a function from X into $\mathbb{C}$. Then $f\restriction X$ is bounded if and only if PreNorms(f) is upper bounded.

Let X be a non empty set. The functor $\mathbb{C}$-BoundedFunctionsNorm X yields a function from $\mathbb{C}$-BoundedFunctions X into $\mathbb{R}$ and is defined by:

(Def. 9) For every set x such that $x \in \mathbb{C}$-BoundedFunctions X holds ($\mathbb{C}$-BoundedFunctionsNorm $X)(x) = \sup \mathrm{PreNorms}(\mathrm{modetrans}(x, X))$.

One can prove the following two propositions:

(13)[1] For every non empty set X and for every function f from X into $\mathbb{C}$ such that $f{\restriction}X$ is bounded holds $\mathrm{modetrans}(f, X) = f$.

(14) For every non empty set X and for every function f from X into $\mathbb{C}$ such that $f{\restriction}X$ is bounded holds ($\mathbb{C}$-BoundedFunctionsNorm $X)(f) = \sup \mathrm{PreNorms}(f)$.

Let X be a non empty set. The $\mathbb{C}$-normed algebra of bounded functions of X yielding a normed complex algebra structure is defined by:

(Def. 10) The $\mathbb{C}$-normed algebra of bounded functions of $X =$
$\langle \mathbb{C}$-BoundedFunctions $X, \mathrm{mult}(\mathbb{C}$-BoundedFunctions $X, \mathrm{CAlgebra}(X))$,
$\mathrm{Add}(\mathbb{C}$-BoundedFunctions $X, \mathrm{CAlgebra}(X))$,
$\mathrm{Mult}(\mathbb{C}$-BoundedFunctions $X, \mathrm{CAlgebra}(X))$,
$\mathrm{One}(\mathbb{C}$-BoundedFunctions $X, \mathrm{CAlgebra}(X))$,
$\mathrm{Zero}(\mathbb{C}$-BoundedFunctions $X, \mathrm{CAlgebra}(X)), \mathbb{C}$-BoundedFunctionsNorm $X\rangle$.

Let X be a non empty set. One can verify that the $\mathbb{C}$-normed algebra of bounded functions of X is non empty.

Let X be a non empty set. One can check that the $\mathbb{C}$-normed algebra of bounded functions of X is unital.

We now state a number of propositions:

(15) Let W be a normed complex algebra structure and V be a complex algebra. Suppose $\langle$the carrier of W, the multiplication of W, the addition of W, the external multiplication of W, the one of W, the zero of $W\rangle = V$. Then W is a complex algebra.

(16) For every non empty set X holds the $\mathbb{C}$-normed algebra of bounded functions of X is a complex algebra.

(17) Let X be a non empty set and F be a point of the $\mathbb{C}$-normed algebra of bounded functions of X.
Then $(\mathrm{Mult}(\mathbb{C}$-BoundedFunctions $X, \mathrm{CAlgebra}(X)))(1_{\mathbb{C}}, F) = F$.

(18) For every non empty set X holds the $\mathbb{C}$-normed algebra of bounded functions of X is a complex linear space.

(19) For every non empty set X holds
$X \longmapsto 0 = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$.

(20) Let X be a non empty set, x be an element of X, f be a function from X into $\mathbb{C}$, and F be a point of the $\mathbb{C}$-normed algebra of bounded functions of X. If $f = F$ and $f{\restriction}X$ is bounded, then $|f(x)| \leq \|F\|$.

[1]The proposition (12) has been removed.

(21) For every non empty set X and for every point F of the $\mathbb{C}$-normed algebra of bounded functions of X holds $0 \leq \|F\|$.

(22) Let X be a non empty set and F be a point of the $\mathbb{C}$-normed algebra of bounded functions of X. Suppose $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$. Then $0 = \|F\|$.

(23) Let X be a non empty set, f, g, h be functions from X into $\mathbb{C}$, and F, G, H be points of the $\mathbb{C}$-normed algebra of bounded functions of X. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F + G$ if and only if for every element x of X holds $h(x) = f(x) + g(x)$.

(24) Let X be a non empty set, a be a complex number, f, g be functions from X into $\mathbb{C}$, and F, G be points of the $\mathbb{C}$-normed algebra of bounded functions of X. Suppose $f = F$ and $g = G$. Then $G = a \cdot F$ if and only if for every element x of X holds $g(x) = a \cdot f(x)$.

(25) Let X be a non empty set, f, g, h be functions from X into $\mathbb{C}$, and F, G, H be points of the $\mathbb{C}$-normed algebra of bounded functions of X. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F \cdot G$ if and only if for every element x of X holds $h(x) = f(x) \cdot g(x)$.

(26) Let X be a non empty set, a be a complex number, and F, G be points of the $\mathbb{C}$-normed algebra of bounded functions of X. Then
   (i) if $\|F\| = 0$, then $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$,
   (ii) if $F = 0_{\text{the } \mathbb{C}\text{-normed algebra of bounded functions of } X}$, then $\|F\| = 0$,
   (iii) $\|a \cdot F\| = |a| \cdot \|F\|$, and
   (iv) $\|F + G\| \leq \|F\| + \|G\|$.

Let X be a non empty set. Note that the $\mathbb{C}$-normed algebra of bounded functions of X is right complementable, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, scalar unital, discernible, reflexive, and complex normed space-like.

We now state two propositions:

(27) Let X be a non empty set, f, g, h be functions from X into $\mathbb{C}$, and F, G, H be points of the $\mathbb{C}$-normed algebra of bounded functions of X. Suppose $f = F$ and $g = G$ and $h = H$. Then $H = F - G$ if and only if for every element x of X holds $h(x) = f(x) - g(x)$.

(28) Let X be a non empty set and s_1 be a sequence of the $\mathbb{C}$-normed algebra of bounded functions of X. If s_1 is CCauchy, then s_1 is convergent.

Let X be a non empty set. Observe that the $\mathbb{C}$-normed algebra of bounded functions of X is complete.

Next we state the proposition

(29) For every non empty set X holds the $\mathbb{C}$-normed algebra of bounded functions of X is a complex Banach algebra.

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