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Mazur-Ulam Theorem

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Summary. The Mazur-Ulam theorem [15] has been formulated as two registrations: `cluster bijective isometric -> midpoints-preserving Function of E,F;` and `cluster isometric midpoints-preserving -> Affine Function of E,F;` A proof given by Jussi Väisälä [23] has been formalized.

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The notation and terminology used in this paper have been introduced in the following papers: [19], [18], [4], [5], [20], [11], [10], [14], [17], [1], [6], [16], [24], [25], [21], [13], [12], [22], [2], [9], [8], [3], and [7].

For simplicity, we use the following convention: E, F, G are real normed spaces, f is a function from E into F, g is a function from F into G, a, b are points of E, and t is a real number.

Let us note that $\mathbb{I}$ is closed.

Next we state four propositions:

(1) DYADIC is a dense subset of $\mathbb{I}$.

(2) $\overline{\text{DYADIC}} = [0, 1]$.

(3) $a + a = 2 \cdot a$.

(4) $(a + b) - b = a$.

Let A be an upper bounded real-membered set and let r be a non negative real number. Observe that $r \circ A$ is upper bounded.

Let A be an upper bounded real-membered set and let r be a non positive real number. Note that $r \circ A$ is lower bounded.

Let A be a lower bounded real-membered set and let r be a non negative real number. Observe that $r \circ A$ is lower bounded.

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Let A be a lower bounded non empty real-membered set and let r be a non positive real number. One can check that $r \circ A$ is upper bounded.

Next we state three propositions:

(5) For every sequence f of real numbers holds $f + (\mathbb{N} \longmapsto t) = t + f$.

(6) For every real number r holds $\lim(\mathbb{N} \longmapsto r) = r$.

(7) For every convergent sequence f of real numbers holds $\lim(t + f) = t + \lim f$.

Let f be a convergent sequence of real numbers and let us consider t. One can check that $t + f$ is convergent.

Next we state three propositions:

(8) For every sequence f of real numbers holds $f \cdot (\mathbb{N} \longmapsto a) = f \cdot a$.

(9) $\lim(\mathbb{N} \longmapsto a) = a$.

(10) For every convergent sequence f of real numbers holds $\lim(f \cdot a) = \lim f \cdot a$.

Let f be a convergent sequence of real numbers and let us consider E, a. Note that $f \cdot a$ is convergent.

Let E, F be non empty normed structures and let f be a function from E into F. We say that f is isometric if and only if:

(Def. 1) For all points a, b of E holds $\|f(a) - f(b)\| = \|a - b\|$.

Let E, F be non empty RLS structures and let f be a function from E into F. We say that f is affine if and only if:

(Def. 2) For all points a, b of E and for every real number t such that $0 \leq t \leq 1$ holds $f((1 - t) \cdot a + t \cdot b) = (1 - t) \cdot f(a) + t \cdot f(b)$.

We say that f preserves midpoints if and only if:

(Def. 3) For all points a, b of E holds $f(\frac{1}{2} \cdot (a + b)) = \frac{1}{2} \cdot (f(a) + f(b))$.

Let E be a non empty normed structure. Observe that id_E is isometric.

Let E be a non empty RLS structure. Note that id_E is affine and preserves midpoints.

Let E be a non empty normed structure. Observe that there exists a unary operation on E which is bijective, isometric, and affine and preserves midpoints.

Next we state the proposition

(11) If f is isometric and g is isometric, then $g \cdot f$ is isometric.

Let us consider E and let f, g be isometric unary operations on E. One can verify that $g \cdot f$ is isometric.

The following proposition is true

(12) If f is bijective and isometric, then f^{-1} is isometric.

Let us consider E and let f be a bijective isometric unary operation on E. One can check that f^{-1} is isometric.

We now state the proposition

(13) If f preserves midpoints and g preserves midpoints, then $g \cdot f$ preserves midpoints.

Let us consider E and let f, g be unary operations on E preserving midpoints. Note that $g \cdot f$ preserves midpoints.

The following proposition is true

(14) If f is bijective and preserves midpoints, then f^{-1} preserves midpoints.

Let us consider E and let f be a bijective unary operation on E preserving midpoints. Observe that f^{-1} preserves midpoints.

Next we state the proposition

(15) If f is affine and g is affine, then $g \cdot f$ is affine.

Let us consider E and let f, g be affine unary operations on E. Observe that $g \cdot f$ is affine.

One can prove the following proposition

(16) If f is bijective and affine, then f^{-1} is affine.

Let us consider E and let f be a bijective affine unary operation on E. Observe that f^{-1} is affine.

Let E be a non empty RLS structure and let a be a point of E. The functor a-reflection yields a unary operation on E and is defined as follows:

(Def. 4) For every point b of E holds $a\text{-reflection}(b) = 2 \cdot a - b$.

The following proposition is true

(17) $a\text{-reflection} \cdot a\text{-reflection} = \mathrm{id}_E$.

Let us consider E, a. Note that a-reflection is bijective.

We now state several propositions:

(18) $a\text{-reflection}(a) = a$ and for every b such that $a\text{-reflection}(b) = b$ holds $a = b$.

(19) $a\text{-reflection}(b) - a = a - b$.

(20) $\|a\text{-reflection}(b) - a\| = \|b - a\|$.

(21) $a\text{-reflection}(b) - b = 2 \cdot (a - b)$.

(22) $\|a\text{-reflection}(b) - b\| = 2 \cdot \|b - a\|$.

(23) $a\text{-reflection}^{-1} = a\text{-reflection}$.

Let us consider E, a. Observe that a-reflection is isometric.

Next we state the proposition

(24) If f is isometric, then f is continuous on $\mathrm{dom}\, f$.

Let us consider E, F. Observe that every function from E into F which is bijective and isometric also preserves midpoints.

Let us consider E, F. One can check that every function from E into F which is isometric and preserves midpoints is also affine.

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Set of Points on Elliptic Curve in Projective Coordinates[1]

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Summary. In this article, we formalize a set of points on an elliptic curve over **GF**(**p**). Elliptic curve cryptography [10], whose security is based on a difficulty of discrete logarithm problem of elliptic curves, is important for information security.

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The notation and terminology used here have been introduced in the following papers: [15], [1], [16], [13], [3], [8], [5], [6], [19], [18], [14], [17], [2], [12], [4], [9], [22], [23], [20], [21], [11], and [7].

1. Finite Prime Field GF(p)

For simplicity, we use the following convention: x is a set, i, j are integers, n, n_1, n_2 are natural numbers, and K, K_1, K_2 are fields.

Let K be a field. A field is called a subfield of K if it satisfies the conditions (Def. 1).

(Def. 1)(i) The carrier of it $\subseteq$ the carrier of K,

(ii) the addition of it = (the addition of K) $\restriction$ (the carrier of it),

(iii) the multiplication of it = (the multiplication of K) $\restriction$ (the carrier of it),

(iv) $1_{\rm it} = 1_K$, and

(v) $0_{\rm it} = 0_K$.

We now state two propositions:

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(1) K is a subfield of K.

(2) Let S_1 be a non empty double loop structure. Suppose that

(i) the carrier of S_1 is a subset of the carrier of K,

(ii) the addition of S_1 = (the addition of K) $\restriction$ (the carrier of S_1),

(iii) the multiplication of S_1 = (the multiplication of K) $\restriction$ (the carrier of S_1),

(iv) $1_{(S_1)} = 1_K$,

(v) $0_{(S_1)} = 0_K$, and

(vi) S_1 is right complementable, commutative, almost left invertible, and non degenerated.

Then S_1 is a subfield of K.

Let K be a field. One can check that there exists a subfield of K which is strict.

In the sequel S_2, S_3 denote subfields of K and e_1, e_2 denote elements of K.

We now state several propositions:

(3) If K_1 is a subfield of K_2, then for every x such that $x \in K_1$ holds $x \in K_2$.

(4) For all strict fields K_1, K_2 such that K_1 is a subfield of K_2 and K_2 is a subfield of K_1 holds $K_1 = K_2$.

(5) Let K_1, K_2, K_3 be strict fields. Suppose K_1 is a subfield of K_2 and K_2 is a subfield of K_3. Then K_1 is a subfield of K_3.

(6) S_2 is a subfield of S_3 iff the carrier of $S_2 \subseteq$ the carrier of S_3.

(7) S_2 is a subfield of S_3 iff for every x such that $x \in S_2$ holds $x \in S_3$.

(8) For all strict subfields S_2, S_3 of K holds $S_2 = S_3$ iff the carrier of S_2 = the carrier of S_3.

(9) For all strict subfields S_2, S_3 of K holds $S_2 = S_3$ iff for every x holds $x \in S_2$ iff $x \in S_3$.

Let K be a finite field. Observe that there exists a subfield of K which is finite. Then $\overline{\overline{K}}$ is an element of $\mathbb{N}$.

Let us mention that there exists a field which is strict and finite.

Next we state the proposition

(10) For every strict finite field K and for every strict subfield S_2 of K such that $\overline{\overline{K}} = \overline{\overline{S_2}}$ holds $S_2 = K$.

Let I_1 be a field. We say that I_1 is prime if and only if:

(Def. 2) If K_1 is a strict subfield of I_1, then $K_1 = I_1$.

Let p be a prime number. We introduce GF(p) as a synonym of $\mathbb{Z}_p^{\mathrm{R}}$. One can check that GF($p$) is finite. One can check that GF(p) is prime.

One can check that there exists a field which is prime.

2. Arithmetic in GF(p)

In the sequel b, c denote elements of GF(p) and F denotes a finite sequence of elements of GF(p).

Next we state a number of propositions:

(11) $0 = 0_{\mathrm{GF}(p)}$.

(12) $1 = 1_{\mathrm{GF}(p)}$.

(13) There exists n_1 such that $a = n_1 \bmod p$.

(14) There exists a such that $a = i \bmod p$.

(15) If $a = i \bmod p$ and $b = j \bmod p$, then $a + b = (i + j) \bmod p$.

(16) If $a = i \bmod p$, then $-a = (p - i) \bmod p$.

(17) If $a = i \bmod p$ and $b = j \bmod p$, then $a - b = (i - j) \bmod p$.

(18) If $a = i \bmod p$ and $b = j \bmod p$, then $a \cdot b = i \cdot j \bmod p$.

(19) If $a = i \bmod p$ and $i \cdot j \bmod p = 1$, then $a^{-1} = j \bmod p$.

(20) $a = 0$ or $b = 0$ iff $a \cdot b = 0$.

(21) $a^0 = \mathbf{1}_{\mathrm{GF}(p)}$ and $a^0 = 1$.

(22) $a^2 = a \cdot a$.

(23) If $a = n_1 \bmod p$, then $a^n = {n_1}^n \bmod p$.

(24) $a^{n+1} = a^n \cdot a$.

(25) If $a \neq 0$, then $a^n \neq 0$.

(26) Let F be an Abelian add-associative right zeroed right complementable associative commutative well unital almost left invertible distributive non empty double loop structure and x, y be elements of F. Then $x \cdot x = y \cdot y$ if and only if $x = y$ or $x = -y$.

(27) For every prime number p and for every element x of GF(p) such that $2 < p$ and $x + x = 0_{\mathrm{GF}(p)}$ holds $x = 0_{\mathrm{GF}(p)}$.

(28) $a^n \cdot b^n = (a \cdot b)^n$.

(29) If $a \neq 0$, then $(a^{-1})^n = (a^n)^{-1}$.

(30) $a^{n_1} \cdot a^{n_2} = a^{n_1 + n_2}$.

(31) $(a^{n_1})^{n_2} = a^{n_1 \cdot n_2}$.

Let us consider p. One can verify that MultGroup(GF(p)) is cyclic.

The following two propositions are true:

(32) Let x be an element of MultGroup(GF(p)), x_1 be an element of GF(p), and n be a natural number. If $x = x_1$, then $x^n = {x_1}^n$.

(33) There exists an element g of GF(p) such that for every element a of GF(p) if $a \neq 0_{\mathrm{GF}(p)}$, then there exists a natural number n such that $a = g^n$.

3. Relation between Legendre Symbol and the Number of Roots in GF(p)

Let us consider p, a. We say that a is quadratic residue if and only if:

(Def. 3) $a \neq 0$ and there exists an element x of $\mathrm{GF}(p)$ such that $x^2 = a$.

We say that a is not quadratic residue if and only if:

(Def. 4) $a \neq 0$ and it is not true that there exists an element x of $\mathrm{GF}(p)$ such that $x^2 = a$.

One can prove the following proposition

(34) If $a \neq 0$, then a^2 is quadratic residue.

Let p be a prime number. Observe that $1_{\mathrm{GF}(p)}$ is quadratic residue.

Let us consider p, a. The functor $\mathrm{Lege}_p\, a$ yields an integer and is defined as follows:

(Def. 5) $\mathrm{Lege}_p\, a = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{if } a \text{ is quadratic residue}, \\ -1, & \text{otherwise.} \end{cases}$

Next we state several propositions:

(35) a is not quadratic residue iff $\mathrm{Lege}_p\, a = -1$.

(36) a is quadratic residue iff $\mathrm{Lege}_p\, a = 1$.

(37) $a = 0$ iff $\mathrm{Lege}_p\, a = 0$.

(38) If $a \neq 0$, then $\mathrm{Lege}_p(a^2) = 1$.

(39) $\mathrm{Lege}_p(a \cdot b) = \mathrm{Lege}_p\, a \cdot \mathrm{Lege}_p\, b$.

(40) If $a \neq 0$ and $n \bmod 2 = 0$, then $\mathrm{Lege}_p(a^n) = 1$.

(41) If $n \bmod 2 = 1$, then $\mathrm{Lege}_p(a^n) = \mathrm{Lege}_p\, a$.

(42) If $2 < p$, then $\overline{\overline{\{b : b^2 = a\}}} = 1 + \mathrm{Lege}_p\, a$.

4. Set of Points on an Elliptic Curve over GF(p)

Let K be a field. The functor $\mathrm{ProjCo}\, K$ yields a non empty subset of (the carrier of K) $\times$ (the carrier of K) $\times$ (the carrier of K) and is defined by:

(Def. 6) $\mathrm{ProjCo}\, K = ((\text{the carrier of } K) \times (\text{the carrier of } K) \times (\text{the carrier of } K)) \setminus \{\langle 0_K, 0_K, 0_K \rangle\}$.

One can prove the following proposition

(43) $\mathrm{ProjCo}\,\mathrm{GF}(p) = ((\text{the carrier of } \mathrm{GF}(p)) \times (\text{the carrier of } \mathrm{GF}(p)) \times (\text{the carrier of } \mathrm{GF}(p))) \setminus \{\langle 0, 0, 0 \rangle\}$.

In the sequel P_1, P_2, P_3 are elements of $\mathrm{GF}(p)$.

Let p be a prime number and let a, b be elements of $\mathrm{GF}(p)$. The functor $\mathrm{Disc}(a, b, p)$ yields an element of $\mathrm{GF}(p)$ and is defined as follows:

(Def. 7) For all elements g_4, g_{27} of GF(p) such that $g_4 = 4 \bmod p$ and $g_{27} = 27 \bmod p$ holds $\mathrm{Disc}(a, b, p) = g_4 \cdot a^3 + g_{27} \cdot b^2$.

Let p be a prime number and let a, b be elements of GF(p). The functor EC WEqProjCo(a, b, p) yielding a function from (the carrier of GF(p)) $\times$ (the carrier of GF(p)) $\times$ (the carrier of GF(p)) into GF(p) is defined by the condition (Def. 8).

(Def. 8) Let P be an element of (the carrier of GF(p)) $\times$ (the carrier of GF(p)) $\times$ (the carrier of GF(p)). Then (EC WEqProjCo(a, b, p))(P) $= (P_{\mathbf{2}})^2 \cdot P_{\mathbf{3}} - ((P_{\mathbf{1}})^3 + a \cdot P_{\mathbf{1}} \cdot (P_{\mathbf{3}})^2 + b \cdot (P_{\mathbf{3}})^3)$.

We now state the proposition

(44) For all elements X, Y, Z of GF(p) holds (EC WEqProjCo(a, b, p))($\langle X, Y, Z \rangle$) $= Y^2 \cdot Z - (X^3 + a \cdot X \cdot Z^2 + b \cdot Z^3)$.

Let p be a prime number and let a, b be elements of GF(p). The functor EC SetProjCo(a, b, p) yielding a non empty subset of ProjCo GF(p) is defined by:

(Def. 9) EC SetProjCo(a, b, p) $= \{P \in$ ProjCo GF(p) : (EC WEqProjCo(a, b, p))(P) $= 0_{\mathrm{GF}(p)}\}$.

One can prove the following two propositions:

(45) $\langle 0, 1, 0 \rangle$ is an element of EC SetProjCo(a, b, p).

(46) Let p be a prime number and a, b, X, Y be elements of GF(p). Then $Y^2 = X^3 + a \cdot X + b$ if and only if $\langle X, Y, 1 \rangle$ is an element of EC SetProjCo(a, b, p).

Let p be a prime number and let P, Q be elements of ProjCo GF(p). We say that P EQ Q if and only if:

(Def. 10) There exists an element a of GF(p) such that $a \neq 0_{\mathrm{GF}(p)}$ and $P_{\mathbf{1}} = a \cdot Q_{\mathbf{1}}$ and $P_{\mathbf{2}} = a \cdot Q_{\mathbf{2}}$ and $P_{\mathbf{3}} = a \cdot Q_{\mathbf{3}}$.

Let us notice that the predicate P EQ Q is reflexive and symmetric.

We now state two propositions:

(47) For every prime number p and for all elements P, Q, R of ProjCo GF(p) such that P EQ Q and Q EQ R holds P EQ R.

(48) Let p be a prime number, a, b be elements of GF(p), P, Q be elements of (the carrier of GF(p)) $\times$ (the carrier of GF(p)) $\times$ (the carrier of GF(p)), and d be an element of GF(p). Suppose $p > 3$ and $\mathrm{Disc}(a, b, p) \neq 0_{\mathrm{GF}(p)}$ and $P \in$ EC SetProjCo(a, b, p) and $d \neq 0_{\mathrm{GF}(p)}$ and $Q_{\mathbf{1}} = d \cdot P_{\mathbf{1}}$ and $Q_{\mathbf{2}} = d \cdot P_{\mathbf{2}}$ and $Q_{\mathbf{3}} = d \cdot P_{\mathbf{3}}$. Then $Q \in$ EC SetProjCo(a, b, p).

Let p be a prime number. The functor $\mathbb{R}$-ProjCo p yielding a binary relation on ProjCo GF(p) is defined by:

(Def. 11) $\mathbb{R}$-ProjCo $p = \{\langle P, Q \rangle; P$ ranges over elements of ProjCo GF(p), Q ranges over elements of ProjCo GF(p) : P EQ $Q\}$.

One can prove the following proposition

(49) For every prime number p and for all elements P, Q of ProjCo GF(p) holds P EQ Q iff $\langle P, Q\rangle \in \mathbb{R}$-ProjCo p.

Let p be a prime number. Note that $\mathbb{R}$-ProjCo p is total, symmetric, and transitive.

Let p be a prime number and let a, b be elements of GF(p). The functor $\mathbb{R}$-EllCur(a,b,p) yielding an equivalence relation of EC SetProjCo(a,b,p) is defined as follows:

(Def. 12) $\mathbb{R}\text{-EllCur}(a,b,p) = \mathbb{R}\text{-ProjCo}\,p \cap \nabla_{\text{EC SetProjCo}(a,b,p)}$.

Next we state a number of propositions:

(50) Let p be a prime number, a, b be elements of GF(p), and P, Q be elements of ProjCo GF(p). Suppose $\text{Disc}(a,b,p) \neq 0_{\text{GF}(p)}$ and P, $Q \in$ EC SetProjCo(a,b,p). Then P EQ Q if and only if $\langle P, Q\rangle \in \mathbb{R}$-EllCur$(a,b,p)$.

(51) Let p be a prime number, a, b be elements of GF(p), and P be an element of ProjCo GF(p). Suppose $p > 3$ and $\text{Disc}(a,b,p) \neq 0_{\text{GF}(p)}$ and $P \in$ EC SetProjCo(a,b,p) and $P_{\mathbf{3}} \neq 0$. Then there exists an element Q of ProjCo GF(p) such that $Q \in$ EC SetProjCo(a,b,p) and Q EQ P and $Q_{\mathbf{3}} = 1$.

(52) Let p be a prime number, a, b be elements of GF(p), and P be an element of ProjCo GF(p). Suppose $p > 3$ and $\text{Disc}(a,b,p) \neq 0_{\text{GF}(p)}$ and $P \in$ EC SetProjCo(a,b,p) and $P_{\mathbf{3}} = 0$. Then there exists an element Q of ProjCo GF(p) such that $Q \in$ EC SetProjCo(a,b,p) and Q EQ P and $Q_{\mathbf{1}} = 0$ and $Q_{\mathbf{2}} = 1$ and $Q_{\mathbf{3}} = 0$.

(53) Let p be a prime number, a, b be elements of GF(p), and x be a set. Suppose $p > 3$ and $\text{Disc}(a,b,p) \neq 0_{\text{GF}(p)}$ and $x \in$ Classes $\mathbb{R}$-EllCur(a,b,p). Then

(i) there exists an element P of ProjCo GF(p) such that $P \in$ EC SetProjCo(a,b,p) and $P = \langle 0, 1, 0\rangle$ and $x = [P]_{\mathbb{R}\text{-EllCur}(a,b,p)}$, or

(ii) there exists an element P of ProjCo GF(p) and there exist elements X, Y of GF(p) such that $P \in$ EC SetProjCo(a,b,p) and $P = \langle X, Y, 1\rangle$ and $x = [P]_{\mathbb{R}\text{-EllCur}(a,b,p)}$.

(54) Let p be a prime number and a, b be elements of GF(p). Suppose $p > 3$ and $\text{Disc}(a,b,p) \neq 0_{\text{GF}(p)}$. Then Classes $\mathbb{R}$-EllCur$(a,b,p) = \{[\langle 0, 1, 0\rangle]_{\mathbb{R}\text{-EllCur}(a,b,p)}\} \cup \{[P]_{\mathbb{R}\text{-EllCur}(a,b,p)}; P$ ranges over elements of ProjCo GF(p) : $P \in$ EC SetProjCo$(a,b,p) \wedge \bigvee_{X,Y\,:\,\text{element of GF}(p)} P = \langle X, Y, 1\rangle\}$.

(55) Let p be a prime number and a, b, d_1, Y_1, d_2, Y_2 be elements of GF(p). Suppose $p > 3$ and $\text{Disc}(a,b,p) \neq 0_{\text{GF}(p)}$ and $\langle d_1, Y_1, 1\rangle$, $\langle d_2, Y_2, 1\rangle \in$ EC SetProjCo(a,b,p). Then $[\langle d_1, Y_1, 1\rangle]_{\mathbb{R}\text{-EllCur}(a,b,p)} = [\langle d_2, Y_2, 1\rangle]_{\mathbb{R}\text{-EllCur}(a,b,p)}$ if and only if $d_1 = d_2$ and $Y_1 = Y_2$.

(56) Let p be a prime number, a, b be elements of $\mathrm{GF}(p)$, and F_1, F_2 be sets. Suppose that

(i) $p > 3$,

(ii) $\mathrm{Disc}(a,b,p) \neq 0_{\mathrm{GF}(p)}$,

(iii) $F_1 = \{[\langle 0, 1, 0\rangle]_{\mathbb{R}\text{-EllCur}(a,b,p)}\}$, and

(iv) $F_2 = \{[P]_{\mathbb{R}\text{-EllCur}(a,b,p)}; P$ ranges over elements of $\mathrm{ProjCo}\,\mathrm{GF}(p)$: $P \in \mathrm{EC\ SetProjCo}(a,b,p) \wedge \bigvee_{X,Y\,:\,\text{element of }\mathrm{GF}(p)} P = \langle X, Y, 1\rangle\}$.

Then F_1 misses F_2.

(57) Let X be a non empty finite set, R be an equivalence relation of X, S be a Classes R-valued function, and i be a set. If $i \in \mathrm{dom}\,S$, then $S(i)$ is a finite subset of X.

(58) Let X be a non empty set, R be an equivalence relation of X, and S be a Classes R-valued function. If S is one-to-one, then S is disjoint valued.

(59) Let X be a non empty set, R be an equivalence relation of X, and S be a Classes R-valued function. If S is onto, then $\bigcup S = X$.

(60) Let X be a non empty finite set, R be an equivalence relation of X, S be a Classes R-valued function, and L be a finite sequence of elements of $\mathbb{N}$. Suppose S is one-to-one and onto and $\mathrm{dom}\,S = \mathrm{dom}\,L$ and for every natural number i such that $i \in \mathrm{dom}\,S$ holds $L(i) = \overline{\overline{S(i)}}$. Then $\overline{\overline{X}} = \sum L$.

(61) Let p be a prime number, a, b, d be elements of $\mathrm{GF}(p)$, and F, G be sets. Suppose that

(i) $p > 3$,

(ii) $\mathrm{Disc}(a,b,p) \neq 0_{\mathrm{GF}(p)}$,

(iii) $F = \{Y \in \mathrm{GF}(p)\colon Y^2 = d^3 + a \cdot d + b\}$,

(iv) $F \neq \emptyset$, and

(v) $G = \{[\langle d, Y, 1\rangle]_{\mathbb{R}\text{-EllCur}(a,b,p)}; Y$ ranges over elements of $\mathrm{GF}(p)$: $\langle d, Y, 1\rangle \in \mathrm{EC\ SetProjCo}(a,b,p)\}$.

Then there exists a function from F into G which is onto and one-to-one.

(62) Let p be a prime number and a, b, d be elements of $\mathrm{GF}(p)$. Suppose $p > 3$ and $\mathrm{Disc}(a,b,p) \neq 0_{\mathrm{GF}(p)}$.

Then $\overline{\overline{\{[\langle d, Y, 1\rangle]_{\mathbb{R}\text{-EllCur}(a,b,p)}; Y \text{ ranges over elements of } \mathrm{GF}(p)\colon \langle d, Y, 1\rangle \in \mathrm{EC\ SetProjCo}(a,b,p)\}}} = 1 + \mathrm{Lege}_p(d^3 + a \cdot d + b)$.

(63) Let p be a prime number and a, b be elements of $\mathrm{GF}(p)$. Suppose $p > 3$ and $\mathrm{Disc}(a,b,p) \neq 0_{\mathrm{GF}(p)}$. Then there exists a finite sequence F of elements of $\mathbb{N}$ such that

(i) $\mathrm{len}\,F = p$,

(ii) for every natural number n such that $n \in \mathrm{Seg}\,p$ there exists an element d of $\mathrm{GF}(p)$ such that $d = n - 1$ and $F(n) = 1 + \mathrm{Lege}_p(d^3 + a \cdot d + b)$, and

(iii) $\overline{\overline{\{[P]_{\mathbb{R}\text{-EllCur}(a,b,p)}; P \text{ ranges over elements of } \mathrm{ProjCo}\,\mathrm{GF}(p) : P \in \mathrm{EC\ SetProjCo}(a,b,p) \wedge \bigvee_{X,Y\,:\,\text{element of }\mathrm{GF}(p)} P = \langle X, Y, 1\rangle\}}} = \sum F$.

(64) Let p be a prime number and a, b be elements of $\mathrm{GF}(p)$. Suppose $p > 3$ and $\mathrm{Disc}(a, b, p) \neq 0_{\mathrm{GF}(p)}$. Then there exists a finite sequence F of elements of $\mathbb{Z}$ such that

(i) $\mathrm{len}\, F = p$,

(ii) for every natural number n such that $n \in \mathrm{Seg}\, p$ there exists an element d of $\mathrm{GF}(p)$ such that $d = n - 1$ and $F(n) = \mathrm{Lege}_p(d^3 + a \cdot d + b)$, and

(iii) $\overline{\overline{\mathrm{Classes}\, \mathbb{R}\text{-}\mathrm{EllCur}(a, b, p)}} = 1 + p + \sum F$.

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Continuity of Barycentric Coordinates in Euclidean Topological Spaces

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Summary. In this paper we present selected properties of barycentric coordinates in the Euclidean topological space. We prove the topological correspondence between a subset of an affine closed space of $\mathcal{E}^n$ and the set of vectors created from barycentric coordinates of points of this subset.

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The terminology and notation used here have been introduced in the following articles: [1], [3], [15], [25], [13], [18], [5], [4], [6], [12], [7], [8], [33], [21], [24], [2], [22], [20], [17], [30], [31], [23], [10], [28], [26], [11], [16], [29], [14], [19], [27], [32], and [9].

1. Preliminaries

For simplicity, we adopt the following rules: x denotes a set, n, m, k denote natural numbers, r denotes a real number, V denotes a real linear space, v, w denote vectors of V, A_1 denotes a finite subset of V, and A_2 denotes a finite affinely independent subset of V.

One can prove the following propositions:

(1) For all real-valued finite sequences f_1, f_2 and for every real number r holds $(\mathrm{Intervals}(f_1, r)) ^\frown \mathrm{Intervals}(f_2, r) = \mathrm{Intervals}(f_1 ^\frown f_2, r)$.

(2) Let f_1, f_2 be finite sequences. Then $x \in \prod(f_1 ^\frown f_2)$ if and only if there exist finite sequences p_1, p_2 such that $x = p_1 ^\frown p_2$ and $p_1 \in \prod f_1$ and $p_2 \in \prod f_2$.

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(3) V is finite dimensional iff Ω_V is finite dimensional.

Let V be a finite dimensional real linear space. One can verify that every affinely independent subset of V is finite.

Let us consider n. One can check that $\mathcal{E}^n_{\rm T}$ is add-continuous and mult-continuous and $\mathcal{E}^n_{\rm T}$ is finite dimensional.

In the sequel p_3 denotes a point of $\mathcal{E}^n_{\rm T}$, A_3 denotes a subset of $\mathcal{E}^n_{\rm T}$, A_4 denotes an affinely independent subset of $\mathcal{E}^n_{\rm T}$, and A_5 denotes a subset of $\mathcal{E}^k_{\rm T}$.

Next we state three propositions:

(4) $\dim(\mathcal{E}^n_{\rm T}) = n$.

(5) Let V be a finite dimensional real linear space and A be an affinely independent subset of V. Then $\overline{\overline{A}} \leq 1 + \dim(V)$.

(6) Let V be a finite dimensional real linear space and A be an affinely independent subset of V. Then $\overline{\overline{A}} = \dim(V)+1$ if and only if $\operatorname{Affin} A = \Omega_V$.

2. Open and Closed Subsets of a Subspace of the Euclidean Topological Space

One can prove the following propositions:

(7) If $k \leq n$ and $A_3 = \{v \in \mathcal{E}^n_{\rm T}\colon v{\restriction}k \in A_5\}$, then A_3 is open iff A_5 is open.

(8) Let A be a subset of $\mathcal{E}^{k+n}_{\rm T}$. Suppose $A = \{v \mathbin{^\frown} (n \mapsto 0) : v$ ranges over elements of $\mathcal{E}^k_{\rm T}\}$. Let B be a subset of $\mathcal{E}^{k+n}_{\rm T}{\restriction}A$. Suppose $B = \{v; v$ ranges over points of $\mathcal{E}^{k+n}_{\rm T}\colon v{\restriction}k \in A_5 \ \wedge \ v \in A\}$. Then A_5 is open if and only if B is open.

(9) For every affinely independent subset A of V and for every subset B of V such that $B \subseteq A$ holds $\operatorname{conv} A \cap \operatorname{Affin} B = \operatorname{conv} B$.

(10) Let V be a non empty RLS structure, A be a non empty set, f be a partial function from A to the carrier of V, and X be a set. Then $(r \cdot f)^\circ X = r \cdot f^\circ X$.

(11) If $\langle \underbrace{0, \ldots, 0}_{n} \rangle \in A_3$, then $\operatorname{Affin} A_3 = \Omega_{\operatorname{Lin}(A_3)}$.

Let V be a non empty additive loop structure, let A be a finite subset of V, and let v be an element of V. Note that $v + A$ is finite.

Let V be a non empty RLS structure, let A be a finite subset of V, and let us consider r. Observe that $r \cdot A$ is finite.

Next we state the proposition

(12) For every subset A of V holds $\overline{\overline{A}} = \overline{\overline{r \cdot A}}$ iff $r \neq 0$ or A is trivial.

Let V be a non empty RLS structure, let f be a finite sequence of elements of V, and let us consider r. Note that $r \cdot f$ is finite sequence-like.

3. The Vector of Barycentric Coordinates

Let X be a finite set. A one-to-one finite sequence is said to be an enumeration of X if:

(Def. 1) $\operatorname{rng} \mathrm{it} = X$.

Let X be a 1-sorted structure and let A be a finite subset of X. We see that the enumeration of A is a one-to-one finite sequence of elements of X.

In the sequel E_1 denotes an enumeration of A_2 and E_2 denotes an enumeration of A_4.

One can prove the following three propositions:

(13) Let V be an Abelian add-associative right zeroed right complementable non empty additive loop structure, A be a finite subset of V, E be an enumeration of A, and v be an element of V. Then $E + \overline{\overline{A}} \mapsto v$ is an enumeration of $v + A$.

(14) For every enumeration E of A_1 holds $r \cdot E$ is an enumeration of $r \cdot A_1$ iff $r \neq 0$ or A_1 is trivial.

(15) Let M be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$. Suppose $\operatorname{rk}(M) = n$. Let A be a finite subset of $\mathcal{E}^n_{\mathrm{T}}$ and E be an enumeration of A. Then $\operatorname{Mx2Tran} M \cdot E$ is an enumeration of $(\operatorname{Mx2Tran} M)^\circ A$.

Let us consider V, A_1, let E be an enumeration of A_1, and let us consider x. The functor $x \rightarrow E$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 2) $x \rightarrow E = (x \rightarrow A_1) \cdot E$.

The following propositions are true:

(16) For every enumeration E of A_1 holds $\operatorname{len}(x \rightarrow E) = \overline{\overline{A_1}}$.

(17) For every enumeration E of $v + A_2$ such that $w \in \operatorname{Affin} A_2$ and $E = E_1 + \overline{\overline{A_2}} \mapsto v$ holds $w \rightarrow E_1 = v + w \rightarrow E$.

(18) For every enumeration r_1 of $r \cdot A_2$ such that $v \in \operatorname{Affin} A_2$ and $r_1 = r \cdot E_1$ and $r \neq 0$ holds $v \rightarrow E_1 = r \cdot v \rightarrow r_1$.

(19) Let M be a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $n \times m$. Suppose $\operatorname{rk}(M) = n$. Let M_1 be an enumeration of $(\operatorname{Mx2Tran} M)^\circ A_4$. If $M_1 = \operatorname{Mx2Tran} M \cdot E_2$, then for every p_3 such that $p_3 \in \operatorname{Affin} A_4$ holds $p_3 \rightarrow E_2 = (\operatorname{Mx2Tran} M)(p_3) \rightarrow M_1$.

(20) Let A be a subset of V. Suppose $A \subseteq A_2$ and $x \in \operatorname{Affin} A_2$. Then $x \in \operatorname{Affin} A$ if and only if for every set y such that $y \in \operatorname{dom}(x \rightarrow E_1)$ and $E_1(y) \notin A$ holds $(x \rightarrow E_1)(y) = 0$.

(21) For every E_1 such that $x \in \operatorname{Affin} A_2$ holds $x \in \operatorname{Affin}(E_1{}^\circ \operatorname{Seg} k)$ iff $x \rightarrow E_1 = ((x \rightarrow E_1){\restriction}k) \frown ((\overline{\overline{A_2}} -' k) \mapsto 0)$.

(22) For every E_1 such that $k \leq \overline{\overline{A_2}}$ and $x \in \operatorname{Affin} A_2$ holds $x \in \operatorname{Affin}(A_2 \setminus E_1{}^\circ \operatorname{Seg} k)$ iff $x \rightarrow E_1 = (k \mapsto 0) \frown ((x \rightarrow E_1)_{\downharpoonleft k})$.

(23) Suppose $\langle\underbrace{0,\ldots,0}_{n}\rangle \in A_4$ and $E_2(\operatorname{len} E_2) = \langle\underbrace{0,\ldots,0}_{n}\rangle$. Then

(i) $\operatorname{rng}(E_2 \restriction (\overline{\overline{A_4}} -' 1)) = A_4 \setminus \{\langle\underbrace{0,\ldots,0}_{n}\rangle\}$, and

(ii) for every subset A of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$ such that $A_4 = A$ holds $E_2 \restriction (\overline{\overline{A_4}} -' 1)$ is an ordered basis of $\operatorname{Lin}(A)$.

(24) Let A be a subset of the n-dimension vector space over $\mathbb{R}_{\mathrm{F}}$. Suppose $A_4 = A$ and $\langle\underbrace{0,\ldots,0}_{n}\rangle \in A_4$ and $E_2(\operatorname{len} E_2) = \langle\underbrace{0,\ldots,0}_{n}\rangle$. Let B be an ordered basis of $\operatorname{Lin}(A)$. If $B = E_2 \restriction (\overline{\overline{A_4}} -' 1)$, then for every element v of $\operatorname{Lin}(A)$ holds $v \to B = (v \to E_2) \restriction (\overline{\overline{A_4}} -' 1)$.

(25) For all E_2, A_3 such that $k \leq n$ and $\overline{\overline{A_4}} = n + 1$ and $A_3 = \{p_3 : (p_3 \to E_2) \restriction k \in A_5\}$ holds A_5 is open iff A_3 is open.

(26) For every E_2 such that $k \leq n$ and $\overline{\overline{A_4}} = n + 1$ and $A_3 = \{p_3 : (p_3 \to E_2) \restriction k \in A_5\}$ holds A_5 is closed iff A_3 is closed.

Let us consider n. One can verify that every subset of $\mathcal{E}^n_{\mathrm{T}}$ which is affine is also closed.

In the sequel p_4 denotes an element of $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Affin} A_4$.

Next we state two propositions:

(27) For every E_2 and for every subset B of $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Affin} A_4$ such that $k < \overline{\overline{A_4}}$ and $B = \{p_4 : (p_4 \to E_2) \restriction k \in A_5\}$ holds A_5 is open iff B is open.

(28) Let given E_2 and B be a subset of $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Affin} A_4$. Suppose $k < \overline{\overline{A_4}}$ and $B = \{p_4 : (p_4 \to E_2) \restriction k \in A_5\}$. Then A_5 is closed if and only if B is closed.

Let us consider n and let p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. Observe that $\operatorname{halfline}(p, q)$ is closed.

4. Continuity of Barycentric Coordinates

Let us consider V, let A be a subset of V, and let us consider x. The functor $\vdash (A, x)$ yielding a function from V into $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def. 3) $(\vdash (A, x))(v) = (v \to A)(x)$.

One can prove the following four propositions:

(29) For every subset A of V such that $x \notin A$ holds $\vdash (A, x) = \Omega_V \longmapsto 0$.

(30) For every affinely independent subset A of V such that $\vdash (A, x) = \Omega_V \longmapsto 0$ holds $x \notin A$.

(31) $\vdash (A_4, x) \restriction \operatorname{Affin} A_4$ is a continuous function from $\mathcal{E}^n_{\mathrm{T}} \restriction \operatorname{Affin} A_4$ into $\mathbb{R}^{\mathbf{1}}$.

(32) If $\overline{\overline{A_4}} = n + 1$, then $\vdash (A_4, x)$ is continuous.

Let us consider n, A_4. Note that $\operatorname{conv} A_4$ is closed.

We now state the proposition

(33) If $\overline{\overline{A_4}} = n+1$, then $\mathrm{Int}\, A_4$ is open.

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Brouwer Fixed Point Theorem for Simplexes

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Summary. In this article we prove the Brouwer fixed point theorem for an arbitrary simplex which is the convex hull of its $n+1$ affinely indepedent vertices of $\mathcal{E}^n$. First we introduce the Lebesgue number, which for an arbitrary open cover of a compact metric space $\mathfrak{M}$ is a positive real number so that any ball of about such radius must be completely contained in a member of the cover. Then we introduce the notion of a bounded simplicial complex and the diameter of a bounded simplicial complex. We also prove the estimation of diameter decrease which is connected with the barycentric subdivision. Finally, we prove the Brouwer fixed point theorem and compute the small inductive dimension of $\mathcal{E}^n$. This article is based on [16].

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The papers [7], [31], [1], [8], [11], [17], [30], [14], [20], [4], [13], [9], [32], [21], [5], [19], [2], [3], [6], [22], [24], [18], [35], [26], [29], [33], [23], [27], [28], [34], [15], [25], [12], and [10] provide the terminology and notation for this paper.

1. The Lebesgue Number

In this paper M is a non empty metric space and F, G are open families of subsets of $M_{\rm top}$.

Let us consider M. Let us assume that $M_{\rm top}$ is compact. Let F be a family of subsets of $M_{\rm top}$. Let us assume that F is open and F is a cover of $M_{\rm top}$. A positive real number is said to be a Lebesgue number of F if:

(Def. 1) For every point p of M there exists a subset A of $M_{\rm top}$ such that $A \in F$ and ${\rm Ball}(p, {\rm it}) \subseteq A$.

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In the sequel L denotes a Lebesgue number of F.

Next we state three propositions:

(1) If $M_{\rm top}$ is compact and F is a cover of $M_{\rm top}$ and $F \subseteq G$, then L is a Lebesgue number of G.

(2) If $M_{\rm top}$ is compact and F is a cover of $M_{\rm top}$ and finer than G, then L is a Lebesgue number of G.

(3) Let L_1 be a positive real number. Suppose $M_{\rm top}$ is compact and F is a cover of $M_{\rm top}$ and $L_1 \leq L$. Then L_1 is a Lebesgue number of F.

2. Bounded Simplicial Complexes

In the sequel n, k denote natural numbers, X denotes a set, and K denotes a simplicial complex structure.

Let us consider M. One can check that every subset of M which is finite is also bounded.

Next we state the proposition

(4) For every finite non empty subset S of M there exist points p, q of M such that p, $q \in S$ and $\rho(p,q) = \varnothing S$.

Let us consider M, K. We say that K is M-bounded if and only if:

(Def. 2) There exists r such that for every A such that $A \in$ the topology of K holds A is bounded and $\varnothing A \leq r$.

The following proposition is true

(5) Let K be a non void simplicial complex structure. If K is M-bounded and A is a simplex of K, then A is bounded.

Let us consider M, X. Note that there exists a simplicial complex of X which is M-bounded and non void.

Let us consider M. Note that there exists a simplicial complex structure which is M-bounded, non void, subset-closed, and finite-membered.

Let us consider M, X and let K be an M-bounded simplicial complex str of X. Note that every sub simplicial complex of K is M-bounded.

Let us consider M, X, let K be an M-bounded subset-closed simplicial complex str of X, and let i be an integer. One can verify that the skeleton of K and i is M-bounded.

The following proposition is true

(6) If K is finite-vertices, then K is M-bounded.

3. The Diameter of a Bounded Simplicial Complex

Let us consider M and let K be a simplicial complex structure. Let us assume that K is M-bounded. The functor diameter(M, K) yielding a real number is defined by:

(Def. 3)(i) For every A such that $A \in$ the topology of K holds $\varnothing A \leq$ diameter(M, K) and for every r such that for every A such that $A \in$ the topology of K holds $\varnothing A \leq r$ holds $r \geq$ diameter(M, K) if the topology of K meets 2^{Ω_M},

(ii) diameter$(M, K) = 0$, otherwise.

One can prove the following three propositions:

(7) If K is M-bounded, then $0 \leq$ diameter(M, K).

(8) For every M-bounded simplicial complex str K of X and for every sub simplicial complex K_1 of K holds diameter$(M, K_1) \leq$ diameter(M, K).

(9) Let K be an M-bounded subset-closed simplicial complex str of X and i be an integer. Then diameter$(M,$ the skeleton of K and $i) \leq$ diameter(M, K).

Let us consider M and let K be an M-bounded non void subset-closed simplicial complex structure. Then diameter(M, K) is a real number and it can be characterized by the condition:

(Def. 4)(i) For every A such that A is a simplex of K holds $\varnothing A \leq$ diameter(M, K), and

(ii) for every r such that for every A such that A is a simplex of K holds $\varnothing A \leq r$ holds $r \geq$ diameter(M, K).

Next we state the proposition

(10) For every finite subset S of M holds diameter$(M,$ the complex of $\{S\}) = \varnothing S$.

Let us consider n and let K be a simplicial complex str of $\mathcal{E}^n_{\mathrm{T}}$. We say that K is bounded if and only if:

(Def. 5) K is $\mathcal{E}^n$-bounded.

The functor $\varnothing K$ yielding a real number is defined as follows:

(Def. 6) $\varnothing K =$ diameter$(\mathcal{E}^n, K)$.

Let us consider n. One can verify the following observations:

* every simplicial complex str of $\mathcal{E}^n_{\mathrm{T}}$ which is bounded is also $\mathcal{E}^n$-bounded,
* there exists a simplicial complex of $\mathcal{E}^n_{\mathrm{T}}$ which is bounded, affinely independent, simplex-join-closed, non void, finite-degree, and total, and
* every simplicial complex str of $\mathcal{E}^n_{\mathrm{T}}$ which is finite-vertices is also bounded.

4. The Estimation of Diameter of the Barycentric Subdivision

In the sequel V is a real linear space.

The following two propositions are true:

(11) Let S be a simplex of BCS K_2 and F be a $\subseteq$-linear finite finite-membered family of subsets of V. Suppose $S =$ (the center of mass $V)^\circ F$ and $\bigcup F$ is a simplex of K_2. Let a_1, a_2 be vectors of V. Suppose $a_1, a_2 \in S$. Then there exist vectors b_1, b_2 of V and there exists a real number r such that $b_1 \in$ Vertices BCS (the complex of $\{\bigcup F\}$) and $b_2 \in$ Vertices BCS (the complex of $\{\bigcup F\}$) and $a_1 - a_2 = r \cdot (b_1 - b_2)$ and $0 \leq r \leq \frac{\overline{\overline{\bigcup F}}-1}{\overline{\overline{\bigcup F}}}$.

(12) Let A be an affinely independent subset of $\mathcal{E}^n_{\rm T}$ and E be an enumeration of A. If $\operatorname{dom} E \setminus X$ is non empty, then $\operatorname{conv} E^\circ X = \bigcap\{\operatorname{conv} A \setminus \{E(k)\}; k$ ranges over elements of $\mathbb{N}$: $k \in \operatorname{dom} E \setminus X\}$.

In the sequel A denotes a subset of $\mathcal{E}^n_{\rm T}$.

The following three propositions are true:

(13) For every bounded subset a of $\mathcal{E}^n$ such that $a = A$ and for every point p of $\mathcal{E}^n$ such that $p \in \operatorname{conv} A$ holds $\operatorname{conv} A \subseteq \overline{\operatorname{Ball}}(p, \varnothing a)$.

(14) A is Bounded iff $\operatorname{conv} A$ is Bounded.

(15) For all bounded subsets a, c_1 of $\mathcal{E}^n$ such that $c_1 = \operatorname{conv} A$ and $a = A$ holds $\varnothing a = \varnothing c_1$.

Let us consider n and let K be a bounded simplicial complex str of $\mathcal{E}^n_{\rm T}$. Observe that every subdivision str of K is bounded.

The following propositions are true:

(16) For every bounded finite-degree non void simplicial complex K of $\mathcal{E}^n_{\rm T}$ such that $|K| \subseteq \Omega_K$ holds $\varnothing\, \mathrm{BCS}\, K \leq \frac{\mathrm{degree}(K)}{\mathrm{degree}(K)+1} \cdot \varnothing K$.

(17) For every bounded finite-degree non void simplicial complex K of $\mathcal{E}^n_{\rm T}$ such that $|K| \subseteq \Omega_K$ holds $\varnothing\, \mathrm{BCS}(k, K) \leq (\frac{\mathrm{degree}(K)}{\mathrm{degree}(K)+1})^k \cdot \varnothing K$.

(18) Let K be a bounded finite-degree non void simplicial complex of $\mathcal{E}^n_{\rm T}$. If $|K| \subseteq \Omega_K$, then for every r such that $r > 0$ there exists k such that $\varnothing\, \mathrm{BCS}(k, K) < r$.

(19) Let i, j be elements of $\mathbb{N}$. Then there exists a function f from $\mathcal{E}^i_{\rm T} \times \mathcal{E}^j_{\rm T}$ into $\mathcal{E}^{i+j}_{\rm T}$ such that f is homeomorphism and for every element f_1 of $\mathcal{E}^i_{\rm T}$ and for every element f_2 of $\mathcal{E}^j_{\rm T}$ holds $f(f_1, f_2) = f_1 \frown f_2$.

(20) Let i, j be elements of $\mathbb{N}$ and f be a function from $\mathcal{E}^i_{\rm T} \times \mathcal{E}^j_{\rm T}$ into $\mathcal{E}^{i+j}_{\rm T}$. Suppose that for every element f_1 of $\mathcal{E}^i_{\rm T}$ and for every element f_2 of $\mathcal{E}^j_{\rm T}$ holds $f(f_1, f_2) = f_1 \frown f_2$. Let given r, f_1 be a point of $\mathcal{E}^i$, f_2 be a point of $\mathcal{E}^j$, and f_3 be a point of $\mathcal{E}^{i+j}$. If $f_3 = f_1 \frown f_2$, then $f^\circ(\mathrm{OpenHypercube}(f_1, r) \times \mathrm{OpenHypercube}(f_2, r)) = \mathrm{OpenHypercube}(f_3, r)$.

(21) A is Bounded iff there exists a point p of $\mathcal{E}^n$ and there exists r such that $A \subseteq \mathrm{OpenHypercube}(p, r)$.

Let us consider n. Observe that every subset of $\mathcal{E}^n_{\rm T}$ which is closed and Bounded is also compact.

Let us consider n and let A be an affinely independent subset of $\mathcal{E}^n_{\rm T}$. One can verify that $\mathrm{conv}\, A$ is compact.

5. Main Theorems

Next we state the proposition

(22) Let A be a non empty affinely independent subset of $\mathcal{E}^n_{\rm T}$, E be an enumeration of A, and F be a finite sequence of elements of $2^{\text{the carrier of } \mathcal{E}^n_{\rm T} \restriction \mathrm{conv}\, A}$. Suppose $\mathrm{len}\, F = \overline{\overline{A}}$ and $\mathrm{rng}\, F$ is closed and for every subset S of $\mathrm{dom}\, F$ holds $\mathrm{conv}\, E^\circ S \subseteq \bigcup (F^\circ S)$. Then $\bigcap \mathrm{rng}\, F$ is non empty.

In the sequel A denotes an affinely independent subset of $\mathcal{E}^n_{\rm T}$.

Next we state four propositions:

(23) Let given A. Suppose $\overline{\overline{A}} = n + 1$. Let f be a continuous function from $\mathcal{E}^n_{\rm T} \restriction \mathrm{conv}\, A$ into $\mathcal{E}^n_{\rm T} \restriction \mathrm{conv}\, A$. Then there exists a point p of $\mathcal{E}^n_{\rm T}$ such that $p \in \mathrm{dom}\, f$ and $f(p) = p$.

(24) For every A such that $\overline{\overline{A}} = n + 1$ holds every continuous function from $\mathcal{E}^n_{\rm T} \restriction \mathrm{conv}\, A$ into $\mathcal{E}^n_{\rm T} \restriction \mathrm{conv}\, A$ has a fixpoint.

(25) If $\overline{\overline{A}} = n + 1$, then $\mathrm{ind}\, \mathrm{conv}\, A = n$.

(26) $\mathrm{ind}(\mathcal{E}^n_{\rm T}) = n$.

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Brouwer Fixed Point Theorem in the General Case

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Summary. In this article we prove the Brouwer fixed point theorem for an arbitrary convex compact subset of $\mathcal{E}^n$ with a non empty interior. This article is based on [15].

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The notation and terminology used here have been introduced in the following papers: [17], [12], [1], [4], [7], [16], [6], [13], [10], [2], [3], [14], [9], [20], [18], [8], [19], [11], [21], and [5].

1. Preliminaries

For simplicity, we adopt the following convention: n is a natural number, p, q, u, w are points of $\mathcal{E}^n_{\mathrm{T}}$, S is a subset of $\mathcal{E}^n_{\mathrm{T}}$, A, B are convex subsets of $\mathcal{E}^n_{\mathrm{T}}$, and r is a real number.

Next we state several propositions:

(1) $(1 - r) \cdot p + r \cdot q = p + r \cdot (q - p)$.

(2) If u, $w \in \mathrm{halfline}(p, q)$ and $|u - p| = |w - p|$, then $u = w$.

(3) Let given S. Suppose $p \in S$ and $p \neq q$ and $S \cap \mathrm{halfline}(p, q)$ is Bounded. Then there exists w such that

(i) $w \in \mathrm{Fr}\, S \cap \mathrm{halfline}(p, q)$,

(ii) for every u such that $u \in S \cap \mathrm{halfline}(p, q)$ holds $|p - u| \leq |p - w|$, and

(iii) for every r such that $r > 0$ there exists u such that $u \in S \cap \mathrm{halfline}(p, q)$ and $|w - u| < r$.

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(4) For every A such that A is closed and $p \in \operatorname{Int} A$ and $p \neq q$ and $A \cap \operatorname{halfline}(p, q)$ is Bounded there exists u such that $\operatorname{Fr} A \cap \operatorname{halfline}(p, q) = \{u\}$.

(5) If $r > 0$, then $\operatorname{Fr} \overline{\operatorname{Ball}}(p, r) = \operatorname{Sphere}(p, r)$.

Let n be an element of $\mathbb{N}$, let A be a Bounded subset of $\mathcal{E}^n_{\mathrm{T}}$, and let p be a point of $\mathcal{E}^n_{\mathrm{T}}$. One can verify that $p + A$ is Bounded.

2. Main Theorems

Next we state four propositions:

(6) Let n be an element of $\mathbb{N}$ and A be a convex subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose A is compact and non boundary. Then there exists a function h from $\mathcal{E}^n_{\mathrm{T}} \restriction A$ into $\operatorname{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ such that h is homeomorphism and $h^\circ \operatorname{Fr} A = \operatorname{Sphere}((0_{\mathcal{E}^n_{\mathrm{T}}}), 1)$.

(7) Let given A, B. Suppose A is compact and non boundary and B is compact and non boundary. Then there exists a function h from $\mathcal{E}^n_{\mathrm{T}} \restriction A$ into $\mathcal{E}^n_{\mathrm{T}} \restriction B$ such that h is homeomorphism and $h^\circ \operatorname{Fr} A = \operatorname{Fr} B$.

(8)[1] For every A such that A is compact and non boundary holds every continuous function from $\mathcal{E}^n_{\mathrm{T}} \restriction A$ into $\mathcal{E}^n_{\mathrm{T}} \restriction A$ has a fixpoint.

(9) Let A be a non empty convex subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose A is compact and non boundary. Let F_1 be a non empty subspace of $\mathcal{E}^n_{\mathrm{T}} \restriction A$. If $\Omega_{(F_1)} = \operatorname{Fr} A$, then F_1 is not a retract of $\mathcal{E}^n_{\mathrm{T}} \restriction A$.

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Preliminaries to Classical First Order Model Theory[1]

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Summary. First of a series of articles laying down the bases for classical first order model theory. These articles introduce a framework for treating arbitrary languages with equality. This framework is kept as generic and modular as possible: both the language and the derivation rule are introduced as a type, rather than a fixed functor; definitions and results regarding syntax, semantics, interpretations and sequent derivation rules, respectively, are confined to separate articles, to mark out the hierarchy of dependences among different definitions and constructions.

As an application limited to countable languages, satisfiability theorem and a full version of the Gödel completeness theorem are delivered, with respect to a fixed, remarkably thrifty, set of correct rules. Besides the self-referential significance for the Mizar project itself of those theorems being formalized with respect to a generic, equality-furnished, countable language, this is the first step to work out other milestones of model theory, such as Lowenheim-Skolem and compactness theorems. Being the receptacle of all results of broader scope stemmed during the various formalizations, this first article stays at a very generic level, with results and registrations about objects already in the Mizar Mathematical Library.

Without introducing the Language structure yet, three fundamental definitions of wide applicability are also given: the 'unambiguous' attribute (see [20], definition on page 5), the functor '-multiCat', which is the iteration of '^ ' over a FinSequence of FinSequence, and the functor SubstWith, which realizes the substitution of a single symbol inside a generic FinSequence.

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The papers [11], [2], [4], [12], [23], [7], [13], [19], [22], [14], [15], [10], [16], [9], [25], [1], [27], [8], [24], [6], [3], [5], [17], [28], [30], [29], [21], [26], and [18] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: U, D are non empty sets, X is a non empty subset of D, d is an element of D, A, B, C, Y, x, y, z are sets, f is a binary operation on D, i, m, n are natural numbers, and g is a function.

Let X be a set and let f be a function. We say that f is X-one-to-one if and only if:

(Def. 1) $f{\restriction}X$ is one-to-one.

Let us consider D, f and let X be a set. We say that X is f-unambiguous if and only if:

(Def. 2) f is $X \times D$-one-to-one.

Let us consider D and let X be a set. We say that X is D-prefix if and only if:

(Def. 3) X is (the concatenation of D)-unambiguous.

Let D be a set. The functor D-pr1 yielding a binary operation on D is defined by:

(Def. 4) $D\text{-pr1} = \pi_1(D \times D)$.

One can prove the following propositions:

(1) $A^m \cap B^* = A^m \cap B^m$.

(2) $A^m \cap B^* = (A \cap B)^m$.

(3) $(A \cap B)^m = A^m \cap B^m$.

(4) For all finite sequences x, y such that x is U-valued and y is U-valued holds (the concatenation of U)$(x, y) = x \frown y$.

(5) For every set x holds x is a non empty finite sequence of elements of D iff $x \in D^* \setminus \{\emptyset\}$.

Let D be a non empty set. One can check that D-pr1 is associative.

Let D be a set. Note that there exists a binary operation on D which is associative.

Let X be a set and let Y be a subset of X. Then Y^* is a non empty subset of X^*.

Let D be a non empty set. Observe that the concatenation of D is associative. Observe that $D^* \setminus \{\emptyset\}$ is non empty.

Let m be a natural number. Note that there exists an element of D^* which is m-element.

Let X be a set and let f be a function. Let us observe that f is X-one-to-one if and only if:

(Def. 5) For all sets x, y such that $x, y \in X \cap \operatorname{dom} f$ and $f(x) = f(y)$ holds $x = y$.

Let us consider D, f. Note that there exists a set which is f-unambiguous.

Let f be a function and let x be a set. Note that $f \restriction \{x\}$ is one-to-one.

One can verify that every set which is empty is also empty-membered. Let e be an empty set. Note that $\{e\}$ is empty-membered.

Let us consider U and let m_1 be a non zero natural number. Observe that U^{m_1} has non empty elements.

Let X be an empty-membered set. Note that every subset of X is empty-membered.

Let us consider A and let m_0 be a zero number. Note that A^{m_0} is empty-membered.

Let e be an empty set and let m_1 be a non zero natural number. Observe that e^{m_1} is empty.

Let us consider D, f and let e be an empty set. One can verify that $e \cap f$ is f-unambiguous.

Let us consider U and let e be an empty set. One can check that $e \cap U$ is U-prefix.

Let us consider U. Observe that there exists a set which is U-prefix.

Let us consider D, f and let x be a finite sequence of elements of D. The functor $\mathrm{MultPlace}(f, x)$ yields a function and is defined by:

(Def. 6) $\mathrm{dom}\,\mathrm{MultPlace}(f, x) = \mathbb{N}$ and $(\mathrm{MultPlace}(f, x))(0) = x(1)$ and for every natural number n holds $(\mathrm{MultPlace}(f, x))(n + 1) = f((\mathrm{MultPlace}(f, x))(n), x(n + 2))$.

Let us consider D, f and let x be an element of $D^* \setminus \{\emptyset\}$. The functor $\mathrm{MultPlace}(f, x)$ yields a function and is defined as follows:

(Def. 7) $\mathrm{MultPlace}(f, x) = \mathrm{MultPlace}(f, (x$ **qua** element of $D^*))$.

Let us consider D, f. The functor $\mathrm{MultPlace}\,f$ yielding a function from $D^* \setminus \{\emptyset\}$ into D is defined as follows:

(Def. 8) For every element x of $D^* \setminus \{\emptyset\}$ holds $(\mathrm{MultPlace}\,f)(x) = (\mathrm{MultPlace}(f, x))(\mathrm{len}\,x - 1)$.

Let us consider D, f and let X be a set. Let us observe that X is f-unambiguous if and only if:

(Def. 9) For all sets x, y, d_1, d_2 such that $x, y \in X \cap D$ and $d_1, d_2 \in D$ and $f(x, d_1) = f(y, d_2)$ holds $x = y$ and $d_1 = d_2$.

Let us consider D. The functor D-firstChar yields a function from $D^* \setminus \{\emptyset\}$ into D and is defined as follows:

(Def. 10) D-firstChar $= \mathrm{MultPlace}(D\text{-pr1})$.

One can prove the following proposition

(6) For every finite sequence p such that p is U-valued and non empty holds U-firstChar$(p) = p(1)$.

Let us consider D. The functor D-multiCat yielding a function is defined as follows:

(Def. 11) D-multiCat $= (\emptyset \,\dot{\longmapsto}\, \emptyset) {+\cdot}$ MultPlace (the concatenation of D).

Let us consider D. Then D-multiCat is a function from $(D^*)^*$ into D^*.

Let us consider D and let e be an empty set. One can check that D-multiCat(e) is empty.

Let us consider D. Observe that every subset of D^1 is D-prefix.

The following propositions are true:

(7) If A is D-prefix, then D-multiCat$^\circ A^m$ is D-prefix.

(8) If A is D-prefix, then D-multiCat is A^m-one-to-one.

(9) $Y^{m+1} \subseteq Y^* \setminus \{\emptyset\}$.

(10) If m is zero, then $Y^m = \{\emptyset\}$.

(11) $Y^i = Y^{\mathrm{Seg}\, i}$.

(12) If $x \in A^m$, then x is a finite sequence of elements of A.

Let A, X be sets. Then $\chi_{A,X}$ is a function from X into *Boolean*.

Next we state three propositions:

(13) (MultPlace f)$(\langle d \rangle) = d$ and for every non empty finite sequence x of elements of D holds (MultPlace f)$(x \frown \langle d \rangle) = f(($MultPlace $f)(x), d)$.

(14) For every non empty element d of $(D^*)^*$ holds D-multiCat(d) = (MultPlace (the concatenation of D))(d).

(15) For all elements d_1, d_2 of D^* holds D-multiCat$(\langle d_1, d_2 \rangle) = d_1 \frown d_2$.

Let f, g be finite sequences. One can verify that $\langle f, g \rangle$ is finite sequence-like.

Let us consider m and let f, g be m-element finite sequences. Note that $\langle f, g \rangle$ is m-element.

Let X, Y be sets, let f be an X-defined function, and let g be a Y-defined function. Observe that $\langle f, g \rangle$ is $X \cap Y$-defined.

Let X be a set and let f, g be X-defined functions. Observe that $\langle f, g \rangle$ is X-defined.

Let X, Y be sets, let f be a total X-defined function, and let g be a total Y-defined function. Note that $\langle f, g \rangle$ is total.

Let X be a set and let f, g be total X-defined functions. Note that $\langle f, g \rangle$ is total.

Let X, Y be sets, let f be an X-valued function, and let g be a Y-valued function. One can verify that $\langle f, g \rangle$ is $X \times Y$-valued.

Let us consider D. Observe that there exists a finite sequence which is D-valued.

Let us consider D, m. Note that there exists a D-valued finite sequence which is m-element.

Let X, Y be non empty sets, let f be a function from X into Y, and let p be an X-valued finite sequence. Observe that $f \cdot p$ is finite sequence-like.

Let us consider m, let f be a function from X into Y, and let p be an m-element X-valued finite sequence. Note that $f \cdot p$ is m-element.

Let us consider D, f and let p, q be elements of D^*.

The functor f AppliedPairwiseTo(p, q) yields a finite sequence of elements of D and is defined by:

(Def. 12) f AppliedPairwiseTo$(p, q) = f \cdot \langle p, q \rangle$.

Let us consider D, f, m and let p, q be m-element elements of D^*. Note that f AppliedPairwiseTo(p, q) is m-element.

Let us consider D, f and let p, q be elements of D^*. We introduce $f_\backslash(p, q)$ as a synonym of f AppliedPairwiseTo(p, q).

$\mathbb{Z}$ can be characterized by the condition:

(Def. 13) $\mathbb{Z} = \mathbb{N} \cup (\{0\} \times \mathbb{N} \setminus \{\langle 0, 0 \rangle\})$.

We now state the proposition

(16) For every finite sequence p such that p is Y-valued and m-element holds $p \in Y^m$.

Let us consider A, B. The functor $A^{\leftarrow} \cap B$ yields a subset of A and is defined by:

(Def. 14) $A^{\leftarrow} \cap B = A \cap B$.

The functor $A \cap^{\rightarrow} B$ yielding a subset of B is defined as follows:

(Def. 15) $A \cap^{\rightarrow} B = A \cap B$.

Let us consider B, A. The functor $A \operatorname{null} B$ is defined by:

(Def. 16) $A \operatorname{null} B = A$.

Let us consider A, B, C. One can check that $(B \setminus A) \cap (A \cap C)$ is empty.

Let us consider A, B. The functor $A \setminus^{\leftarrow} B$ yields a subset of A and is defined as follows:

(Def. 17) $A \setminus^{\leftarrow} B = A \setminus B$.

Let us consider A, B. The functor $A \cup^{\leftrightarrow} B$ yielding a subset of $A \cup B$ is defined by:

(Def. 18) $A \cup^{\leftrightarrow} B = A$.

For simplicity, we adopt the following convention: X is a set, P, Q, R are binary relations, f is a function, p, q are finite sequences, and U_1, U_2 are non empty sets.

Let R be a binary relation. Note that R^* is transitive and R^* is reflexive.

The function plus from $\mathbb{C}$ into $\mathbb{C}$ is defined as follows:

(Def. 19) For every complex number z holds $\operatorname{plus}(z) = z + 1$.

The following two propositions are true:

(17) If $\operatorname{rng} f \subseteq \operatorname{dom} f$, then $f^* = \bigcup\{f^{m_2} : m_2$ ranges over elements of $\mathbb{N}\}$.

(18) If $f \subseteq g$, then $f^m \subseteq g^m$.

Let X be a functional set. Note that $\bigcup X$ is relation-like.

Next we state the proposition

(19) If $Y \subseteq B^A$, then $\bigcup Y \subseteq A \times B$.

Let us consider Y. Observe that $Y \setminus Y$ is empty.

Let us consider D, d. One can check that $\{\mathrm{id}_D(d)\} \setminus \{d\}$ is empty.

One can prove the following propositions:

(20) $f = \{\langle x, f(x)\rangle; x$ ranges over elements of $\mathrm{dom}\, f : x \in \mathrm{dom}\, f\}$.

(21) For every total Y-defined binary relation R holds $\mathrm{id}_Y \subseteq R \cdot R^{\smile}$.

(22) $D^{m+n} =$ (the concatenation of $D)^{\circ}(D^m \times D^n)$.

(23) For all binary relations P, Q holds $(P \cup Q)^{-1}(Y) = P^{-1}(Y) \cup Q^{-1}(Y)$.

(24) $(\chi_{A,B})^{-1}(\{0\}) = B \setminus A$ and $(\chi_{A,B})^{-1}(\{1\}) = A \cap B$.

(25) For every non empty set y holds $y = f(x)$ iff $x \in f^{-1}(\{y\})$.

(26) If f is Y-valued and finite sequence-like, then f is a finite sequence of elements of Y.

Let us consider Y and let X be a subset of Y. Observe that every binary relation which is X-valued is also Y-valued.

Let us consider A, U. One can verify that every relation between A and U which is quasi total is also total.

The following propositions are true:

(27) Let Q be a quasi total relation between B and U_1, R be a quasi total relation between B and U_2, and P be a relation between A and B. If $P \cdot Q \cdot Q^{\smile} \cdot R$ is function-like, then $P \cdot Q \cdot Q^{\smile} \cdot R = P \cdot R$.

(28) For all finite sequences p, q such that p is non empty holds $(p \frown q)(1) = p(1)$.

Let us consider U and let p, q be U-valued finite sequences. One can check that $p \frown q$ is U-valued.

Let X be a set. We see that the finite sequence of elements of X is an element of X^*.

Let us consider U, X. Let us observe that X is U-prefix if and only if:

(Def. 20) For all U-valued finite sequences p_1, q_1, p_2, q_2 such that p_1, $p_2 \in X$ and $p_1 \frown q_1 = p_2 \frown q_2$ holds $p_1 = p_2$ and $q_1 = q_2$.

Let X be a set. Observe that every element of X^* is X-valued.

Let us consider U, m and let X be a U-prefix set. Observe that U-multiCat$^{\circ}\, X^m$ is U-prefix.

Next we state the proposition

(29) $X \dot{-} Y = \emptyset$ iff $X = Y$.

Let us consider x. Note that $\mathrm{id}_{\{x\}} \dot{-} \{\langle x, x\rangle\}$ is empty.

Let us consider x, y. Observe that $(x \dot{\longmapsto} y) \dot{-} \{\langle x, y\rangle\}$ is empty.

Let us consider x. Note that $\mathrm{id}_{\{x\}} \dot{-} (x \dot{\longmapsto} x)$ is empty.

The following proposition is true

(30) $x \in D^* \setminus \{\emptyset\}$ iff x is a D-valued finite sequence and non empty.

In the sequel f denotes a binary operation on D.

The following proposition is true

(31) $(\mathrm{MultPlace}\, f)(\langle d \rangle) = d$ and for every D-valued finite sequence x such that x is non empty holds $(\mathrm{MultPlace}\, f)(x \frown \langle d \rangle) = f((\mathrm{MultPlace}\, f)(x), d)$.

For simplicity, we adopt the following rules: A, B, C, X, Y, Z, x, x_1, y, y_1, y_2 are sets, U, U_1, U_2, U_3 are non empty sets, u, u_1, u_2 are elements of U, P, R are binary relations, f, g are functions, k, m, n are natural numbers, k_1, m_2, n_1 are elements of $\mathbb{N}$, m_1, n_2 are non zero natural numbers, p, p_1, p_2 are finite sequences, and q, q_1, q_2 are U-valued finite sequences.

Let us consider p, x, y. Note that $p \widetilde{+} (x, y)$ is finite sequence-like.

Let us consider x, y, p. The functor (x, y)-SymbolSubstIn p yielding a finite sequence is defined by:

(Def. 21) $(x, y)\text{-SymbolSubstIn}\, p = p \widetilde{+} (x, y)$.

Let us consider x, y, m and let p be an m-element finite sequence. Observe that (x, y)-SymbolSubstIn p is m-element.

Let us consider X. Observe that there exists a finite sequence which is X-valued.

Let us consider x, U, u and let p be a U-valued finite sequence. Observe that (x, u)-SymbolSubstIn p is U-valued.

Let us consider X, x, y and let p be an X-valued finite sequence. Then (x, y)-SymbolSubstIn p can be characterized by the condition:

(Def. 22) $(x, y)\text{-SymbolSubstIn}\, p = (\mathrm{id}_X +\cdot (x, y)) \cdot p$.

Let us consider U, x, u, q. Then (x, u)-SymbolSubstIn q is a finite sequence of elements of U.

Let us consider U, x, u. The functor x SubstWith u yielding a function from U^* into U^* is defined as follows:

(Def. 23) For every q holds $(x\, \mathrm{SubstWith}\, u)(q) = (x, u)\text{-SymbolSubstIn}\, q$.

Let us consider U, x, u. Note that x SubstWith u is finite sequence-yielding.

Let F be a finite sequence-yielding function and let x be a set. Observe that $F(x)$ is finite sequence-like.

Let us consider U, x, u, m and let p be a U-valued m-element finite sequence. Note that $(x\, \mathrm{SubstWith}\, u)(p)$ is m-element.

Let e be an empty set. One can verify that $(x\, \mathrm{SubstWith}\, u)(e)$ is empty.

Let us consider U. Note that U-multiCat is finite sequence-yielding.

One can verify that there exists a U-valued finite sequence which is non empty.

Let us consider U, m_1, n and let p be an $m_1 + n$-element U-valued finite sequence. Observe that $\{p(m_1)\} \setminus U$ is empty.

Let us consider U, m, n and let p be an $m + 1 + n$-element element of U^*. One can check that $\{p(m + 1)\} \setminus U$ is empty.

Let us consider x. Note that $\langle x \rangle \dot{-} \{\langle 1, x \rangle\}$ is empty.

Let us consider m and let p be an $m+1$-element finite sequence. Observe that $(p\restriction \operatorname{Seg} m) \frown \langle p(m+1)\rangle \dot{-} p$ is empty.

Let us consider m, n and let p be an $m+n$-element finite sequence. One can verify that $p\restriction \operatorname{Seg} m$ is m-element.

Let us observe that every binary relation which is $\{\emptyset\}$-valued is also empty yielding and every binary relation which is empty yielding is also $\{\emptyset\}$-valued.

The following two propositions are true:

(32) U-multiCat$(x) = (\text{MultPlace}\,(\text{the concatenation of } U))(x)$.

(33) If p is U^*-valued, then U-multiCat$(p \frown \langle q\rangle) = U$-multiCat$(p) \frown q$.

Let us consider Y, let X be a subset of Y, and let R be a total Y-defined binary relation. One can check that $R\restriction X$ is total.

The following propositions are true:

(34) If $u = u_1$, then (u_1, x_2)-SymbolSubstIn$\langle u\rangle = \langle x_2\rangle$ and if $u \neq u_1$, then (u_1, x_2)-SymbolSubstIn$\langle u\rangle = \langle u\rangle$.

(35) If $u = u_1$, then $(u_1 \operatorname{SubstWith} u_2)(\langle u\rangle) = \langle u_2\rangle$ and if $u \neq u_1$, then $(u_1 \operatorname{SubstWith} u_2)(\langle u\rangle) = \langle u\rangle$.

(36) $(x \operatorname{SubstWith} u)(q_1 \frown q_2) = (x \operatorname{SubstWith} u)(q_1) \frown (x \operatorname{SubstWith} u)(q_2)$.

(37) If p is U^*-valued, then $(x \operatorname{SubstWith} u)(U\text{-multiCat}(p)) = U\text{-multiCat}((x \operatorname{SubstWith} u) \cdot p)$.

(38) (The concatenation of $U)^\circ(\mathrm{id}_{U^1}) = \{\langle u, u\rangle : u$ ranges over elements of $U\}$.

Let us consider f, U, u. One can verify that $(f\restriction U)(u) \dot{-} f(u)$ is empty.

Let us consider f, U_1, U_2, let u be an element of U_1, and let g be a function from U_1 into U_2. Observe that $(f \cdot g)(u) \dot{-} f(g(u))$ is empty.

One can verify that every integer number which is non negative is also natural.

Let x, y be real numbers. One can verify that $\max(x, y) - x$ is non negative.

The following proposition is true

(39) If x is boolean, then $x = 1$ iff $x \neq 0$.

Let us consider Y and let X be a subset of Y. Note that $X \setminus Y$ is empty.

Let us consider x, y. Observe that $\{x\} \setminus \{x, y\}$ is empty and $\langle x, y\rangle_{\mathbf{1}} \dot{-} x$ is empty.

Let us consider x, y. Observe that $\langle x, y\rangle_{\mathbf{2}} \dot{-} y$ is empty.

Let n be a positive natural number and let X be a non empty set. Note that there exists an element of $X^* \setminus \{\emptyset\}$ which is n-element.

Let us consider m_1. One can verify that every finite sequence which is m_1+0-element is also non empty.

Let us consider R, x. Note that $R \operatorname{null} x$ is relation-like.

Let f be a function-like set and let us consider x. One can check that $f \operatorname{null} x$ is function-like.

Let p be a finite sequence-like binary relation and let us consider x. One can check that $p\,\mathrm{null}\,x$ is finite sequence-like.

Let us consider p, x. Observe that $p\,\mathrm{null}\,x$ is $\mathrm{len}\,p$-element.

Let p be a non empty finite sequence. Note that $\mathrm{len}\,p$ is non zero.

Let R be a binary relation and let X be a set. Observe that $R{\upharpoonright}X$ is X-defined.

Let us consider x and let e be an empty set. Observe that $e\,\mathrm{null}\,x$ is empty.

Let us consider X and let e be an empty set. One can verify that $e\,\mathrm{null}\,X$ is X-valued.

Let Y be a non empty finite sequence-membered set. One can check that every function which is Y-valued is also finite sequence-yielding.

Let us consider X, Y. Note that every element of $(Y^*)^X$ is finite sequence-yielding.

We now state the proposition

(40) If f is X^*-valued, then $f(x) \in X^*$.

Let us consider m, n and let p be an m-element finite sequence. Observe that $p\,\mathrm{null}\,n$ is $\mathrm{Seg}\,m+n$-defined.

Let us consider m, n, let p be an m-element finite sequence, and let q be an n-element finite sequence. Observe that $p \frown q$ is $m+n$-element.

The following two propositions are true:

(41) Let p_1, p_2, q_1, q_2 be finite sequences. Suppose p_1 is m-element but q_1 is m-element but $p_1 \frown p_2 = q_1 \frown q_2$ or $p_2 \frown p_1 = q_2 \frown q_1$. Then $p_1 = q_1$ and $p_2 = q_2$.

(42) If U-multiCat(x) is U_1-valued and $x \in (U^*)^*$, then x is a finite sequence of elements of ${U_1}^*$.

Let us consider U. One can verify that there exists a reflexive binary relation on U which is total.

Let us consider m. Note that every finite sequence which is $m+1$-element is also non empty.

Let us consider U, u. Note that $\mathrm{id}_U(u) \dot{-} u$ is empty.

Let us consider U and let p be a U-valued non empty finite sequence. Observe that $\{p(1)\} \setminus U$ is empty.

Next we state the proposition

(43) If $x_1 = x_2$, then $f {+\cdot} (x_1 \dot{\longmapsto} y_1) {+\cdot} (x_2 \dot{\longmapsto} y_2) = f {+\cdot} (x_2 \dot{\longmapsto} y_2)$ and if $x_1 \neq x_2$, then $f {+\cdot} (x_1 \dot{\longmapsto} y_1) {+\cdot} (x_2 \dot{\longmapsto} y_2) = f {+\cdot} (x_2 \dot{\longmapsto} y_2) {+\cdot} (x_1 \dot{\longmapsto} y_1)$.

Let us consider X, U. Note that there exists an X-defined function which is U-valued and total.

Let us consider X, U, let P be a U-valued total X-defined binary relation, and let Q be a total U-defined binary relation. One can verify that $P \cdot Q$ is total.

We now state the proposition

(44) If $p \frown p_1 \frown p_2$ is X-valued, then p_2 is X-valued and p_1 is X-valued and p is X-valued.

Let us consider X and let R be a binary relation. One can check that $R\,\mathrm{null}\,X$ is $X \cup \mathrm{rng}\,R$-valued.

Let X, Y be functional sets. One can verify that $X \cup Y$ is functional.

Let us note that every set which is finite sequence-membered is also finite-membered.

Let X be a functional set. The functor SymbolsOf X is defined by:

(Def. 24) $\mathrm{SymbolsOf}\,X = \bigcup\{\mathrm{rng}\,x; x \text{ ranges over elements of } X \cup \{\emptyset\} : x \in X\}$.

Let us observe that there exists a set which is trivial, finite sequence-membered, and non empty.

Let X be a functional finite finite-membered set. Note that SymbolsOf X is finite.

Let X be a finite finite sequence-membered set. One can verify that SymbolsOf X is finite.

The following proposition is true

(45) $\mathrm{SymbolsOf}\{f\} = \mathrm{rng}\,f$.

Let z be a non zero complex number. One can check that $|z|$ is positive.

The scheme *Sc1* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\{\mathcal{F}(x); x \text{ ranges over elements of } \mathcal{A} : x \in \mathcal{A}\} = \{\mathcal{F}(x); x \text{ ranges over elements of } \mathcal{B} : x \in \mathcal{A}\}$

provided the following condition is satisfied:

- $\mathcal{A} \subseteq \mathcal{B}$.

Let X be a functional set. Then SymbolsOf X can be characterized by the condition:

(Def. 25) $\mathrm{SymbolsOf}\,X = \bigcup\{\mathrm{rng}\,x; x \text{ ranges over elements of } X: x \in X\}$.

One can prove the following propositions:

(46) For every functional set B and for every subset A of B holds $\mathrm{SymbolsOf}\,A \subseteq \mathrm{SymbolsOf}\,B$.

(47) For all functional sets A, B holds $\mathrm{SymbolsOf}(A \cup B) = \mathrm{SymbolsOf}\,A \cup \mathrm{SymbolsOf}\,B$.

Let us consider X and let F be a subset of 2^X. One can verify that $\bigcup F \setminus X$ is empty.

The following four propositions are true:

(48) $X = (X \setminus Y) \cup X \cap Y$.

(49) If A^m meets B^n, then $m = n$.

(50) If B is D-prefix and $A \subseteq B$, then A is D-prefix.

(51) $f \subseteq g$ iff for every x such that $x \in \mathrm{dom}\,f$ holds $x \in \mathrm{dom}\,g$ and $f(x) = g(x)$.

Let us consider U. One can verify that every element of $(U^* \setminus \{\emptyset\})^*$ which is non empty is also non empty yielding.

Let e be an empty set. One can verify that every element of e^* is empty.

The following proposition is true

(52)(i) If U_1-multiCat$(x) \neq \emptyset$ and U_2-multiCat$(x) \neq \emptyset$, then U_1-multiCat$(x) = U_2$-multiCat(x),

(ii) if p is $\emptyset^*$-valued, then U_1-multiCat$(p) = \emptyset$, and

(iii) if U_1-multiCat$(p) = \emptyset$ and p is ${U_1}^*$-valued, then p is $\emptyset^*$-valued.

Let us consider U, x. Note that U-multiCat(x) is U-valued.

Let us consider x. The functor $x\,\mathrm{null}$ is defined by:

(Def. 26) $x\,\mathrm{null} = x$.

Let Y be a set with non empty elements. Observe that every Y-valued binary relation which is non empty is also non empty yielding.

Let us consider X. Observe that $X \setminus \{\emptyset\}$ has non empty elements.

Let X be a set with non empty elements. One can check that every subset of X has non empty elements.

Let us consider U. Note that U^* is infinite. Observe that U^* has a non-empty element.

Let X be a set with a non-empty element. Note that there exists a subset of X which is non empty and has non empty elements.

One can prove the following propositions:

(53) If $U_1 \subseteq U_2$ and $Y \subseteq {U_1}^*$ and p is Y-valued and $p \neq \emptyset$ and Y has non empty elements, then U_1-multiCat$(p) = U_2$-multiCat(p).

(54) If there exists p such that $x = p$ and p is X^*-valued, then U-multiCat(x) is X-valued.

Let us consider X, m. Observe that $X^m \setminus X^*$ is empty.

The following two propositions are true:

(55) $(A \cap B)^* = A^* \cap B^*$.

(56) $(P \cup Q){\restriction}X = P{\restriction}X \cup Q{\restriction}X$.

Let us consider X. One can check that $2^X \setminus X$ is non empty.

Let us consider X and let R be a binary relation. One can verify that $R\,\mathrm{null}\,X$ is $X \cup \mathrm{dom}\,R$-defined.

Next we state the proposition

(57) $f{\restriction}X{+\cdot}g = f{\restriction}(X \setminus \mathrm{dom}\,g) \cup g$.

We now state the proposition

(58) If $y \notin \pi_2(X)$, then $A \times \{y\}$ misses X.

Let us consider X. The functor X-freeCountableSet is defined by:

(Def. 27) X-freeCountableSet $= \mathbb{N} \times \{$the element of $2^{\,\pi_2(X)} \setminus \pi_2(X)\}$.

Next we state the proposition

(59) X-freeCountableSet $\cap X = \emptyset$ and X-freeCountableSet is infinite.

Let us consider X. Observe that X-freeCountableSet is infinite. Observe that X-freeCountableSet $\cap X$ is empty. One can verify that X-freeCountableSet is countable.

One can check that $\mathbb{N} \setminus \mathbb{Z}$ is empty.

Let us consider x, p. Observe that $(\langle x \rangle \frown p)(1) \dot{-} x$ is empty.

Let us consider m, let m_0 be a zero number, and let p be an m-element finite sequence. Note that p null m_0 is total.

Let us consider U, q_1, q_2. One can check that U-multiCat$(\langle q_1, q_2 \rangle) \dot{-} q_1 \frown q_2$ is empty.

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Definition of First Order Language with Arbitrary Alphabet. Syntax of Terms, Atomic Formulas and their Subterms[1]

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Summary. Second of a series of articles laying down the bases for classical first order model theory. A language is defined basically as a tuple made of an integer-valued function (adicity), a symbol of equality and a symbol for the NOR logical connective. The only requests for this tuple to be a language is that the value of the adicity in = is -2 and that its preimage (i.e. the variables set) in 0 is infinite. Existential quantification will be rendered (see [11]) by mere prefixing a formula with a letter. Then the hierarchy among symbols according to their adicity is introduced, taking advantage of attributes and clusters.

The strings of symbols of a language are depth-recursively classified as terms using the standard approach (see for example [16], definition 1.1.2); technically, this is done here by deploying the '-multiCat' functor and the 'unambiguous' attribute previously introduced in [10], and the set of atomic formulas is introduced. The set of all terms is shown to be unambiguous with respect to concatenation; we say that it is a prefix set. This fact is exploited to uniquely define the subterms both of a term and of an atomic formula without resorting to a parse tree.

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The papers [1], [3], [18], [5], [6], [12], [10], [7], [8], [9], [19], [14], [13], [2], [17], [4], [21], [22], [15], and [20] provide the terminology and notation for this paper.

We follow the rules: m, n are natural numbers, m_1, n_1 are elements of $\mathbb{N}$, and X, x, z are sets.

Let z be a zero integer number. One can check that $|z|$ is zero.

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Let us observe that there exists a real number which is negative and integer and every integer number which is positive is also natural.

Let S be a non degenerated zero-one structure. Observe that (the carrier of S) $\setminus$ {the one of S} is non empty.

We introduce languages-like which are extensions of zero-one structure and are systems

⟨ a carrier, a zero, a one, an adicity ⟩,

where the carrier is a set, the zero and the one are elements of the carrier, and the adicity is a function from the carrier $\setminus${the one} into $\mathbb{Z}$.

Let S be a language-like. The functor AllSymbolsOf S is defined by:

(Def. 1) AllSymbolsOf S = the carrier of S.

The functor LettersOf S is defined as follows:

(Def. 2) LettersOf S = (the adicity of $S)^{-1}(\{0\})$.

The functor OpSymbolsOf S is defined by:

(Def. 3) OpSymbolsOf S = (the adicity of $S)^{-1}(\mathbb{N} \setminus \{0\})$.

The functor RelSymbolsOf S is defined by:

(Def. 4) RelSymbolsOf S = (the adicity of $S)^{-1}(\mathbb{Z} \setminus \mathbb{N})$.

The functor TermSymbolsOf S is defined as follows:

(Def. 5) TermSymbolsOf S = (the adicity of $S)^{-1}(\mathbb{N})$.

The functor LowerCompoundersOf S is defined as follows:

(Def. 6) LowerCompoundersOf S = (the adicity of $S)^{-1}(\mathbb{Z} \setminus \{0\})$.

The functor TheEqSymbOf S is defined as follows:

(Def. 7) TheEqSymbOf S = the zero of S.

The functor TheNorSymbOf S is defined as follows:

(Def. 8) TheNorSymbOf S = the one of S.

The functor OwnSymbolsOf S is defined by:

(Def. 9) OwnSymbolsOf S = (the carrier of S) $\setminus$ {the zero of S, the one of S}.

Let S be a language-like. An element of S is an element of AllSymbolsOf S. The functor AtomicFormulaSymbolsOf S is defined by:

(Def. 10) AtomicFormulaSymbolsOf S = AllSymbolsOf S $\setminus$ {TheNorSymbOf S}.

The functor AtomicTermsOf S is defined by:

(Def. 11) AtomicTermsOf S = (LettersOf $S)^1$.

We say that S is operational if and only if:

(Def. 12) OpSymbolsOf S is non empty.

We say that S is relational if and only if:

(Def. 13) RelSymbolsOf S $\setminus$ {TheEqSymbOf S} is non empty.

Let S be a language-like and let s be an element of S. We say that s is literal if and only if:

(Def. 14) $s \in$ LettersOf S.

We say that s is low-compounding if and only if:

(Def. 15) $s \in$ LowerCompoundersOf S.

We say that s is operational if and only if:

(Def. 16) $s \in$ OpSymbolsOf S.

We say that s is relational if and only if:

(Def. 17) $s \in$ RelSymbolsOf S.

We say that s is termal if and only if:

(Def. 18) $s \in$ TermSymbolsOf S.

We say that s is own if and only if:

(Def. 19) $s \in$ OwnSymbolsOf S.

We say that s is of-atomic-formula if and only if:

(Def. 20) $s \in$ AtomicFormulaSymbolsOf S.

Let S be a zero-one structure and let s be an element of (the carrier of S) $\setminus$ {the one of S}. The functor TrivialArity s yields an integer number and is defined by:

(Def. 21) $\text{TrivialArity}\, s = \begin{cases} -2, & \text{if } s = \text{the zero of } S, \\ 0, & \text{otherwise.} \end{cases}$

Let S be a zero-one structure and let s be an element of (the carrier of S) $\setminus$ {the one of S}. Then TrivialArity s is an element of $\mathbb{Z}$.

Let S be a non degenerated zero-one structure. The functor S TrivialArity yielding a function from (the carrier of S) $\setminus$ {the one of S} into $\mathbb{Z}$ is defined by:

(Def. 22) For every element s of (the carrier of S) $\setminus$ {the one of S} holds (S TrivialArity)(s) = TrivialArity s.

Let us observe that there exists a non degenerated zero-one structure which is infinite.

Let S be an infinite non degenerated zero-one structure.

Observe that $(S\,\text{TrivialArity})^{-1}(\{0\})$ is infinite.

Let S be a language-like. We say that S is eligible if and only if:

(Def. 23) LettersOf S is infinite and (the adicity of S)(TheEqSymbOf S) $= -2$.

One can check that there exists a language-like which is non degenerated.

One can check that there exists a non degenerated language-like which is eligible.

A language is an eligible non degenerated language-like.

We follow the rules: S, S_1, S_2 are languages and s, s_1, s_2 are elements of S.

Let S be a non empty language-like. Then AllSymbolsOf S is a non empty set.

Let S be an eligible language-like. Note that LettersOf S is infinite.

Let S be a language.

Then LettersOf S is a non empty subset of AllSymbolsOf S. Note that TheEqSymbOf S is relational.

Let S be a non degenerated language-like. Then AtomicFormulaSymbolsOf S is a non empty subset of AllSymbolsOf S.

Let S be a non degenerated language-like. Then TheEqSymbOf S is an element of AtomicFormulaSymbolsOf S.

We now state the proposition

(1) Let S be a language. Then LettersOf $S \cap$ OpSymbolsOf $S = \emptyset$ and TermSymbolsOf $S \cap$ LowerCompoundersOf $S =$ OpSymbolsOf S and RelSymbolsOf $S \setminus$ OwnSymbolsOf $S = \{$TheEqSymbOf $S\}$ and OwnSymbolsOf $S \subseteq$ AtomicFormulaSymbolsOf S and RelSymbolsOf $S \subseteq$ LowerCompoundersOf S and OpSymbolsOf $S \subseteq$ TermSymbolsOf S and LettersOf $S \subseteq$ TermSymbolsOf $S \subseteq$ OwnSymbolsOf S and OpSymbolsOf $S \subseteq$ LowerCompoundersOf $S \subseteq$ AtomicFormulaSymbolsOf S.

Let S be a language. One can verify the following observations:

* TermSymbolsOf S is non empty,
* every element of S which is own is also of-atomic-formula,
* every element of S which is relational is also low-compounding,
* every element of S which is operational is also termal,
* every element of S which is literal is also termal,
* every element of S which is termal is also own,
* every element of S which is operational is also low-compounding,
* every element of S which is low-compounding is also of-atomic-formula,
* every element of S which is termal is also non relational,
* every element of S which is literal is also non relational, and
* every element of S which is literal is also non operational.

Let S be a language. Note that there exists an element of S which is relational and there exists an element of S which is literal. Observe that every low-compounding element of S which is termal is also operational. One can check that there exists an element of S which is of-atomic-formula.

Let s be an of-atomic-formula element of S. The functor ar s yielding an element of $\mathbb{Z}$ is defined by:

(Def. 24) ar $s =$ (the adicity of S)(s).

Let S be a language and let s be a literal element of S. Note that ar s is zero. The functor S-cons yielding a binary operation on (AllSymbolsOf $S)^*$ is defined as follows:

(Def. 25) S-cons $=$ the concatenation of AllSymbolsOf S.

Let S be a language.

The functor S-multiCat yields a function from $((\text{AllSymbolsOf}\,S)^*)^*$ into $(\text{AllSymbolsOf}\,S)^*$ and is defined by:

(Def. 26) S-multiCat = (AllSymbolsOf S)-multiCat.

Let S be a language. The functor S-firstChar yielding a function from $(\text{AllSymbolsOf}\,S)^* \setminus \{\emptyset\}$ into AllSymbolsOf S is defined as follows:

(Def. 27) S-firstChar = (AllSymbolsOf S)-firstChar.

Let S be a language and let X be a set. We say that X is S-prefix if and only if:

(Def. 28) X is AllSymbolsOf S-prefix.

Let S be a language. Note that every set which is S-prefix is also AllSymbolsOf S-prefix and every set which is AllSymbolsOf S-prefix is also S-prefix. A string of S is an element of $(\text{AllSymbolsOf}\,S)^* \setminus \{\emptyset\}$.

Let us consider S. One can check that $(\text{AllSymbolsOf}\,S)^* \setminus \{\emptyset\}$ is non empty. Note that every string of S is non empty.

Let us note that every language is infinite. Observe that AllSymbolsOf S is infinite.

Let s be an of-atomic-formula element of S, and let S_3 be a set. The functor Compound(s, S_3) is defined by:

(Def. 29) $\text{Compound}(s, S_3) = \{\langle s \rangle \frown S\text{-multiCat}(S_4); S_4$ ranges over elements of $((\text{AllSymbolsOf}\,S)^*)^*$: $\text{rng}\, S_4 \subseteq S_3 \ \wedge\ S_4$ is $|\text{ar}\, s|$-element$\}$.

Let S be a language, let s be an of-atomic-formula element of S, and let S_3 be a set. Then Compound(s, S_3) is an element of $2^{(\text{AllSymbolsOf}\,S)^* \setminus \{\emptyset\}}$. The functor S-termsOfMaxDepth yields a function and is defined by the conditions (Def. 30).

(Def. 30)(i) dom(S-termsOfMaxDepth) = $\mathbb{N}$,

(ii) S-termsOfMaxDepth(0) = AtomicTermsOf S, and

(iii) for every natural number n holds S-termsOfMaxDepth$(n + 1) = \bigcup\{\text{Compound}(s, S\text{-termsOfMaxDepth}(n)); s$ ranges over of-atomic-formula elements of S: s is operational$\} \cup S$-termsOfMaxDepth(n).

Let us consider S. Then AtomicTermsOf S is a subset of $(\text{AllSymbolsOf}\,S)^*$.

Let S be a language. The functor AllTermsOf S is defined as follows:

(Def. 31) AllTermsOf $S = \bigcup \text{rng}(S\text{-termsOfMaxDepth})$.

One can prove the following proposition

(2) S-termsOfMaxDepth$(m_1) \subseteq$ AllTermsOf S.

Let S be a language and let w be a string of S. We say that w is termal if and only if:

(Def. 32) $w \in$ AllTermsOf S.

Let m be a natural number, let S be a language, and let w be a string of S. We say that w is m-termal if and only if:

(Def. 33) $w \in S$-termsOfMaxDepth(m).

Let m be a natural number and let S be a language. Note that every string of S which is m-termal is also termal.

Let us consider S. Then S-termsOfMaxDepth is a function from $\mathbb{N}$ into $2^{(\text{AllSymbolsOf}\, S)^*}$. Then AllTermsOf S is a non empty subset of $(\text{AllSymbolsOf}\, S)^*$. Note that AllTermsOf S is non empty.

Let us consider m. One can verify that S-termsOfMaxDepth(m) is non empty. Observe that every element of S-termsOfMaxDepth(m) is non empty. Observe that every element of AllTermsOf S is non empty.

Let m be a natural number and let S be a language. Note that there exists a string of S which is m-termal. Observe that every string of S which is 0-termal is also 1-element.

Let S be a language and let w be a 0-termal string of S. Observe that S-firstChar(w) is literal.

Let us consider S and let w be a termal string of S. Note that S-firstChar(w) is termal.

Let us consider S and let t be a termal string of S. The functor ar t yielding an element of $\mathbb{Z}$ is defined as follows:

(Def. 34) $\text{ar}\, t = \text{ar}\, S\text{-firstChar}(t)$.

Next we state the proposition

(3) For every $m_1 + 1$-termal string w of S there exists an element T of S-termsOfMaxDepth$(m_1)^*$ such that T is $|\text{ar}\, S\text{-firstChar}(w)|$-element and $w = \langle S\text{-firstChar}(w)\rangle \frown S\text{-multiCat}(T)$.

Let us consider S, m. Note that S-termsOfMaxDepth(m) is S-prefix.

Let us consider S and let V be an element of $(\text{AllTermsOf}\, S)^*$. Observe that S-multiCat(V) is relation-like.

Let us consider S and let V be an element of $(\text{AllTermsOf}\, S)^*$. One can verify that S-multiCat(V) is function-like.

Let us consider S and let p_1 be a string of S. We say that p_1 is 0-w.f.f. if and only if:

(Def. 35) There exists a relational element s of S and there exists an $|\text{ar}\, s|$-element element V of $(\text{AllTermsOf}\, S)^*$ such that $p_1 = \langle s\rangle \frown S\text{-multiCat}(V)$.

Let us consider S. Note that there exists a string of S which is 0-w.f.f..

Let p_1 be a 0-w.f.f. string of S. Observe that S-firstChar(p_1) is relational. The functor AtomicFormulasOf S is defined as follows:

(Def. 36) AtomicFormulasOf $S = \{p_1; p_1$ ranges over strings of S: p_1 is 0-w.f.f.$\}$.

Let us consider S. Then AtomicFormulasOf S is a subset of $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$. Note that AtomicFormulasOf S is non empty. Observe that every element of AtomicFormulasOf S is 0-w.f.f.. Observe that AllTermsOf S is S-prefix.

Let us consider S and let t be a termal string of S. The functor SubTerms t yields an element of $(\text{AllTermsOf } S)^*$ and is defined by:

(Def. 37) SubTerms t is $|\text{ar } S\text{-firstChar}(t)|$-element and $t = \langle S\text{-firstChar}(t)\rangle \frown S\text{-multiCat}(\text{SubTerms } t)$.

Let us consider S and let t be a termal string of S. One can verify that SubTerms t is $|\text{ar } t|$-element.

Let t_0 be a 0-termal string of S. Note that SubTerms t_0 is empty.

Let us consider m_1, S and let t be an $m_1 + 1$-termal string of S. One can verify that SubTerms t is S-termsOfMaxDepth(m_1)-valued.

Let us consider S and let p_1 be a 0-w.f.f. string of S. The functor SubTerms p_1 yields an $|\text{ar } S\text{-firstChar}(p_1)|$-element element of $(\text{AllTermsOf } S)^*$ and is defined as follows:

(Def. 38) $p_1 = \langle S\text{-firstChar}(p_1)\rangle \frown S\text{-multiCat}(\text{SubTerms } p_1)$.

Let us consider S and let p_1 be a 0-w.f.f. string of S. Note that SubTerms p_1 is $|\text{ar } S\text{-firstChar}(p_1)|$-element.

Then AllTermsOf S is an element of $2^{(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}}$. Note that every element of AllTermsOf S is termal. The functor S-subTerms yielding a function from AllTermsOf S into $(\text{AllTermsOf } S)^*$ is defined by:

(Def. 39) For every element t of AllTermsOf S holds $S\text{-subTerms}(t) = \text{SubTerms } t$.

We now state several propositions:

(4) $S\text{-termsOfMaxDepth}(m) \subseteq S\text{-termsOfMaxDepth}(m + n)$.

(5) If $x \in \text{AllTermsOf } S$, then there exists n_1 such that $x \in S\text{-termsOfMaxDepth}(n_1)$.

(6) $\text{AllTermsOf } S \subseteq (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$.

(7) AllTermsOf S is S-prefix.

(8) If $x \in \text{AllTermsOf } S$, then x is a string of S.

(9) $\text{AtomicFormulaSymbolsOf } S \setminus \text{OwnSymbolsOf } S = \{\text{TheEqSymbOf } S\}$.

(10) $\text{TermSymbolsOf } S \setminus \text{LettersOf } S = \text{OpSymbolsOf } S$.

(11) $\text{AtomicFormulaSymbolsOf } S \setminus \text{RelSymbolsOf } S = \text{TermSymbolsOf } S$.

Let us consider S. Observe that every of-atomic-formula element of S which is non relational is also termal.

Then OwnSymbolsOf S is a subset of AllSymbolsOf S. Observe that every termal element of S which is non literal is also operational.

Next we state three propositions:

(12) x is a string of S iff x is a non empty element of $(\text{AllSymbolsOf } S)^*$.

(13) x is a string of S iff x is a non empty finite sequence of elements of AllSymbolsOf S.

(14) S-termsOfMaxDepth is a function from $\mathbb{N}$ into $2^{(\text{AllSymbolsOf } S)^*}$.

Let us consider S. Note that every element of LettersOf S is literal. One can check that TheNorSymbOf S is non low-compounding.

Observe that TheNorSymbOf S is non own.

Next we state the proposition

(15) If $s \neq$ TheNorSymbOf S and $s \neq$ TheEqSymbOf S, then $s \in$ OwnSymbolsOf S.

For simplicity, we use the following convention: l, l_1, l_2 denote literal elements of S, a denotes an of-atomic-formula element of S, r denotes a relational element of S, w, w_1 denote strings of S, and t_2 denotes an element of AllTermsOf S.

Let us consider S, t. The functor Depth t yielding a natural number is defined by:

(Def. 40) t is Depth t-termal and for every n such that t is n-termal holds Depth $t \leq n$.

Let us consider S, let m_0 be a zero number, and let t be an m_0-termal string of S. Note that Depth t is zero.

Let us consider S and let s be a low-compounding element of S. Note that ar s is non zero.

Let us consider S and let s be a termal element of S. Observe that ar s is non negative and extended real.

Let us consider S and let s be a relational element of S. Note that ar s is negative and extended real.

Next we state the proposition

(16) If t is non 0-termal, then S-firstChar(t) is operational and SubTerms $t \neq \emptyset$.

Let us consider S. Observe that S-multiCat is finite sequence-yielding.

Let us consider S and let W be a non empty AllSymbolsOf $S^* \setminus \{\emptyset\}$-valued finite sequence. One can verify that S-multiCat(W) is non empty.

Let us consider S, l. Note that $\langle l \rangle$ is 0-termal.

Let us consider S, m, n. One can check that every string of S which is $m + 0 \cdot n$-termal is also $m + n$-termal.

Let us consider S. One can check that every own element of S which is non low-compounding is also literal.

Let us consider S, t. One can check that SubTerms t is rng t^*-valued.

Let p_0 be a 0-w.f.f. string of S. Observe that SubTerms p_0 is rng $p_0{}^*$-valued. Then S-termsOfMaxDepth is a function from $\mathbb{N}$ into $2^{(\mathrm{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}}$.

Let us consider S, m_1. Observe that S-termsOfMaxDepth(m_1) has non empty elements.

Let us consider S, m and let t be a termal string of S. One can verify that t null m is Depth $t + m$-termal. One can check that every string of S which is termal is also TermSymbolsOf S-valued. Observe that AllTermsOf $S \setminus$ (TermSymbolsOf S)* is empty.

Let p_0 be a 0-w.f.f. string of S. Observe that SubTerms p_0 is TermSymbolsOf S^*-valued. One can verify that every string of S which is 0-w.f.f. is also

AtomicFormulaSymbolsOf S-valued. One can check that OwnSymbolsOf S is non empty.

In the sequel p_0 is a 0-w.f.f. string of S.

The following proposition is true

(17) If S-firstChar$(p_0) \neq$ TheEqSymbOf S, then p_0 is OwnSymbolsOf S-valued.

Let us observe that there exists a language-like which is strict and non empty.

Let S_1, S_2 be languages-like. We say that S_2 is S_1-extending if and only if:

(Def. 41) The adicity of $S_1 \subseteq$ the adicity of S_2 and TheEqSymbOf S_1 = TheEqSymbOf S_2 and TheNorSymbOf S_1 = TheNorSymbOf S_2.

Let us consider S. One can verify that S null is S-extending. Observe that there exists a language which is S-extending.

Let us consider S_1 and let S_2 be an S_1-extending language. Observe that OwnSymbolsOf $S_1 \setminus$ OwnSymbolsOf S_2 is empty.

Let f be a $\mathbb{Z}$-valued function and let L be a non empty language-like. The functor L extendWith f yields a strict non empty language-like and is defined by the conditions (Def. 42).

(Def. 42)(i) The adicity of L extendWith $f = f\restriction(\mathrm{dom}\, f \setminus \{$the one of $L\})+\cdot$the adicity of L,

(ii) the zero of L extendWith f = the zero of L, and

(iii) the one of L extendWith f = the one of L.

Let S be a non empty language-like and let f be a $\mathbb{Z}$-valued function. Note that S extendWith f is S-extending.

Let S be a non degenerated language-like. Observe that every language-like which is S-extending is also non degenerated.

Let S be an eligible language-like. One can check that every language-like which is S-extending is also eligible.

Let E be an empty binary relation and let us consider X. Note that $X\restriction E$ is empty.

Let us consider X and let m be an integer number. Note that $X \longmapsto m$ is $\mathbb{Z}$-valued.

Let us consider S and let X be a functional set.

The functor S addLettersNotIn X yields an S-extending language and is defined as follows:

(Def. 43) S addLettersNotIn X = S extendWith((AllSymbolsOf $S \cup$ SymbolsOf X)-freeCountableSet $\longmapsto$ 0 **qua** $\mathbb{Z}$-valued function).

Let us consider S_1 and let X be a functional set.

Note that $\mathrm{LettersOf}(S_1\,\mathrm{addLettersNotIn}\,X) \setminus \mathrm{SymbolsOf}\,X$ is infinite.

Let us note that there exists a language which is countable.

Let S be a countable language. Observe that $\mathrm{AllSymbolsOf}\,S$ is countable. One can verify that $(\mathrm{AllSymbolsOf}\,S)^* \setminus \{\emptyset\}$ is countable.

Let L be a non empty language-like and let f be a $\mathbb{Z}$-valued function. Note that $\mathrm{AllSymbolsOf}(L\,\mathrm{extendWith}\,f) \dot{-} (\mathrm{dom}\,f \cup \mathrm{AllSymbolsOf}\,L)$ is empty.

Let S be a countable language and let X be a functional set. One can check that $S\,\mathrm{addLettersNotIn}\,X$ is countable.

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First Order Languages: Further Syntax and Semantics[1]

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Summary. Third of a series of articles laying down the bases for classical first order model theory. Interpretation of a language in a universe set. Evaluation of a term in a universe. Truth evaluation of an atomic formula. Reassigning the value of a symbol in a given interpretation. Syntax and semantics of a non atomic formula are then defined concurrently (this point is explained in [16], 4.2.1). As a consequence, the evaluation of any w.f.f. string and the relation of logical implication are introduced. Depth of a formula. Definition of satisfaction and entailment (aka entailment or logical implication) relations, see [18] III.3.2 and III.4.1 respectively.

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The terminology and notation used in this paper have been introduced in the following papers: [7], [1], [23], [6], [8], [17], [14], [15], [22], [9], [10], [11], [2], [21], [26], [24], [5], [3], [4], [12], [27], [28], [19], [20], [25], and [13].

For simplicity, we follow the rules: m, n denote natural numbers, m_1 denotes an element of $\mathbb{N}$, A, B, X, Y, Z, x, y denote sets, S, S_1, S_2 denote languages, s denotes an element of S, w, w_1, w_2 denote strings of S, U denotes a non empty set, f, g denote functions, and p, p_2 denote finite sequences.

Let us consider S. Then TheNorSymbOf S is an element of S.

Let U be a non empty set. The functor U-deltaInterpreter yielding a function from U^2 into *Boolean* is defined by:

(Def. 1) U-deltaInterpreter $= \chi_{(\text{the concatenation of } U)^\circ(\mathrm{id}_{U^1}), U^2}$.

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Let X be a set. Then id_X is an equivalence relation of X.

Let S be a language, let U be a non empty set, and let s be an of-atomic-formula element of S. Interpreter of s and U is defined as follows:

(Def. 2)(i) It is a function from $U^{|\mathrm{ar}\, s|}$ into *Boolean* if s is relational,

(ii) it is a function from $U^{|\mathrm{ar}\, s|}$ into U, otherwise.

Let us consider S, U and let s be an of-atomic-formula element of S. We see that the interpreter of s and U is a function from $U^{|\mathrm{ar}\, s|}$ into $U \cup \mathit{Boolean}$.

Let us consider S, U and let s be a termal element of S. One can verify that every interpreter of s and U is U-valued.

Let S be a language. Note that every element of S which is literal is also own.

Let us consider S, U. A function is called an interpreter of S and U if:

(Def. 3) For every own element s of S holds $\mathrm{it}(s)$ is an interpreter of s and U.

Let us consider S, U, f. We say that f is (S,U)-interpreter-like if and only if:

(Def. 4) f is an interpreter of S and U and function yielding.

Let us consider S and let U be a non empty set. One can verify that every function which is (S,U)-interpreter-like is also function yielding.

Let us consider S, U and let s be an own element of S. Observe that every interpreter of s and U is non empty.

Let S be a language and let U be a non empty set. Note that there exists a function which is (S,U)-interpreter-like.

Let us consider S, U, let I be an (S,U)-interpreter-like function, and let s be an own element of S. Then $I(s)$ is an interpreter of s and U.

Let S be a language, let U be a non empty set, let I be an (S,U)-interpreter-like function, let x be an own element of S, and let f be an interpreter of x and U. One can check that $I{+\cdot}(x \mapsto\!\!\!\!\dot{}\!\!\longrightarrow f)$ is (S,U)-interpreter-like.

Let us consider f, x, y. The functor $(x,y)\,\mathrm{ReassignIn}\,f$ yields a function and is defined by:

(Def. 5) $(x,y)\,\mathrm{ReassignIn}\,f = f{+\cdot}(x \dot{\longmapsto} (\emptyset \dot{\longmapsto} y))$.

Let S be a language, let U be a non empty set, let I be an (S,U)-interpreter-like function, let x be a literal element of S, and let u be an element of U. One can verify that $(x,u)\,\mathrm{ReassignIn}\,I$ is (S,U)-interpreter-like.

Let S be a language. One can check that $\mathrm{AllSymbolsOf}\,S$ is non empty.

Let Y be a set and let X, Z be non empty sets. Observe that every function from X into Z^Y is function yielding.

Let X, Y, Z be non empty sets. One can verify that there exists a function from X into Z^Y which is function yielding.

Let f be a function yielding function and let g be a function. The functor $[g,f]$ yields a function and is defined by:

(Def. 6) $\mathrm{dom}[g, f] = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $[g, f](x) = g \cdot f(x)$.

Let f be an empty function and let g be a function. One can verify that $[g, f]$ is empty.

Let f be a function yielding function and let g be a function. The functor $[f, g]$ yielding a function is defined as follows:

(Def. 7) $\mathrm{dom}[f, g] = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and for every set x such that $x \in \mathrm{dom}[f, g]$ holds $[f, g](x) = f(x)(g(x))$.

Let f be a function yielding function and let g be an empty function. One can check that $[f, g]$ is empty.

Let X be a finite sequence-membered set. Observe that every function which is X-valued is also function yielding.

Let E, D be non empty sets, let p be a D-valued finite sequence, and let h be a function from D into E. Note that $h \cdot p$ is $\mathrm{len}\, p$-element.

Let X, Y be non empty sets, let f be a function from X into Y, and let p be an X-valued finite sequence. One can verify that $f \cdot p$ is finite sequence-like.

Let E, D be non empty sets, let n be a natural number, let p be an n-element D-valued finite sequence, and let h be a function from D into E. Observe that $h \cdot p$ is n-element.

We now state the proposition

(1) For every 0-termal string t_0 of S holds $t_0 = \langle S\text{-firstChar}(t_0) \rangle$.

Let us consider S, let U be a non empty set, let u be an element of U, and let I be an (S, U)-interpreter-like function. The functor (I, u)-TermEval yields a function from $\mathbb{N}$ into $U^{\mathrm{AllTermsOf}\, S}$ and is defined as follows:

(Def. 8) (I, u)-TermEval$(0) = \mathrm{AllTermsOf}\, S \longmapsto u$ and for every m_1 holds (I, u)-TermEval$(m_1 + 1) = [I \cdot S\text{-firstChar}, [((I, u)\text{-TermEval}(m_1)$ **qua** function$), S\text{-subTerms}]]$.

Let us consider S, U, let I be an (S, U)-interpreter-like function, and let t be an element of AllTermsOf S. The functor I-TermEval t yields an element of U and is defined as follows:

(Def. 9) For every element u_1 of U and for every m_1 such that $t \in S$-termsOfMaxDepth(m_1) holds I-TermEval $t = (I, u_1)$-TermEval$(m_1 + 1)(t)$.

Let us consider S, U and let I be an (S, U)-interpreter-like function. The functor I-TermEval yielding a function from AllTermsOf S into U is defined by:

(Def. 10) For every element t of AllTermsOf S holds I-TermEval$(t) = I$-TermEval t.

Let us consider S, U and let I be an (S, U)-interpreter-like function. The functor I === yielding a function is defined as follows:

(Def. 11) I ==== $I +\cdot$ (TheEqSymbOf $S \dot{\longmapsto} U$-deltaInterpreter).

Let us consider S, U, let I be an (S,U)-interpreter-like function, and let x be a set. We say that x is I-extension if and only if:

(Def. 12) $x = I ===$.

Let us consider S, U and let I be an (S,U)-interpreter-like function. Note that $I ===$ is I-extension and every set which is I-extension is also function-like. Observe that there exists a function which is I-extension. Observe that $I ===$ is (S,U)-interpreter-like.

Let f be an I-extension function, and let s be an of-atomic-formula element of S. Then $f(s)$ is an interpreter of s and U.

Let p_1 be a 0-w.f.f. string of S. The functor I-AtomicEval p_1 is defined as follows:

(Def. 13) $I\text{-AtomicEval}\, p_1 = (I === (S\text{-firstChar}(p_1)))(I\text{-TermEval} \cdot \text{SubTerms}\, p_1)$.

Let us consider S, U, let I be an (S,U)-interpreter-like function, and let p_1 be a 0-w.f.f. string of S. Then I-AtomicEval p_1 is an element of *Boolean*. Note that $I \restriction \text{OwnSymbolsOf}\, S$ is $(U^* \dot{\rightarrow} (U \cup \mathit{Boolean}))$-valued and $I \restriction \text{OwnSymbolsOf}\, S$ is (S,U)-interpreter-like.

Let us consider S, U and let I be an (S,U)-interpreter-like function. Observe that $I \restriction \text{OwnSymbolsOf}\, S$ is total.

Let us consider S, U. The functor U-InterpretersOf S is defined by:

(Def. 14) $U\text{-InterpretersOf}\, S = \{f \in (U^* \dot{\rightarrow} (U \cup \mathit{Boolean}))^{\text{OwnSymbolsOf}\, S} : f$ is (S,U)-interpreter-like$\}$.

Let us consider S, U. Then U-InterpretersOf S is a subset of $(U^* \dot{\rightarrow} (U \cup \mathit{Boolean}))^{\text{OwnSymbolsOf}\, S}$. Observe that U-InterpretersOf S is non empty. One can verify that every element of U-InterpretersOf S is (S,U)-interpreter-like. The functor S-TruthEval U yields a function from

$(U\text{-InterpretersOf}\, S) \times \text{AtomicFormulasOf}\, S$ into *Boolean* and is defined by:

(Def. 15) For every element I of U-InterpretersOf S and for every element p_1 of AtomicFormulasOf S holds $(S\text{-TruthEval}\, U)(I, p_1) = I\text{-AtomicEval}\, p_1$.

Let us consider S, U, let I be an element of U-InterpretersOf S, let f be a partial function from $(U\text{-InterpretersOf}\, S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\})$ to *Boolean*, and let p_1 be an element of $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$. The functor f-ExFunctor(I, p_1) yielding an element of *Boolean* is defined as follows:

(Def. 16) $f\text{-ExFunctor}(I,p_1) = \begin{cases} \mathit{true}, & \text{if there exists an element } u \text{ of } U \text{ and there exists a literal element } v \text{ of } S \text{ such that } p_1(1) = v \text{ and } f((v,u)\,\text{ReassignIn}\, I,\ (p_1)_{\restriction 1}) = \mathit{true}, \\ \mathit{false}, & \text{otherwise.} \end{cases}$

Let us consider S, U and let g be an element of $(U\text{-InterpretersOf}\, S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}) \dot{\rightarrow} \mathit{Boolean}$. The functor ExIterator g yields a partial function from $(U\text{-InterpretersOf}\, S) \times ((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\})$ to *Boolean* and is defined by the conditions (Def. 17).

(Def. 17)(i) For every element x of U-InterpretersOf S and for every element y of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ holds $\langle x, y\rangle \in \operatorname{dom} \text{ExIterator } g$ iff there exists a literal element v of S and there exists a string w of S such that $\langle x, w\rangle \in \operatorname{dom} g$ and $y = \langle v\rangle \frown w$, and

(ii) for every element x of U-InterpretersOf S and for every element y of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ such that $\langle x, y\rangle \in \operatorname{dom} \text{ExIterator } g$ holds $(\text{ExIterator } g)(x, y) = g\text{-ExFunctor}(x, y)$.

Let us consider S, U, let f be a partial function from $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\})$ to *Boolean*, let I be an element of U-InterpretersOf S, and let p_1 be an element of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$.

The functor f-NorFunctor(I, p_1) yielding an element of *Boolean* is defined by:

(Def. 18) $$f\text{-NorFunctor}(I, p_1) = \begin{cases} \mathit{true}, & \text{if there exist elements } w_1, w_2 \text{ of } (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\} \text{ such that } \langle I, w_1\rangle \in \operatorname{dom} f \text{ and } f(I, w_1) = \mathit{false} \text{ and } f(I, w_2) = \mathit{false} \text{ and } p_1 = \langle \text{TheNorSymbOf } S\rangle \frown w_1 \frown w_2, \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

Let us consider S, U and let g be an element of $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \dot{\to} \mathit{Boolean}$. The functor NorIterator g yielding a partial function from $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\})$ to *Boolean* is defined by the conditions (Def. 19).

(Def. 19)(i) For every element x of U-InterpretersOf S and for every element y of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ holds $\langle x, y\rangle \in \operatorname{dom} \text{NorIterator } g$ iff there exist elements p_3, p_4 of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ such that $y = \langle \text{TheNorSymbOf } S\rangle \frown p_3 \frown p_4$ and $\langle x, p_3\rangle$, $\langle x, p_4\rangle \in \operatorname{dom} g$, and

(ii) for every element x of U-InterpretersOf S and for every element y of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ such that $\langle x, y\rangle \in \operatorname{dom} \text{NorIterator } g$ holds $(\text{NorIterator } g)(x, y) = g\text{-NorFunctor}(x, y)$.

Let us consider S, U. The functor (S, U)-TruthEval yields a function from $\mathbb{N}$ into $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \dot{\to} \mathit{Boolean}$ and is defined as follows:

(Def. 20) $(S, U)\text{-TruthEval}(0) = S\text{-TruthEval } U$ and for every m_1 holds $(S, U)\text{-TruthEval}(m_1+1) = \text{ExIterator}(S, U)\text{-TruthEval}(m_1) {+}\!\cdot \text{NorIterator } (S, U)\text{-TruthEval}(m_1) {+}\!\cdot (S, U)\text{-TruthEval}(m_1)$.

Next we state the proposition

(2) For every (S, U)-interpreter-like function I holds $I \restriction \text{OwnSymbolsOf } S \in U\text{-InterpretersOf } S$.

Let S be a language, let m be a natural number, and let U be a non empty set.

The functor (S, U)-TruthEval m yielding an element of (U-InterpretersOf S) $\times$ ((AllSymbolsOf $S)^* \setminus \{\emptyset\}) \dot{\rightarrow}$ *Boolean* is defined as follows:

(Def. 21) For every m_1 such that $m = m_1$ holds (S, U)-TruthEval m = (S, U)-TruthEval(m_1).

Let us consider S, U, m and let I be an element of U-InterpretersOf S. The functor (I, m)-TruthEval yields an element of ((AllSymbolsOf $S)^* \setminus \{\emptyset\}) \dot{\rightarrow}$ *Boolean* and is defined by:

(Def. 22) (I, m)-TruthEval = (curry((S, U)-TruthEval m))(I).

Let us consider S, m. The functor S-formulasOfMaxDepth m yielding a subset of (AllSymbolsOf $S)^* \setminus \{\emptyset\}$ is defined as follows:

(Def. 23) For every non empty set U and for every element I of U-InterpretersOf S and for every element m_1 of $\mathbb{N}$ such that $m = m_1$ holds S-formulasOfMaxDepth m = dom((I, m_1)-TruthEval).

Let us consider S, m, w. We say that w is m-w.f.f. if and only if:

(Def. 24) $w \in S$-formulasOfMaxDepth m.

Let us consider S, w. We say that w is w.f.f. if and only if:

(Def. 25) There exists m such that w is m-w.f.f..

Let us consider S. Note that every string of S which is 0-w.f.f. is also 0-w.f.f. and every string of S which is 0-w.f.f. is also 0-w.f.f.. Let us consider m. One can check that every string of S which is m-w.f.f. is also w.f.f.. Let us consider n. One can check that every string of S which is $m + 0 \cdot n$-w.f.f. is also $m + n$-w.f.f..

Let us consider S, m. Observe that there exists a string of S which is m-w.f.f.. Note that S-formulasOfMaxDepth m is non empty. One can verify that there exists a string of S which is w.f.f..

Let us consider S, U, let I be an element of U-InterpretersOf S, and let w be a w.f.f. string of S. The functor I-TruthEval w yields an element of *Boolean* and is defined as follows:

(Def. 26) For every natural number m such that w is m-w.f.f. holds I-TruthEval $w = (I, m)$-TruthEval(w).

Let us consider S. The functor AllFormulasOf S is defined by:

(Def. 27) AllFormulasOf $S = \{w; w$ ranges over strings of S: $\bigvee_m$ w is m-w.f.f.$\}$.

Let us consider S. One can check that AllFormulasOf S is non empty.

For simplicity, we follow the rules: u, u_1, u_2 are elements of U, t is a termal string of S, I is an (S, U)-interpreter-like function, l, l_1, l_2 are literal elements of S, m_2, n_1 are non zero natural numbers, p_0 is a 0-w.f.f. string of S, and p_5, p_1, p_3, p_4 are w.f.f. strings of S.

The following propositions are true:

(3) (I, u)-TermEval$(m + 1)(t) = I(S$-firstChar$(t))((I, u)$-TermEval$(m) \cdot$ SubTerms $t)$ and if t is 0-termal, then (I, u)-TermEval$(m + 1)(t) = I(S$-firstChar$(t))(\emptyset)$.

(4) For every m-termal string t of S holds (I, u_1)-TermEval$(m + 1)(t) = (I, u_2)$-TermEval$(m + 1 + n)(t)$.

(5) curry$((S, U)$-TruthEval $m)$ is a function from U-InterpretersOf S into $((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \dot{\rightarrow} \mathit{Boolean}$.

(6) $x \in X \cup Y \cup Z$ iff $x \in X$ or $x \in Y$ or $x \in Z$.

(7) S-formulasOfMaxDepth 0 = AtomicFormulasOf S.

Let us consider S, m. Then S-formulasOfMaxDepth m can be characterized by the condition:

(Def. 28) For every non empty set U and for every element I of U-InterpretersOf S holds S-formulasOfMaxDepth $m = \text{dom}((I, m)\text{-TruthEval})$.

Next we state the proposition

(8) (S, U)-TruthEval $m \in \mathit{Boolean}^{(U\text{-InterpretersOf } S) \times (S\text{-formulasOfMaxDepth } m)}$ and (S, U)-TruthEval$(m) \in \mathit{Boolean}^{(U\text{-InterpretersOf } S) \times (S\text{-formulasOfMaxDepth } m)}$.

Let us consider S, m. The functor m-ExFormulasOf S is defined by:

(Def. 29) m-ExFormulasOf $S = \{\langle v \rangle ^\frown p_1 : v$ ranges over elements of LettersOf S, p_1 ranges over elements of S-formulasOfMaxDepth $m\}$.

The functor m-NorFormulasOf S is defined as follows:

(Def. 30) m-NorFormulasOf $S = \{\langle$TheNorSymbOf $S\rangle ^\frown p_3 ^\frown p_4 : p_3$ ranges over elements of S-formulasOfMaxDepth m, p_4 ranges over elements of S-formulasOfMaxDepth $m\}$.

Let us consider S and let w_1, w_2 be strings of S. Then $w_1 ^\frown w_2$ is a string of S.

Let us consider S, s. Then $\langle s \rangle$ is a string of S.

One can prove the following two propositions:

(9) S-formulasOfMaxDepth$(m + 1) = (m\text{-ExFormulasOf } S) \cup (m\text{-NorFormulasOf } S) \cup (S\text{-formulasOfMaxDepth } m)$.

(10) AtomicFormulasOf S is S-prefix.

Let us consider S. Note that AtomicFormulasOf S is S-prefix. Observe that S-formulasOfMaxDepth 0 is S-prefix.

Let us consider p_1. The functor Depth p_1 yielding a natural number is defined by:

(Def. 31) p_1 is Depth p_1-w.f.f. and for every n such that p_1 is n-w.f.f. holds Depth $p_1 \leq n$.

Let us consider S, m and let p_3, p_4 be m-w.f.f. strings of S. Note that $\langle$TheNorSymbOf $S\rangle ^\frown p_3 ^\frown p_4$ is $m + 1$-w.f.f..

Let us consider S, p_3, p_4. Observe that $\langle$TheNorSymbOf $S\rangle ^\frown p_3 ^\frown p_4$ is w.f.f..

Let us consider S, m, let p_1 be an m-w.f.f. string of S, and let v be a literal element of S. Note that $\langle v \rangle ^\frown p_1$ is $m + 1$-w.f.f..

Let us consider S, l, p_1. Note that $\langle l\rangle \frown p_1$ is w.f.f..

Let us consider S, w and let s be a non relational element of S. One can check that $\langle s\rangle \frown w$ is non 0-w.f.f..

Let us consider S, w_1, w_2 and let s be a non relational element of S. Observe that $\langle s\rangle \frown w_1 \frown w_2$ is non 0-w.f.f..

Let us consider S. Observe that TheNorSymbOf S is non relational.

Let us consider S, w. Observe that $\langle \mathrm{TheNorSymbOf}\, S\rangle \frown w$ is non 0-w.f.f..

Let us consider S, l, w. Note that $\langle l\rangle \frown w$ is non 0-w.f.f..

Let us consider S, w. We say that w is exal if and only if:

(Def. 32) S-firstChar(w) is literal.

Let us consider S, w, l. One can verify that $\langle l\rangle \frown w$ is exal.

Let us consider S, m_2. Observe that there exists an m_2-w.f.f. string of S which is exal.

Let us consider S. Note that every string of S which is exal is also non 0-w.f.f..

Let us consider S, m_2. One can check that there exists an exal m_2-w.f.f. string of S which is non 0-w.f.f..

Let us consider S. One can verify that there exists an exal w.f.f. string of S which is non 0-w.f.f..

Let us consider S and let p_1 be a non 0-w.f.f. w.f.f. string of S. Note that Depth p_1 is non zero.

Let us consider S and let w be a non 0-w.f.f. w.f.f. string of S. Observe that S-firstChar(w) is non relational.

Let us consider S, m. Observe that S-formulasOfMaxDepth m is S-prefix. Then AllFormulasOf S is a subset of $(\mathrm{AllSymbolsOf}\, S)^{*} \setminus \{\emptyset\}$. Observe that every element of AllFormulasOf S is w.f.f.. Note that AllFormulasOf S is S-prefix.

We now state three propositions:

(11) $\mathrm{dom}\,\mathrm{NorIterator}((S, U)\text{-TruthEval}\, m) = (U\text{-InterpretersOf}\, S) \times (m\text{-NorFormulasOf}\, S)$.

(12) $\mathrm{dom}\,\mathrm{ExIterator}((S, U)\text{-TruthEval}\, m) = (U\text{-InterpretersOf}\, S) \times (m\text{-ExFormulasOf}\, S)$.

(13) $U\text{-deltaInterpreter}^{-1}(\{1\}) = \{\langle u, u\rangle : u$ ranges over elements of $U\}$.

Let us consider S. Then TheEqSymbOf S is an element of S.

Let us consider S. One can verify that $\mathrm{ar}\,\mathrm{TheEqSymbOf}\, S + 2$ is zero and $|\mathrm{ar}\,\mathrm{TheEqSymbOf}\, S| - 2$ is zero.

We now state two propositions:

(14) Let p_1 be a 0-w.f.f. string of S and I be an (S, U)-interpreter-like function. Then

(i) if S-firstChar$(p_1) \neq$ TheEqSymbOf S, then $I\text{-AtomicEval}\, p_1 = I(S\text{-firstChar}(p_1))(I\text{-TermEval} \cdot \mathrm{SubTerms}\, p_1)$, and

(ii) if S-firstChar(p_1) = TheEqSymbOf S, then I-AtomicEval p_1 = U-deltaInterpreter(I-TermEval $\cdot$ SubTerms p_1).

(15) Let I be an (S, U)-interpreter-like function and p_1 be a 0-w.f.f. string of S. If S-firstChar(p_1) = TheEqSymbOf S, then I-AtomicEval $p_1 = 1$ iff I-TermEval((SubTerms p_1)(1)) = I-TermEval((SubTerms p_1)(2)).

Let us consider S, m. One can check that m-ExFormulasOf S is non empty. Note that m-NorFormulasOf S is non empty. Then m-NorFormulasOf S is a subset of (AllSymbolsOf S)$^* \setminus \{\emptyset\}$.

Let us consider S and let w be an exal string of S. One can verify that S-firstChar(w) is literal.

Let us consider S, m. Observe that every element of m-NorFormulasOf S is non exal. Then m-ExFormulasOf S is a subset of (AllSymbolsOf S)$^* \setminus \{\emptyset\}$.

Let us consider S, m. One can check that every element of m-ExFormulasOf S is exal.

Let us consider S. One can check that there exists an element of S which is non literal.

Let us consider S, w and let s be a non literal element of S. Note that $\langle s \rangle \frown w$ is non exal.

Let us consider S, w_1, w_2 and let s be a non literal element of S. Observe that $\langle s \rangle \frown w_1 \frown w_2$ is non exal.

Let us consider S. Note that TheNorSymbOf S is non literal.

Next we state the proposition

(16) $p_1 \in$ AllFormulasOf S.

Let us consider S, m, w. We introduce w is m-non-w.f.f. as an antonym of w is m-w.f.f..

Let us consider m, S. One can verify that every string of S which is non m-w.f.f. is also m-non-w.f.f..

Let us consider S, p_3, p_4. Observe that $\langle$TheNorSymbOf $S\rangle \frown p_3 \frown p_4$ is max(Depth p_3, Depth p_4)-non-w.f.f..

Let us consider S, p_1, l. Note that $\langle l \rangle \frown p_1$ is Depth p_1-non-w.f.f..

Let us consider S, p_1, l. One can check that $\langle l \rangle \frown p_1$ is $1 +$ Depth p_1-w.f.f..

Let us consider S, U. Observe that every element of U-InterpretersOf S is OwnSymbolsOf S-defined.

Let us consider S, U. Note that there exists an element of U-InterpretersOf S which is OwnSymbolsOf S-defined.

Let us consider S, U. Note that every OwnSymbolsOf S-defined element of U-InterpretersOf S is total.

Let us consider S, U, let I be an element of U-InterpretersOf S, let x be a literal element of S, and let u be an element of U. Then (x, u) ReassignIn I is an element of U-InterpretersOf S.

In the sequel I denotes an element of U-InterpretersOf S.

Let us consider S, w. The functor $\mathrm{xnot}\, w$ yields a string of S and is defined as follows:

(Def. 33) $\mathrm{xnot}\, w = \langle \mathrm{TheNorSymbOf}\, S \rangle \frown w \frown w$.

Let us consider S, m and let p_1 be an m-w.f.f. string of S. Observe that $\mathrm{xnot}\, p_1$ is $m+1$-w.f.f..

Let us consider S, p_1. Note that $\mathrm{xnot}\, p_1$ is w.f.f..

Let us consider S. One can verify that $\mathrm{TheEqSymbOf}\, S$ is non own.

Let us consider S, X. We say that X is S-mincover if and only if:

(Def. 34) For every p_1 holds $p_1 \in X$ iff $\mathrm{xnot}\, p_1 \notin X$.

One can prove the following propositions:

(17) $\mathrm{Depth}(\langle \mathrm{TheNorSymbOf}\, S \rangle \frown p_3 \frown p_4) = 1 + \max(\mathrm{Depth}\, p_3, \mathrm{Depth}\, p_4)$ and $\mathrm{Depth}(\langle l \rangle \frown p_3) = \mathrm{Depth}\, p_3 + 1$.

(18) If $\mathrm{Depth}\, p_1 = m + 1$, then p_1 is exal iff $p_1 \in m\text{-}\mathrm{ExFormulasOf}\, S$ and p_1 is non exal iff $p_1 \in m\text{-}\mathrm{NorFormulasOf}\, S$.

(19) $I\text{-}\mathrm{TruthEval}\langle l \rangle \frown p_1 = \mathit{true}$ iff there exists u such that $((l, u)\, \mathrm{ReassignIn}\, I)\text{-}\mathrm{TruthEval}\, p_1 = 1$ and $I\text{-}\mathrm{TruthEval}\langle \mathrm{TheNorSymbOf}\, S \rangle \frown p_3 \frown p_4 = \mathit{true}$ iff $I\text{-}\mathrm{TruthEval}\, p_3 = \mathit{false}$ and $I\text{-}\mathrm{TruthEval}\, p_4 = \mathit{false}$.

In the sequel I denotes an (S, U)-interpreter-like function.

One can prove the following two propositions:

(20) $(I, u)\text{-}\mathrm{TermEval}(m + 1) \restriction S\text{-}\mathrm{termsOfMaxDepth}(m) = I\text{-}\mathrm{TermEval} \restriction S\text{-}\mathrm{termsOfMaxDepth}(m)$.

(21) $I\text{-}\mathrm{TermEval}(t) = I(S\text{-}\mathrm{firstChar}(t))(I\text{-}\mathrm{TermEval} \cdot \mathrm{SubTerms}\, t)$.

Let us consider S, p_1. The functor $\mathrm{SubWffsOf}\, p_1$ is defined as follows:

(Def. 35)(i) There exist p_3, p such that p is $\mathrm{AllSymbolsOf}\, S$-valued and $\mathrm{SubWffsOf}\, p_1 = \langle\!\langle p_3, p \rangle\!\rangle$ and $p_1 = \langle S\text{-}\mathrm{firstChar}(p_1) \rangle \frown p_3 \frown p$ if p_1 is non 0-w.f.f.,

(ii) $\mathrm{SubWffsOf}\, p_1 = \langle\!\langle p_1, \emptyset \rangle\!\rangle$, otherwise.

Let us consider S, p_1. The functor $\mathrm{head}\, p_1$ yields a w.f.f. string of S and is defined as follows:

(Def. 36) $\mathrm{head}\, p_1 = (\mathrm{SubWffsOf}\, p_1)_{\mathbf{1}}$.

The functor $\mathrm{tail}\, p_1$ yields an element of $(\mathrm{AllSymbolsOf}\, S)^*$ and is defined by:

(Def. 37) $\mathrm{tail}\, p_1 = (\mathrm{SubWffsOf}\, p_1)_{\mathbf{2}}$.

Let us consider S, m. One can verify that $(S\text{-}\mathrm{formulasOfMaxDepth}\, m) \setminus \mathrm{AllFormulasOf}\, S$ is empty.

Let us consider S. Observe that $\mathrm{AtomicFormulasOf}\, S \setminus \mathrm{AllFormulasOf}\, S$ is empty.

We now state two propositions:

(22) $\mathrm{Depth}(\langle l \rangle \frown p_3) > \mathrm{Depth}\, p_3$ and $\mathrm{Depth}(\langle \mathrm{TheNorSymbOf}\, S \rangle \frown p_3 \frown p_4) > \mathrm{Depth}\, p_3$ and $\mathrm{Depth}(\langle \mathrm{TheNorSymbOf}\, S \rangle \frown p_3 \frown p_4) > \mathrm{Depth}\, p_4$.

(23) If p_1 is not 0-w.f.f., then $p_1 = \langle x \rangle \frown p_4 \frown p_2$ iff $x = S$-firstChar(p_1) and $p_4 = \mathrm{head}\, p_1$ and $p_2 = \mathrm{tail}\, p_1$.

Let us consider S, m_2. Observe that there exists a non 0-w.f.f. m_2-w.f.f. string of S which is non exal.

Let us consider S and let p_1 be an exal w.f.f. string of S. One can verify that tail p_1 is empty.

Let us consider S and let p_1 be a non exal non 0-w.f.f. w.f.f. string of S. Then tail p_1 is a w.f.f. string of S.

Let us consider S and let p_1 be a non exal non 0-w.f.f. w.f.f. string of S. One can check that tail p_1 is w.f.f..

Let us consider S and let p_1 be a non 0-w.f.f. non exal w.f.f. string of S. One can verify that S-firstChar$(p_1) \dot{-}$ TheNorSymbOf S is empty.

Let us consider m, S and let p_1 be an $m+1$-w.f.f. string of S. Note that head p_1 is m-w.f.f..

Let us consider m, S and let p_1 be an $m+1$-w.f.f. non exal non 0-w.f.f. string of S. Observe that tail p_1 is m-w.f.f..

One can prove the following proposition

(24) For every element I of U-InterpretersOf S holds (I, m)-TruthEval $\in$ $\mathit{Boolean}^{S\text{-formulasOfMaxDepth}\, m}$.

Let us consider S. One can check that there exists an of-atomic-formula element of S which is non literal.

One can prove the following proposition

(25) If $l_2 \notin \mathrm{rng}\, p$, then $((l_2, u)$ ReassignIn $I)$-TermEval$(p) = I$-TermEval(p).

Let us consider X, S, s. We say that s is X-occurring if and only if:

(Def. 38) $s \in$ SymbolsOf$(((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}) \cap X)$.

Let us consider S, s and let us consider X. We say that X is s-containing if and only if:

(Def. 39) $s \in$ SymbolsOf$((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\} \cap X)$.

Let us consider X, S, s. We introduce s is X-absent as an antonym of s is X-occurring.

Let us consider S, s, X. We introduce X is s-free as an antonym of X is s-containing.

Let X be a finite set and let us consider S. Observe that there exists a literal element of S which is X-absent.

Let us consider S, t. Note that $\mathrm{rng}\, t \cap$ LettersOf S is non empty.

Let us consider S, p_1. One can verify that $\mathrm{rng}\, p_1 \cap$ LettersOf S is non empty.

Let us consider B, S and let A be a subset of B. Note that every element of S which is A-occurring is also B-occurring.

Let us consider A, B, S. Observe that every element of S which is A null B-absent is also $A \cap B$-absent.

Let F be a finite set and let us consider A, S. Note that every F-absent element of S which is A-absent is also $A \cup F$-absent.

Let us consider S, U and let I be an (S, U)-interpreter-like function. One can check that OwnSymbolsOf $S \setminus \operatorname{dom} I$ is empty.

One can prove the following proposition

(26) There exists u such that $u = I(l)(\emptyset)$ and (l, u) ReassignIn $I = I$.

Let us consider S, X. We say that X is S-covering if and only if:

(Def. 40) For every p_1 holds $p_1 \in X$ or xnot $p_1 \in X$.

Let us consider S. One can check that every set which is S-mincover is also S-covering.

Let us consider U, let p_1 be a non 0-w.f.f. non exal w.f.f. string of S, and let I be an element of U-InterpretersOf S.

One can verify that (I-TruthEval p_1)$\dot{-}$((I-TruthEval head p_1) 'nor' (I-TruthEval tail p_1)) is empty.

The functor ExFormulasOf S yielding a subset of (AllSymbolsOf $S)^* \setminus \{\emptyset\}$ is defined by:

(Def. 41) ExFormulasOf $S = \{p_1; p_1$ ranges over strings of S: p_1 is w.f.f. $\wedge$ p_1 is exal$\}$.

Let us consider S. Note that ExFormulasOf S is non empty.

Let us consider S. One can check that every element of ExFormulasOf S is exal and w.f.f..

Let us consider S. Note that every element of ExFormulasOf S is w.f.f..

Let us consider S. Observe that every element of ExFormulasOf S is exal.

Let us consider S. Observe that ExFormulasOf $S \setminus$AllFormulasOf S is empty.

Let us consider U, S_1 and let S_2 be an S_1-extending language. Note that every function which is (S_2, U)-interpreter-like is also (S_1, U)-interpreter-like.

Let us consider U, S_1, let S_2 be an S_1-extending language, and let I be an (S_2, U)-interpreter-like function. Observe that $I \restriction$ OwnSymbolsOf S_1 is (S_1, U)-interpreter-like.

Let us consider U, S_1, let S_2 be an S_1-extending language, let I_1 be an element of U-InterpretersOf S_1, and let I_2 be an (S_2, U)-interpreter-like function. Note that $I_2 {+\cdot} I_1$ is (S_2, U)-interpreter-like.

Let us consider U, S, let I be an element of U-InterpretersOf S, and let us consider X. We say that X is I-satisfied if and only if:

(Def. 42) For every p_1 such that $p_1 \in X$ holds I-TruthEval $p_1 = 1$.

Let us consider S, U, X and let I be an element of U-InterpretersOf S. We say that I satisfies X if and only if:

(Def. 43) X is I-satisfied.

Let us consider U, S, let e be an empty set, and let I be an element of U-InterpretersOf S. Observe that e null I is I-satisfied.

Let us consider X, U, S and let I be an element of U-InterpretersOf S. Observe that there exists a subset of X which is I-satisfied.

Let us consider U, S and let I be an element of U-InterpretersOf S. One can check that there exists a set which is I-satisfied.

Let us consider U, S, let I be an element of U-InterpretersOf S, and let X be an I-satisfied set. One can check that every subset of X is I-satisfied.

Let us consider U, S, let I be an element of U-InterpretersOf S, and let X, Y be I-satisfied sets. One can verify that $X \cup Y$ is I-satisfied.

Let us consider U, S, let I be an element of U-InterpretersOf S, and let X be an I-satisfied set. Observe that I null X satisfies X .

Let us consider S, X. We say that X is S-correct if and only if the condition (Def. 44) is satisfied.

(Def. 44) Let U be a non empty set, I be an element of U-InterpretersOf S, x be an I-satisfied set, and given p_1. If $\langle x, p_1 \rangle \in X$, then I-TruthEval $p_1 = 1$.

Let us consider S. Note that $\emptyset$ null S is S-correct.

Let us consider S, X. Observe that there exists a subset of X which is S-correct.

Next we state two propositions:

(27) For every element I of U-InterpretersOf S holds I-TruthEval $p_1 = 1$ iff $\{p_1\}$ is I-satisfied.

(28) s is $\{w\}$-occurring iff $s \in \operatorname{rng} w$.

Let us consider U, S, let us consider p_3, p_4, and let I be an element of U-InterpretersOf S. Observe that $(I\text{-TruthEval}\langle \text{TheNorSymbOf } S\rangle \frown p_3 \frown p_4) \dot{-} ((I\text{-TruthEval}\, p_3)\ \text{'nor'}\ (I\text{-TruthEval}\, p_4))$ is empty.

Let us consider S, p_1, U and let I be an element of U-InterpretersOf S. Note that $(I\text{-TruthEval xnot}\, p_1) \dot{-} \neg(I\text{-TruthEval}\, p_1)$ is empty.

Let us consider X, S, p_1. We say that p_1 is X-implied if and only if:

(Def. 45) For every non empty set U and for every element I of U-InterpretersOf S such that X is I-satisfied holds I-TruthEval $p_1 = 1$.

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Free Interpretation, Quotient Interpretation and Substitution of a Letter with a Term for First Order Languages[1]

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Summary. Fourth of a series of articles laying down the bases for classical first order model theory. This paper supplies a toolkit of constructions to work with languages and interpretations, and results relating them. The free interpretation of a language, having as a universe the set of terms of the language itself, is defined.

The quotient of an interpreteation with respect to an equivalence relation is built, and shown to remain an interpretation when the relation respects it. Both the concepts of quotient and of respecting relation are defined in broadest terms, with respect to objects as general as possible.

Along with the trivial symbol substitution generally defined in [11], the more complex substitution of a letter with a term is defined, basing right on the free interpretation just introduced, which is a novel approach, to the author's knowledge. A first important result shown is that the quotient operation commute in some sense with term evaluation and reassignment functors, both introduced in [13] (theorem 3, theorem 15). A second result proved is substitution lemma (theorem 10, corresponding to III.8.3 of [15]). This will be vital for proving satisfiability theorem and correctness of a certain sequent derivation rule in [14]. A third result supplied is that if two given languages coincide on the letters of a given FinSequence, their evaluation of it will also coincide. This too will be instrumental in [14] for proving correctness of another rule. Also, the Depth functor is shown to be invariant with respect to term substitution in a formula.

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The notation and terminology used in this paper are introduced in the following articles: [1], [20], [17], [4], [5], [11], [12], [13], [19], [6], [7], [8], [16], [22], [2], [3], [9], [23], [25], [24], [18], [21], and [10].

For simplicity, we adopt the following rules: X, Y, x are sets, U, U_1, U_2 are non empty sets, u, u_1 are elements of U, R is a binary relation, f is a function, m, n are natural numbers, m_1, n_1 are elements of $\mathbb{N}$, S, S_1, S_2 are languages, s is an element of S, l, l_1, l_2 are literal elements of S, a is an of-atomic-formula element of S, r is a relational element of S, w is a string of S, t is a termal string of S, p_0 is a 0-w.f.f. string of S, p_1, p_2 are w.f.f. strings of S, I is an (S, U)-interpreter-like function, and t_1, t_0 are elements of AllTermsOf S.

Let us consider S, s and let V be an element of $((\mathrm{AllSymbolsOf}\,S)^* \setminus \{\emptyset\})^*$. The functor s-compound V yields a string of S and is defined by:

(Def. 1) $s\text{-compound}\,V = \langle s\rangle \frown S\text{-multiCat}(V)$.

Let us consider S, m_1, let s be a termal element of S, and let V be an $|\mathrm{ar}\,s|$-element element of $S\text{-termsOfMaxDepth}(m_1)^*$. One can verify that s-compound V is $m_1 + 1$-termal.

Let us consider S, let s be a termal element of S, and let V be an $|\mathrm{ar}\,s|$-element element of $(\mathrm{AllTermsOf}\,S)^*$. Observe that s-compound V is termal.

Let us consider S, let s be a relational element of S, and let V be an $|\mathrm{ar}\,s|$-element element of $(\mathrm{AllTermsOf}\,S)^*$. One can check that s-compound V is 0-w.f.f..

Let us consider S, s. The functor s-compound yielding a function from $((\mathrm{AllSymbolsOf}\,S)^* \setminus \{\emptyset\})^*$ into $(\mathrm{AllSymbolsOf}\,S)^* \setminus \{\emptyset\}$ is defined by:

(Def. 2) For every element V of $((\mathrm{AllSymbolsOf}\,S)^* \setminus \{\emptyset\})^*$ holds $s\text{-compound}(V) = s\text{-compound}\,V$.

Let us consider S and let s be a termal element of S.

Observe that $s\text{-compound} \restriction (\mathrm{AllTermsOf}\,S)^{|\mathrm{ar}\,s|}$ is AllTermsOf S-valued.

Let us consider S and let s be a relational element of S.

Note that $s\text{-compound} \restriction (\mathrm{AllTermsOf}\,S)^{|\mathrm{ar}\,s|}$ is AtomicFormulasOf S-valued.

Let us consider S, let s be an of-atomic-formula element of S, and let X be a set. The functor X-freeInterpreter s is defined as follows:

(Def. 3) $X\text{-freeInterpreter}\,s = \begin{cases} s\text{-compound} \restriction (\mathrm{AllTermsOf}\,S)^{|\mathrm{ar}\,s|}, & \text{if } s \text{ is not relational,} \\ (s\text{-compound} \restriction (\mathrm{AllTermsOf}\,S)^{|\mathrm{ar}\,s|}) \cdot (\chi_{X,\mathrm{AtomicFormulasOf}\,S}\ \mathbf{qua}\ \text{binary relation}), & \text{otherwise.} \end{cases}$

Let us consider S, let s be an of-atomic-formula element of S, and let X be a set. Then X-freeInterpreter s is an interpreter of s and AllTermsOf S.

Let us consider S, X. The functor (S, X)-freeInterpreter yields a function and is defined as follows:

(Def. 4) dom((S, X)-freeInterpreter) = OwnSymbolsOf S and for every own element s of S holds (S, X)-freeInterpreter(s) = X-freeInterpreter s.

Let us consider S, X. Note that (S, X)-freeInterpreter is function yielding.

Let us consider S, X. Then (S, X)-freeInterpreter is an interpreter of S and AllTermsOf S.

Let us consider S, X. Note that (S, X)-freeInterpreter is $(S, \text{AllTermsOf}\, S)$-interpreter-like.

Then (S, X)-freeInterpreter is an element of AllTermsOf S-InterpretersOf S.

Let X, Y be non empty sets, let R be a relation between X and Y, and let n be a natural number. The functor n-placesOf R yielding a relation between X^n and Y^n is defined as follows:

(Def. 5) n-placesOf $R = \{\langle p, q\rangle; p$ ranges over elements of X^n, q ranges over elements of Y^n: $\bigwedge_{j\,:\,\text{set}} (j \in \text{Seg}\, n \;\Rightarrow\; \langle p(j), q(j)\rangle \in R)\}$.

Let X, Y be non empty sets, let R be a total relation between X and Y, and let n be a non zero natural number. Observe that n-placesOf R is total.

Let X, Y be non empty sets, let R be a total relation between X and Y, and let n be a natural number. Observe that n-placesOf R is total.

Let X, Y be non empty sets, let R be a relation between X and Y, and let n be a zero natural number. One can check that n-placesOf R is function-like.

Let X be a non empty set, let R be a binary relation on X, and let n be a natural number. The functor n-placesOf R yielding a binary relation on X^n is defined by:

(Def. 6) n-placesOf R = n-placesOf(R **qua** relation between X and X).

Let X be a non empty set, let R be a binary relation on X, and let n be a zero natural number. Then n-placesOf R is a binary relation on X^n and it can be characterized by the condition:

(Def. 7) n-placesOf $R = \{\langle \emptyset, \emptyset\rangle\}$.

Let X be a non empty set, let R be a symmetric total binary relation on X, and let us consider n. One can check that n-placesOf R is total.

Let X be a non empty set, let R be a symmetric total binary relation on X, and let us consider n. Observe that n-placesOf R is symmetric.

Let X be a non empty set, let R be a symmetric total binary relation on X, and let us consider n. Observe that n-placesOf R is symmetric and total.

Let X be a non empty set, let R be a transitive total binary relation on X, and let us consider n. Observe that n-placesOf R is transitive and total.

Let X be a non empty set, let R be an equivalence relation of X, and let us consider n. Observe that n-placesOf R is total, symmetric, and transitive.

Let X, Y be non empty sets, let E be an equivalence relation of X, let F be an equivalence relation of Y, and let R be a binary relation. The functor R quotient(E, F) is defined by:

(Def. 8) $R\,\mathrm{quotient}(E,F) = \{\langle e,\ f\rangle; e$ ranges over elements of Classes E, f ranges over elements of Classes F : $\bigvee_{x,y\,:\,\mathrm{set}}\ (x \in e\ \wedge\ y \in f\ \wedge\ \langle x,\ y\rangle \in R)\}$.

Let X, Y be non empty sets, let E be an equivalence relation of X, let F be an equivalence relation of Y, and let R be a binary relation. Then $R\,\mathrm{quotient}(E,F)$ is a relation between Classes E and Classes F.

Let E be a binary relation, let F be a binary relation, and let f be a function. We say that f is (E,F)-respecting if and only if:

(Def. 9) For all sets x_1, x_2 such that $\langle x_1,\ x_2\rangle \in E$ holds $\langle f(x_1),\ f(x_2)\rangle \in F$.

Let us consider S, U, let s be an of-atomic-formula element of S, let E be a binary relation on U, and let f be an interpreter of s and U. We say that f is E-respecting if and only if:

(Def. 10)(i) f is $(|\mathrm{ar}\,s|\text{-placesOf}\,E, E)$-respecting if s is not relational,

(ii) f is $(|\mathrm{ar}\,s|\text{-placesOf}\,E, \mathrm{id}_{Boolean})$-respecting, otherwise.

Let X, Y be non empty sets, let E be an equivalence relation of X, and let F be an equivalence relation of Y. Observe that there exists a function from X into Y which is (E,F)-respecting.

Let us consider S, U, let s be an of-atomic-formula element of S, and let E be an equivalence relation of U. Note that there exists an interpreter of s and U which is E-respecting.

Let X, Y be non empty sets, let E be an equivalence relation of X, and let F be an equivalence relation of Y. One can verify that there exists a function which is (E,F)-respecting.

Let X be a non empty set, let E be an equivalence relation of X, and let us consider n. Then n-placesOf E is an equivalence relation of X^n.

Let X be a non empty set and let x be an element of SmallestPartition(X). The functor DeTrivial x yielding an element of X is defined as follows:

(Def. 11) $x = \{\mathrm{DeTrivial}\,x\}$.

Let X be a non empty set. The functor peeler X yielding a function from $\{\{*\} : * \in X\}$ into X is defined as follows:

(Def. 12) For every element x of $\{\{*\} : * \in X\}$ holds $(\mathrm{peeler}\,X)(x) = \mathrm{DeTrivial}\,x$.

Let X be a non empty set and let E_1 be an equivalence relation of X. Note that every element of Classes E_1 is non empty.

Let X, Y be non empty sets, let E be an equivalence relation of X, let F be an equivalence relation of Y, and let f be an (E,F)-respecting function. One can check that $f\,\mathrm{quotient}(E,F)$ is function-like.

Let X, Y be non empty sets, let E be an equivalence relation of X, let F be an equivalence relation of Y, and let R be a total relation between X and Y. One can check that $R\,\mathrm{quotient}(E,F)$ is total.

Let X, Y be non empty sets, let E be an equivalence relation of X, let F be an equivalence relation of Y, and let f be an (E,F)-respecting function from X

into Y. Then f quotient(E, F) is a function from Classes E into Classes F.

Let X be a non empty set and let E be an equivalence relation of X. The functor E-class yields a function from X into Classes E and is defined by:

(Def. 13) For every element x of X holds E-class$(x) = \mathrm{EqClass}(E, x)$.

Let X be a non empty set and let E be an equivalence relation of X. Observe that E-class is onto.

Let X, Y be non empty sets. Note that there exists a relation between X and Y which is onto.

Let Y be a non empty set. Observe that there exists a Y-valued binary relation which is onto.

Let Y be a non empty set and let R be a Y-valued binary relation. Note that $R^{\smile}$ is Y-defined.

Let Y be a non empty set and let R be an onto Y-valued binary relation. Note that $R^{\smile}$ is total.

Let X, Y be non empty sets and let R be an onto relation between X and Y. One can check that $R^{\smile}$ is total.

Let Y be a non empty set and let R be an onto Y-valued binary relation. Note that $R^{\smile}$ is total.

Let us consider U, n and let E be an equivalence relation of U. The functor n-tuple2Class E yields a relation between $(\mathrm{Classes}\,E)^n$ and Classes(n-placesOf E) and is defined as follows:

(Def. 14) n-tuple2Class E = (n-placesOf(E-class **qua** relation between U and Classes E)$^{\smile}$) $\cdot$ (n-placesOf E)-class .

Let us consider U, n and let E be an equivalence relation of U. Observe that n-tuple2Class E is function-like.

Let us consider U, n and let E be an equivalence relation of U. Note that n-tuple2Class E is total.

Let us consider U, n and let E be an equivalence relation of U. Then n-tuple2Class E is a function from $(\mathrm{Classes}\,E)^n$ into Classes(n-placesOf E).

Let us consider S, U, let s be an of-atomic-formula element of S, let E be an equivalence relation of U, and let f be an interpreter of s and U. The functor f quotient E is defined by:

(Def. 15) $$f\,\mathrm{quotient}\,E = \begin{cases} (|\mathrm{ar}\,s|\text{-tuple2Class}\,E)\cdot \\ \quad (f\,\mathrm{quotient}(|\mathrm{ar}\,s|\text{-placesOf}\,E, E)), \\ \quad \text{if } s \text{ is not relational}, \\ (|\mathrm{ar}\,s|\text{-tuple2Class}\,E)\cdot \\ \quad (f\,\mathrm{quotient}(|\mathrm{ar}\,s|\text{-placesOf}\,E, \mathrm{id}_{Boolean}))\cdot \\ \quad \text{peeler}\,Boolean, \text{ otherwise.} \end{cases}$$

Let us consider S, U, let s be an of-atomic-formula element of S, let E be an equivalence relation of U, and let f be an E-respecting interpreter of s and U. Then f quotient E is an interpreter of s and Classes E.

The following proposition is true

(1) Let X be a non empty set, E be an equivalence relation of X, and C_1, C_2 be elements of Classes E. If C_1 meets C_2, then $C_1 = C_2$.

Let us consider S. Observe that every element of OwnSymbolsOf S is own and every element of OwnSymbolsOf S is of-atomic-formula.

Let us consider S, U, let o be a non relational of-atomic-formula element of S, and let E be a binary relation on U. One can check that every interpreter of o and U which is E-respecting is also ($|\mathrm{ar}\, o|$-placesOf E, E)-respecting.

Let us consider S, U, let r be a relational element of S, and let E be a binary relation on U. Observe that every interpreter of r and U which is E-respecting is also ($|\mathrm{ar}\, r|$-placesOf $E, \mathrm{id}_{Boolean}$)-respecting.

Let us consider n, let U_1, U_2 be non empty sets, and let f be a function-like relation between U_1 and U_2. Note that n-placesOf f is function-like.

Let us consider U_1, U_2, let n be a zero natural number, and let R be a relation between U_1 and U_2. Note that (n-placesOf R)$\dot{-}\mathrm{id}_{\{\emptyset\}}$ is empty.

Let us consider X and let Y be a functional set. Observe that $X \cap Y$ is functional.

We now state the proposition

(2) For every element V of $(\text{AllTermsOf}\, S)^*$ there exists an element m_1 of $\mathbb{N}$ such that V is an element of S-termsOfMaxDepth$(m_1)^*$.

Let us consider S, U, let E be an equivalence relation of U, and let I be an (S, U)-interpreter-like function. We say that I is E-respecting if and only if:

(Def. 16) For every own element s of S holds $I(s)$ **qua** interpreter of s and U is E-respecting.

Let us consider S, U, let E be an equivalence relation of U, and let I be an (S, U)-interpreter-like function. The functor I quotient E yielding a function is defined as follows:

(Def. 17) $\mathrm{dom}(I \text{ quotient } E) = $ OwnSymbolsOf S and for every element o of OwnSymbolsOf S holds $(I \text{ quotient } E)(o) = I(o) \text{ quotient } E$.

Let us consider S, U, let E be an equivalence relation of U, and let I be an (S, U)-interpreter-like function. Then I quotient E can be characterized by the condition:

(Def. 18) $\mathrm{dom}(I \text{ quotient } E) = $ OwnSymbolsOf S and for every own element o of S holds $(I \text{ quotient } E)(o) = I(o) \text{ quotient } E$.

Let us consider S, U, let I be an (S, U)-interpreter-like function, and let E be an equivalence relation of U. Note that I quotient E is OwnSymbolsOf S-defined.

Let us consider S, U and let E be an equivalence relation of U. Note that there exists an element of U-InterpretersOf S which is E-respecting.

Let us consider S, U and let E be an equivalence relation of U. Observe that there exists an (S,U)-interpreter-like function which is E-respecting.

Let us consider S, U, let E be an equivalence relation of U, let o be an own element of S, and let I be an E-respecting (S,U)-interpreter-like function. One can check that $I(o)$ is E-respecting.

Let us consider S, U, let E be an equivalence relation of U, and let I be an E-respecting (S,U)-interpreter-like function. Observe that $I\,\mathrm{quotient}\,E$ is $(S, \mathrm{Classes}\,E)$-interpreter-like.

Let us consider S, U, let E be an equivalence relation of U, and let I be an E-respecting (S,U)-interpreter-like function. Then $I\,\mathrm{quotient}\,E$ is an element of $\mathrm{Classes}\,E$-InterpretersOf S.

The following propositions are true:

(3) Let E be an equivalence relation of U and I be an E-respecting (S,U)-interpreter-like function.
Then $(I\,\mathrm{quotient}\,E)$-TermEval $= E$-class $\cdot I$-TermEval .

(4) (S,X)-freeInterpreter-TermEval $= \mathrm{id}_{\mathrm{AllTermsOf}\,S}$.

(5) Let R be an equivalence relation of U_1, p_2 be a 0-w.f.f. string of S, and i be an R-respecting (S,U_1)-interpreter-like function. If S-firstChar$(p_2) \neq$ TheEqSymbOf S, then $(i\,\mathrm{quotient}\,R)$-AtomicEval $p_2 = i$-AtomicEval p_2.

Let us consider S, x, s, w. Then (x,s)-SymbolSubstIn w is a string of S.

Let us consider S, l_1, l_2, m and let t be an m-termal string of S. Note that (l_1,l_2)-SymbolSubstIn t is m-termal.

Let us consider S, t, l_1, l_2. One can check that (l_1,l_2)-SymbolSubstIn t is termal.

Let us consider S, l_1, l_2 and let p_2 be a 0-w.f.f. string of S. One can check that (l_1,l_2)-SymbolSubstIn p_2 is 0-w.f.f..

Let us consider S, let m_0 be a zero number, and let p_2 be an m_0-w.f.f. string of S. One can verify that Depth p_2 is zero.

Let us consider S, m, w. Then $w\,\mathrm{null}\,m$ is a string of S.

Let us consider S, p_2, m. Note that $p_2\,\mathrm{null}\,m$ is Depth $p_2 + m$-w.f.f..

Let us consider S, m and let p_2 be an m-w.f.f. string of S. Note that $m -$ Depth p_2 is non negative.

Let us consider S, l_1, l_2, m and let p_2 be an m-w.f.f. string of S. Observe that (l_1,l_2)-SymbolSubstIn p_2 is m-w.f.f..

Let us consider S, l_1, l_2, p_2. One can verify that (l_1,l_2)-SymbolSubstIn p_2 is w.f.f.. Observe that Depth$((l_1,l_2)$-SymbolSubstIn $p_2) \dot{-}$ Depth p_2 is empty.

The following proposition is true

(6) Let T be an $|\mathrm{ar}\,a|$-element element of $(\mathrm{AllTermsOf}\,S)^*$. Then
(i) if a is not relational, then $(X\text{-freeInterpreter}\,a)(T) = a$-compound T, and
(ii) if a is relational, then $(X\text{-freeInterpreter}\,a)(T) =$

$\chi_{X,\mathrm{AtomicFormulasOf}\,S}(a\text{-compound}\,T)$.

Let S be a language. One can verify that there exists a string of S which is termal and there exists a string of S which is 0-w.f.f..

One can prove the following proposition

(7) $(I\text{-TermEval}\cdot((l,t_0)\,\mathrm{ReassignIn}(S,X)\text{-freeInterpreter},t_0)\text{-TermEval}(n))\restriction S\text{-termsOfMaxDepth}(n) = ((l,I\text{-TermEval}(t_0))\,\mathrm{ReassignIn}\,I,I\text{-TermEval}(t_0))\text{-TermEval}(n)\restriction S\text{-termsOfMaxDepth}(n)$.

Let us consider S, l, t_1, p_0. The functor $(l,t_1)\,\mathrm{AtomicSubst}\,p_0$ yielding a finite sequence is defined by:

(Def. 19) $(l,t_1)\,\mathrm{AtomicSubst}\,p_0 = \langle S\text{-firstChar}(p_0)\rangle ^\frown S\text{-multiCat}(((l,t_1)\,\mathrm{ReassignIn}\,(S,\emptyset)\text{-freeInterpreter})\text{-TermEval}\cdot \mathrm{SubTerms}\,p_0)$.

Let us consider S, l, t_1, p_0. Then $(l,t_1)\,\mathrm{AtomicSubst}\,p_0$ is a string of S.

Let us consider S, l, t_1, p_0. Observe that $(l,t_1)\,\mathrm{AtomicSubst}\,p_0$ is 0-w.f.f..

We now state the proposition

(8) $I\text{-AtomicEval}((l,t_1)\,\mathrm{AtomicSubst}\,p_0) = ((l,I\text{-TermEval}(t_1))\,\mathrm{ReassignIn}\,I)\text{-AtomicEval}\,p_0$.

Let us consider S, l_1, l_2, m. One can check that $(l_1\,\mathrm{SubstWith}\,l_2)\restriction S\text{-termsOfMaxDepth}(m)$ is S-termsOfMaxDepth(m)-valued.

Note that $(l_1\,\mathrm{SubstWith}\,l_2)\restriction \mathrm{AllTermsOf}\,S$ is AllTermsOf S-valued.

One can prove the following proposition

(9) If $l_2 \notin \mathrm{rng}\,p_1$, then for every element I of U-InterpretersOf S holds $((l_1,u_1)\,\mathrm{ReassignIn}\,I)\text{-TruthEval}\,p_1 = ((l_2,u_1)\,\mathrm{ReassignIn}\,I)\text{-TruthEval}((l_1,l_2)\text{-SymbolSubstIn}\,p_1)$.

Let us consider S, let us consider l, t, n, let f be a finite sequence-yielding function, and let us consider p_2. The functor $(l,t,n,f)\,\mathrm{Subst2}\,p_2$ yielding a finite sequence is defined by:

(Def. 20) $$(l,t,n,f)\,\mathrm{Subst2}\,p_2 = \begin{cases} \langle \mathrm{TheNorSymbOf}\,S\rangle ^\frown f(\mathrm{head}\,p_2) ^\frown f(\mathrm{tail}\,p_2), \\ \quad \text{if } \mathrm{Depth}\,p_2 = n+1 \text{ and } p_2 \text{ is not exal}, \\ \langle \text{the element of LettersOf } S \setminus (\mathrm{rng}\,t \cup \mathrm{rng} \\ \quad \mathrm{head}\,p_2 \cup \{l\})\rangle ^\frown f((S\text{-firstChar}(p_2), \\ \quad \text{the element of LettersOf } S \setminus (\mathrm{rng}\,t \cup \mathrm{rng} \\ \quad \mathrm{head}\,p_2 \cup \{l\}))\text{-SymbolSubstIn}\,\mathrm{head}\,p_2), \\ \quad \text{if } \mathrm{Depth}\,p_2 = n+1 \text{ and } p_2 \text{ is exal and} \\ \quad S\text{-firstChar}(p_2) \neq l, \\ f(p_2), \text{ otherwise.} \end{cases}$$

Let us consider S. One can verify that every element of $(\mathrm{AllFormulasOf}\,S)^{\mathrm{AllFormulasOf}\,S}$ is finite sequence-yielding.

Let us consider l, t, n, let f be an element of $(\mathrm{AllFormulasOf}\,S)^{\mathrm{AllFormulasOf}\,S}$, and let us consider p_2. Then $(l,t,n,f)\,\mathrm{Subst2}\,p_2$ is a w.f.f. string of S. Let f be an element of $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$, and let us consider p_2. Observe that $(l, t, n, f)\,\text{Subst2}\,p_2$ is w.f.f..

Let us consider n_1, let f be an element of $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$, and let us consider p_2. Then $(l, t, n_1, f)\,\text{Subst2}\,p_2$ is an element of AllFormulasOf S.

Let us consider S, l, t, n and let f be an element of $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$. The functor $(l, t, n, f)\,\text{Subst3}$ yields an element of $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$ and is defined as follows:

(Def. 21) For every p_2 holds $(l, t, n, f)\,\text{Subst3}(p_2) = (l, t, n, f)\,\text{Subst2}\,p_2$.

Let us consider S, l, t and let f be an element of $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$. The functor $(l, t)\,\text{Subst4}\,f$ yields a function from $\mathbb{N}$ into $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$ and is defined by:

(Def. 22) $((l, t)\,\text{Subst4}\,f)(0) = f$ and for every m holds $((l, t)\,\text{Subst4}\,f)(m + 1) = (l, t, m, ((l, t)\,\text{Subst4}\,f)(m))\,\text{Subst3}$.

Let us consider S, l, t. The functor $l\,\text{AtomicSubst}\,t$ yields a function from AtomicFormulasOf S into AtomicFormulasOf S and is defined by:

(Def. 23) For all p_0, t_1 such that $t_1 = t$ holds $(l\,\text{AtomicSubst}\,t)(p_0) = (l, t_1)\,\text{AtomicSubst}\,p_0$.

Let us consider S, l, t. The functor $l\,\text{Subst1}\,t$ yielding a function is defined as follows:

(Def. 24) $l\,\text{Subst1}\,t = \text{id}_{\text{AllFormulasOf } S} {+\cdot} (l\,\text{AtomicSubst}\,t)$.

Let us consider S, l, t. Then $l\,\text{Subst1}\,t$ is an element of $((\text{AllSymbolsOf } S)^*)^{\text{AllFormulasOf } S}$. Then $l\,\text{Subst1}\,t$ is an element of $(\text{AllFormulasOf } S)^{\text{AllFormulasOf } S}$.

Let us consider S, l, t, p_2. The functor $(l, t)\,\text{SubstIn}\,p_2$ yielding a w.f.f. string of S is defined as follows:

(Def. 25) $(l, t)\,\text{SubstIn}\,p_2 = ((l, t)\,\text{Subst4}(l\,\text{Subst1}\,t))(\text{Depth}\,p_2)(p_2)$.

Let us consider S, l, t, p_2. Note that $(l, t)\,\text{SubstIn}\,p_2$ is w.f.f..

One can prove the following proposition

(10) $\text{Depth}((l, t_1)\,\text{SubstIn}\,p_1) = \text{Depth}\,p_1$ and for every element I of U-InterpretersOf S holds $I\text{-TruthEval}((l, t_1)\,\text{SubstIn}\,p_1) = ((l, I\text{-TermEval}(t_1))\,\text{ReassignIn}\,I)\text{-TruthEval}\,p_1$.

Let us consider m, S, l, t and let p_2 be an m-w.f.f. string of S. Observe that $(l, t)\,\text{SubstIn}\,p_2$ is m-w.f.f..

The following propositions are true:

(11) Let I_1 be an element of U-InterpretersOf S_1 and I_2 be an element of U-InterpretersOf S_2. Suppose $I_1{\restriction}X = I_2{\restriction}X$ and (the adicity of $S_1){\restriction}X =$ (the adicity of $S_2){\restriction}X$. Then $I_1\text{-TermEval}{\restriction}X^* = I_2\text{-TermEval}{\restriction}X^*$.

(12) Suppose TheNorSymbOf S_1 = TheNorSymbOf S_2 and TheEqSymbOf S_1 = TheEqSymbOf S_2 and (the adicity of S_1)$\restriction$OwnSymbolsOf S_1 = (the adicity of S_2)$\restriction$OwnSymbolsOf S_1. Let I_1 be an element of U-InterpretersOf S_1, I_2 be an element of U-InterpretersOf S_2, and p_4 be a w.f.f. string of S_1. Suppose $I_1\restriction$OwnSymbolsOf $S_1 = I_2\restriction$OwnSymbolsOf S_1. Then there exists a w.f.f. string p_3 of S_2 such that $p_3 = p_4$ and I_2-TruthEval p_3 = I_1-TruthEval p_4.

(13) For all elements I_1, I_2 of U-InterpretersOf S such that $I_1\restriction$(rng p_2 $\cap$ OwnSymbolsOf S) = $I_2\restriction$(rng $p_2\cap$OwnSymbolsOf S) holds I_1-TruthEval p_2 = I_2-TruthEval p_2.

(14) For every element I of U-InterpretersOf S such that l is X-absent and X is I-satisfied holds X is (l, u) ReassignIn I-satisfied.

(15) For every equivalence relation E of U and for every E-respecting element i of U-InterpretersOf S holds $(l, E\text{-class}(u))$ ReassignIn(i quotient E) = $((l, u)$ ReassignIn $i)$ quotient E.

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Sequent Calculus, Derivability, Provability. Gödel's Completeness Theorem[1]

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Summary. Fifth of a series of articles laying down the bases for classical first order model theory. This paper presents multiple themes: first it introduces sequents, rules and sets of rules for a first order language L as L-dependent types. Then defines derivability and provability according to a set of rules, and gives several technical lemmas binding all those concepts. Following that, it introduces a fixed set D of derivation rules, and proceeds to convert them to Mizar functorial cluster registrations to give the user a slick interface to apply them.

The remaining goals summon all the definitions and results introduced in this series of articles. First: D is shown to be correct and having the requisites to deliver a sensible definition of Henkin model (see [18]). Second: as a particular application of all the machinery built thus far, the satisfiability and Gödel completeness theorems are shown when restricting to countable languages. The techniques used to attain this are inspired from [18], then heavily modified with the twofold goal of embedding them into the more flexible framework of a variable ruleset here introduced, and of proving completeness of a set of rules more sparing than the one there used; in particular the simpler ruleset allowed to avoid the definition and tractation of free occurence of a literal, a fact which, along with shortening proofs, is remarkable in its own right. A preparatory account of some of the ideas used in the proofs given here can be found in [15].

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The notation and terminology used here have been introduced in the following papers: [1], [3], [23], [22], [4], [6], [17], [11], [12], [13], [14], [7], [8], [5], [19], [16], [24], [2], [21], [9], [26], [28], [27], [20], [25], and [10].

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1. Formalization of the Notion of Derivability and Provability. Henkin's Theorem for Arbitrary Languages

For simplicity, we adopt the following convention: k, m, n denote natural numbers, m_1 denotes an element of $\mathbb{N}$, U denotes a non empty set, A, B, X, Y, Z, x, y, z denote sets, S denotes a language, s denotes an element of S, f, g denote functions, p_1, p_2, p_3, p_4 denote w.f.f. strings of S, P_1, P_2, P_3 denote subsets of AllFormulasOf S, t, t_1, t_2 denote termal strings of S, a denotes an of-atomic-formula element of S, l, l_1, l_2 denote literal elements of S, p denotes a finite sequence, and m_2 denotes a non zero natural number.

Let S be a language. The functor S-sequents is defined as follows:

(Def. 1) S-sequents $= \{\langle p_5, c_1\rangle; p_5$ ranges over subsets of AllFormulasOf S, c_1 ranges over w.f.f. strings of S: p_5 is finite$\}$.

Let S be a language. Note that S-sequents is non empty.

Let us consider S. Observe that S-sequents is relation-like.

Let S be a language and let x be a set. We say that x is S-sequent-like if and only if:

(Def. 2) $x \in S$-sequents.

Let us consider S, X. We say that X is S-sequents-like if and only if:

(Def. 3) $X \subseteq S$-sequents.

Let us consider S. One can check that every subset of S-sequents is S-sequents-like and every element of S-sequents is S-sequent-like.

Let S be a language. One can verify that there exists an element of S-sequents which is S-sequent-like and there exists a subset of S-sequents which is S-sequents-like.

Let us consider S. One can check that there exists a set which is S-sequent-like and there exists a set which is S-sequents-like.

Let S be a language. A rule of S is an element of $(2^{S\text{-sequents}})^{2^{S\text{-sequents}}}$.

Let S be a language. A rule set of S is a subset of $(2^{S\text{-sequents}})^{2^{S\text{-sequents}}}$.

For simplicity, we adopt the following rules: D, D_1, D_2, D_3 denote rule sets of S, R denotes a rule of S, S_1, S_2, S_3 denote subsets of S-sequents, s_1, s_2, s_3 denote elements of S-sequents, S_4, S_5 denote S-sequents-like sets, and S_6, S_7 denote S-sequent-like sets.

Let us consider A, B and let X be a subset of B^A. One can check that $\bigcup X$ is relation-like.

Let S be a language and let D be a rule set of S. One can check that $\bigcup D$ is relation-like.

Let us consider S, D. The functor OneStep D yielding a rule of S is defined as follows:

(Def. 4) For every element S_8 of $2^{S\text{-sequents}}$ holds $(\text{OneStep}\,D)(S_8) = \bigcup((\bigcup D)^\circ\{S_8\})$.

Let us consider S, D, m. The functor (m, D)-derivables yields a rule of S and is defined by:

(Def. 5) (m, D)-derivables $= (\text{OneStep}\, D)^m$.

Let S be a language, let D be a rule set of S, and let S_9, S_{10} be sets. We say that S_{10} is (S_9, D)-derivable if and only if:

(Def. 6) $S_{10} \subseteq \bigcup(((\text{OneStep}\, D)^*)^\circ\{S_9\})$.

Let us consider m, S, D and let S_1, s_1 be sets. We say that s_1 is (m, S_1, D)-derivable if and only if:

(Def. 7) $s_1 \in (m, D)$-derivables(S_1).

Let us consider S, D. The functor D-iterators yielding a family of subsets of $2^{S\text{-sequents}} \times 2^{S\text{-sequents}}$ is defined as follows:

(Def. 8) D-iterators $= \{(\text{OneStep}\, D)^{m_1}\}$.

Let us consider S, R. We say that R is isotone if and only if:

(Def. 9) If $S_2 \subseteq S_3$, then $R(S_2) \subseteq R(S_3)$.

Let us consider S. Observe that there exists a rule of S which is isotone.

Let us consider S, D. We say that D is isotone if and only if:

(Def. 10) For all S_2, S_3, f such that $S_2 \subseteq S_3$ and $f \in D$ there exists g such that $g \in D$ and $f(S_2) \subseteq g(S_3)$.

Let us consider S and let M be an isotone rule of S. One can verify that $\{M\}$ is isotone.

Let us consider S. One can verify that there exists a rule set of S which is isotone.

In the sequel K, K_1 are isotone rule sets of S.

Let S be a language, let D be a rule set of S, and let S_1 be a set. We say that S_1 is D-derivable if and only if:

(Def. 11) S_1 is $(\emptyset, D)$-derivable.

Let us consider S, D. One can verify that every set which is D-derivable is also $(\emptyset, D)$-derivable and every set which is $(\emptyset, D)$-derivable is also D-derivable.

Let us consider S, D and let S_1 be an empty set. One can verify that every set which is (S_1, D)-derivable is also D-derivable.

Let us consider S, D, X and let p_2 be a set. We say that p_2 is (X, D)-provable if and only if:

(Def. 12) $\{\langle X, p_2\rangle\}$ is D-derivable or there exists a set s_1 such that $(s_1)_{\mathbf{1}} \subseteq X$ and $(s_1)_{\mathbf{2}} = p_2$ and $\{s_1\}$ is D-derivable.

Let us consider S, D, X, x. Let us observe that x is (X, D)-provable if and only if:

(Def. 13) There exists a set s_1 such that $(s_1)_{\mathbf{1}} \subseteq X$ and $(s_1)_{\mathbf{2}} = x$ and $\{s_1\}$ is D-derivable.

Let us consider S, D, R. We say that R is D-macro if and only if:

(Def. 14) For every subset S_8 of S-sequents holds $R(S_8)$ is (S_8, D)-derivable.

Let us consider S, D and let P_1 be a set. The functor (P_1, D)-termEq is defined as follows:

(Def. 15) (P_1, D)-termEq $= \{\langle t_1, t_2\rangle; t_1$ ranges over termal strings of S, t_2 ranges over termal strings of S: $\langle$TheEqSymbOf $S\rangle \frown t_1 \frown t_2$ is (P_1, D)-provable$\}$.

Let us consider S, D and let P_1 be a set. We say that P_1 is D-expanded if and only if:

(Def. 16) If x is (P_1, D)-provable, then $\{x\} \subseteq P_1$.

Let us consider S, x. We say that x is S-null if and only if:

(Def. 17) Not contradiction.

Let us consider S, D and let P_1 be a set. Then (P_1, D)-termEq is a binary relation on AllTermsOf S.

Let us consider S, p_2 and let P_2, P_3 be finite subsets of AllFormulasOf S. One can check that $\langle P_2 \cup P_3, p_2\rangle$ is S-sequent-like.

Let us consider S, let x be an empty set, and let p_2 be a w.f.f. string of S. Then $\langle x, p_2\rangle$ is an element of S-sequents.

Let us consider S. Note that $\emptyset \cap S$ is S-sequents-like.

Let us consider S. One can verify that there exists a set which is S-null.

Let us consider S. One can check that every set which is S-sequent-like is also S-null.

Let us consider S. One can check that every element of S-sequents is S-null.

Let us consider m, S, D, X. One can verify that (m, D)-derivables(X) is S-sequents-like.

Let us consider S, Y and let X be an S-sequents-like set. One can verify that $X \cap Y$ is S-sequents-like.

Let us consider S, D, m, X. Note that every set which is (m, X, D)-derivable is also S-sequent-like.

Let us consider S, D and let P_2, P_3 be sets. Observe that every set which is $(P_2 \setminus P_3, D)$-provable is also (P_2, D)-provable.

Let us consider S, D and let P_2, P_3 be sets. Observe that every set which is $(P_2 \setminus P_3, D)$-provable is also $(P_2 \cup P_3, D)$-provable.

Let us consider S, D and let P_2, P_3 be sets. Observe that every set which is $(P_2 \cap P_3, D)$-provable is also (P_2, D)-provable.

Let us consider S, D, let X be a set, and let x be a subset of X. Note that every set which is (x, D)-provable is also (X, D)-provable.

Let us consider S, let p_5 be a finite subset of AllFormulasOf S, and let p_2 be a w.f.f. string of S. One can check that $\langle p_5, p_2\rangle$ is S-sequent-like.

Let us consider S and let p_3, p_4 be w.f.f. strings of S. Observe that $\langle\{p_3\}, p_4\rangle$ is S-sequent-like. Let p_6 be a w.f.f. string of S. Note that $\langle\{p_3, p_4\}, p_6\rangle$ is S-sequent-like.

Let us consider S, p_3, p_4 and let P_1 be a finite subset of AllFormulasOf S. One can verify that $\langle P_1 \cup \{p_3\}, p_4\rangle$ is S-sequent-like.

Let us consider S, D. Note that there exists a subset of AllFormulasOf S which is D-expanded.

Let us consider S, D. Observe that there exists a set which is D-expanded.

Let S_1 be a set, let S be a language, and let s_1 be an S-null set. We say that s_1 rule 0 S_1 if and only if:

(Def. 18) $(s_1)_{\mathbf{2}} \in (s_1)_{\mathbf{1}}$.

We say that s_1 rule 1 S_1 if and only if:

(Def. 19) There exists a set y such that $y \in S_1$ and $y_{\mathbf{1}} \subseteq (s_1)_{\mathbf{1}}$ and $(s_1)_{\mathbf{2}} = y_{\mathbf{2}}$.

We say that s_1 rule 2 S_1 if and only if:

(Def. 20) $(s_1)_{\mathbf{1}}$ is empty and there exists a termal string t of S such that $(s_1)_{\mathbf{2}} = \langle \text{TheEqSymbOf}\, S\rangle \frown t \frown t$.

We say that s_1 rule 3a S_1 if and only if the condition (Def. 21) is satisfied.

(Def. 21) There exist termal strings t, t_1, t_2 of S and there exists a set x such that $x \in S_1$ and $(s_1)_{\mathbf{1}} = x_{\mathbf{1}} \cup \{\langle \text{TheEqSymbOf}\, S\rangle \frown t_1 \frown t_2\}$ and $x_{\mathbf{2}} = \langle \text{TheEqSymbOf}\, S\rangle \frown t \frown t_1$ and $(s_1)_{\mathbf{2}} = \langle \text{TheEqSymbOf}\, S\rangle \frown t \frown t_2$.

We say that s_1 rule 3b S_1 if and only if:

(Def. 22) There exist termal strings t_1, t_2 of S such that $(s_1)_{\mathbf{1}} = \{\langle \text{TheEqSymbOf}\, S\rangle \frown t_1 \frown t_2\}$ and $(s_1)_{\mathbf{2}} = \langle \text{TheEqSymbOf}\, S\rangle \frown t_2 \frown t_1$.

We say that s_1 rule 3d S_1 if and only if the condition (Def. 23) is satisfied.

(Def. 23) There exists a low-compounding element s of S and there exist $|\text{ar}\, s|$-element elements T, U of $(\text{AllTermsOf}\, S)^*$ such that

(i) s is operational,

(ii) $(s_1)_{\mathbf{1}} = \{\langle \text{TheEqSymbOf}\, S\rangle \frown T_1(j) \frown U_1(j); j$ ranges over elements of $\text{Seg}|\text{ar}\, s|, T_1$ ranges over functions from $\text{Seg}|\text{ar}\, s|$ into $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}, U_1$ ranges over functions from $\text{Seg}|\text{ar}\, s|$ into $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\} : T_1 = T \ \wedge\ U_1 = U\}$, and

(iii) $(s_1)_{\mathbf{2}} = \langle \text{TheEqSymbOf}\, S\rangle \frown (s\text{-compound}\, T) \frown (s\text{-compound}\, U)$.

We say that s_1 rule 3e S_1 if and only if the condition (Def. 24) is satisfied.

(Def. 24) There exists a relational element s of S and there exist $|\text{ar}\, s|$-element elements T, U of $(\text{AllTermsOf}\, S)^*$ such that

(i) $(s_1)_{\mathbf{1}} = \{s\text{-compound}\, T\} \cup \{\langle \text{TheEqSymbOf}\, S\rangle \frown T_1(j) \frown U_1(j); j$ ranges over elements of $\text{Seg}|\text{ar}\, s|, T_1$ ranges over functions from $\text{Seg}|\text{ar}\, s|$ into $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}, U_1$ ranges over functions from $\text{Seg}|\text{ar}\, s|$ into $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\} : T_1 = T \ \wedge\ U_1 = U\}$, and

(ii) $(s_1)_{\mathbf{2}} = s\text{-compound}\, U$.

We say that s_1 rule 4 S_1 if and only if the condition (Def. 25) is satisfied.

(Def. 25) There exists a literal element l of S and there exists a w.f.f. string p_2 of S and there exists a termal string t of S such that $(s_1)_{\mathbf{1}} = \{(l, t) \operatorname{SubstIn} p_2\}$ and $(s_1)_{\mathbf{2}} = \langle l \rangle \frown p_2$.

We say that s_1 rule 5 S_1 if and only if:

(Def. 26) There exist literal elements v_1, v_2 of S and there exist x, p such that $(s_1)_{\mathbf{1}} = x \cup \{\langle v_1 \rangle \frown p\}$ and v_2 is $x \cup \{p\} \cup \{s_{1\mathbf{2}}\}$-absent and $\langle x \cup \{(v_1 \operatorname{SubstWith} v_2)(p)\}, (s_1)_{\mathbf{2}} \rangle \in S_1$.

We say that s_1 rule 6 S_1 if and only if the condition (Def. 27) is satisfied.

(Def. 27) There exist sets y_1, y_2 and there exist w.f.f. strings p_3, p_4 of S such that y_1, $y_2 \in S_1$ and $(y_1)_{\mathbf{1}} = (y_2)_{\mathbf{1}} = (s_1)_{\mathbf{1}}$ and $(y_1)_{\mathbf{2}} = \langle \operatorname{TheNorSymbOf} S \rangle \frown p_3 \frown p_3$ and $(y_2)_{\mathbf{2}} = \langle \operatorname{TheNorSymbOf} S \rangle \frown p_4 \frown p_4$ and $(s_1)_{\mathbf{2}} = \langle \operatorname{TheNorSymbOf} S \rangle \frown p_3 \frown p_4$.

We say that s_1 rule 7 S_1 if and only if:

(Def. 28) There exists a set y and there exist w.f.f. strings p_3, p_4 of S such that $y \in S_1$ and $y_{\mathbf{1}} = (s_1)_{\mathbf{1}}$ and $y_{\mathbf{2}} = \langle \operatorname{TheNorSymbOf} S \rangle \frown p_3 \frown p_4$ and $(s_1)_{\mathbf{2}} = \langle \operatorname{TheNorSymbOf} S \rangle \frown p_4 \frown p_3$.

We say that s_1 rule 8 S_1 if and only if the condition (Def. 29) is satisfied.

(Def. 29) There exist sets y_1, y_2 and there exist w.f.f. strings p_2, p_3, p_4 of S such that y_1, $y_2 \in S_1$ and $(y_1)_{\mathbf{1}} = (y_2)_{\mathbf{1}}$ and $(y_1)_{\mathbf{2}} = p_3$ and $(y_2)_{\mathbf{2}} = \langle \operatorname{TheNorSymbOf} S \rangle \frown p_3 \frown p_4$ and $\{p_2\} \cup (s_1)_{\mathbf{1}} = (y_1)_{\mathbf{1}}$ and $(s_1)_{\mathbf{2}} = \langle \operatorname{TheNorSymbOf} S \rangle \frown p_2 \frown p_2$.

We say that s_1 rule 9 S_1 if and only if:

(Def. 30) There exists a set y and there exists a w.f.f. string p_2 of S such that $y \in S_1$ and $(s_1)_{\mathbf{2}} = p_2$ and $y_{\mathbf{1}} = (s_1)_{\mathbf{1}}$ and $y_{\mathbf{2}} = \operatorname{xnot} \operatorname{xnot} p_2$.

Let S be a language. The functor $\mathrm{P0}\, S$ yielding a relation between $2^{S\text{-sequents}}$ and S-sequents is defined by:

(Def. 31) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P0}\, S$ iff s_1 rule 0 S_1.

The functor $\mathrm{P1}\, S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined as follows:

(Def. 32) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P1}\, S$ iff s_1 rule 1 S_1.

The functor $\mathrm{P2}\, S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined as follows:

(Def. 33) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P2}\, S$ iff s_1 rule 2 S_1.

The functor $\mathrm{P3a}\, S$ yielding a relation between $2^{S\text{-sequents}}$ and S-sequents is defined as follows:

(Def. 34) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P3a}\, S$ iff s_1 rule 3a S_1.

The functor $\mathrm{P3b}\, S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined as follows:

(Def. 35) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P3b}\, S$ iff s_1 rule 3b S_1.

The functor $\mathrm{P3d}\, S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined as follows:

(Def. 36) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P3d}\, S$ iff s_1 rule 3d S_1.

The functor $\mathrm{P3e}\, S$ yielding a relation between $2^{S\text{-sequents}}$ and S-sequents is defined by:

(Def. 37) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P3e}\, S$ iff s_1 rule 3e S_1.

The functor $\mathrm{P4}\, S$ yielding a relation between $2^{S\text{-sequents}}$ and S-sequents is defined by:

(Def. 38) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P4}\, S$ iff s_1 rule 4 S_1.

The functor $\mathrm{P5}\, S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined by:

(Def. 39) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P5}\, S$ iff s_1 rule 5 S_1.

The functor $\mathrm{P6}\, S$ yielding a relation between $2^{S\text{-sequents}}$ and S-sequents is defined by:

(Def. 40) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P6}\, S$ iff s_1 rule 6 S_1.

The functor $\mathrm{P7}\, S$ yielding a relation between $2^{S\text{-sequents}}$ and S-sequents is defined as follows:

(Def. 41) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P7}\, S$ iff s_1 rule 7 S_1.

The functor $\mathrm{P8}\, S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined as follows:

(Def. 42) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P8}\, S$ iff s_1 rule 8 S_1.

The functor $\mathrm{P9}\, S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined as follows:

(Def. 43) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1 \rangle \in \mathrm{P9}\, S$ iff s_1 rule 9 S_1.

Let us consider S and let R be a relation between $2^{S\text{-sequents}}$ and S-sequents. The functor FuncRule R yields a rule of S and is defined by:

(Def. 44) For every set i_1 such that $i_1 \in 2^{S\text{-sequents}}$ holds $(\mathrm{FuncRule}\,R)(i_1) = \{x \in S\text{-sequents}: \langle i_1, x\rangle \in R\}$.

Let us consider S. The functor R0 S yielding a rule of S is defined as follows:

(Def. 45) $\mathrm{R0}\,S = \mathrm{FuncRule}\,\mathrm{P0}\,S$.

The functor R1 S yielding a rule of S is defined as follows:

(Def. 46) $\mathrm{R1}\,S = \mathrm{FuncRule}\,\mathrm{P1}\,S$.

The functor R2 S yielding a rule of S is defined by:

(Def. 47) $\mathrm{R2}\,S = \mathrm{FuncRule}\,\mathrm{P2}\,S$.

The functor R3a S yielding a rule of S is defined by:

(Def. 48) $\mathrm{R3a}\,S = \mathrm{FuncRule}\,\mathrm{P3a}\,S$.

The functor R3b S yielding a rule of S is defined as follows:

(Def. 49) $\mathrm{R3b}\,S = \mathrm{FuncRule}\,\mathrm{P3b}\,S$.

The functor R3d S yielding a rule of S is defined as follows:

(Def. 50) $\mathrm{R3d}\,S = \mathrm{FuncRule}\,\mathrm{P3d}\,S$.

The functor R3e S yielding a rule of S is defined by:

(Def. 51) $\mathrm{R3e}\,S = \mathrm{FuncRule}\,\mathrm{P3e}\,S$.

The functor R4 S yields a rule of S and is defined as follows:

(Def. 52) $\mathrm{R4}\,S = \mathrm{FuncRule}\,\mathrm{P4}\,S$.

The functor R5 S yielding a rule of S is defined as follows:

(Def. 53) $\mathrm{R5}\,S = \mathrm{FuncRule}\,\mathrm{P5}\,S$.

The functor R6 S yields a rule of S and is defined by:

(Def. 54) $\mathrm{R6}\,S = \mathrm{FuncRule}\,\mathrm{P6}\,S$.

The functor R7 S yields a rule of S and is defined by:

(Def. 55) $\mathrm{R7}\,S = \mathrm{FuncRule}\,\mathrm{P7}\,S$.

The functor R8 S yielding a rule of S is defined as follows:

(Def. 56) $\mathrm{R8}\,S = \mathrm{FuncRule}\,\mathrm{P8}\,S$.

The functor R9 S yields a rule of S and is defined by:

(Def. 57) $\mathrm{R9}\,S = \mathrm{FuncRule}\,\mathrm{P9}\,S$.

Let us consider S and let t be a termal string of S.

Note that $\{\langle\emptyset, \langle\mathrm{TheEqSymbOf}\,S\rangle \frown t \frown t\rangle\}$ is $\{\mathrm{R2}\,S\}$-derivable. Note that R2 S is isotone. One can verify that R3b S is isotone.

Let t, t_1, t_2 be termal strings of S, and let p_5 be a finite subset of AllFormulasOf S. Observe that $\langle p_5 \cup \{\langle\mathrm{TheEqSymbOf}\,S\rangle \frown t_1 \frown t_2\}, \langle\mathrm{TheEqSymbOf}\,S\rangle \frown t \frown t_2\rangle$ is $(1, \{\langle p_5, \langle\mathrm{TheEqSymbOf}\,S\rangle \frown t \frown t_1\rangle\}, \{\mathrm{R3a}\,S\})$-derivable.

Let us consider S, let t, t_1, t_2 be termal strings of S, and let p_2 be a w.f.f. string of S. Note that $\langle\{p_2, \langle \text{TheEqSymbOf}\, S\rangle \frown t_1 \frown t_2\}, \langle \text{TheEqSymbOf}\, S\rangle \frown t \frown t_2\rangle$ is $(1, \{\langle\{p_2\}, \langle \text{TheEqSymbOf}\, S\rangle \frown t \frown t_1\rangle\}, \{\text{R3a}\, S\})$-derivable.

Let us consider S, let p_2 be a w.f.f. string of S, and let P_1 be a finite subset of AllFormulasOf S. One can verify that $\langle P_1 \cup \{p_2\}, p_2\rangle$ is $(1, \emptyset, \{\text{R0}\, S\})$-derivable.

Let us consider S and let p_3, p_4 be w.f.f. strings of S. One can check that $\langle\{p_3, p_4\}, p_3\rangle$ is $(1, \emptyset, \{\text{R0}\, S\})$-derivable.

Let us consider S, p_2. Note that $\langle\{p_2\}, p_2\rangle$ is $(1, \emptyset, \{\text{R0}\, S\})$-derivable.

Let us consider S and let p_2 be a w.f.f. string of S. Observe that $\{\langle\{p_2\}, p_2\rangle\}$ is $(\emptyset, \{\text{R0}\, S\})$-derivable.

Let us consider S. One can verify the following observations:

* R0 S is isotone,
* R3a S is isotone,
* R3d S is isotone, and
* R3e S is isotone.

Let us consider K_1, K_2. One can verify that $K_1 \cup K_2$ is isotone.

Let us consider S and let t_1, t_2 be termal strings of S.

Observe that $\langle \text{TheEqSymbOf}\, S\rangle \frown t_1 \frown t_2$ is 0-w.f.f..

Let us consider S, let m be a non zero natural number, and let T, U be m-element elements of $(\text{AllTermsOf}\, S)^*$. The functor PairWiseEq$(T, U)$ is defined by the condition (Def. 58).

(Def. 58) $\text{PairWiseEq}(T, U) = \{\langle \text{TheEqSymbOf}\, S\rangle \frown T_1(j) \frown U_1(j); j$ ranges over elements of $\text{Seg}\, m, T_1$ ranges over functions from $\text{Seg}\, m$ into $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}, U_1$ ranges over functions from $\text{Seg}\, m$ into $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\} : T_1 = T \ \wedge \ U_1 = U\}$.

Let us consider S, let m be a non zero natural number, and let T_2, T_3 be m-element elements of $(\text{AllTermsOf}\, S)^*$. Then PairWiseEq$(T_2, T_3)$ is a subset of AllFormulasOf S.

Let us consider S, let m be a non zero natural number, and let T, U be m-element elements of $(\text{AllTermsOf}\, S)^*$. Observe that PairWiseEq(T, U) is finite.

Let us consider S, let s be a relational element of S, and let T_2, T_3 be $|\text{ar}\, s|$-element elements of $(\text{AllTermsOf}\, S)^*$. Observe that $\{\langle \text{PairWiseEq}(T_2, T_3) \cup \{s\text{-compound}\, T_2\}, s\text{-compound}\, T_3\rangle\}$ is $(\emptyset, \{\text{R3e}\, S\})$-derivable.

Let us consider m, S, D. We say that D is m-ranked if and only if:

(Def. 59)(i) R0 S, R2 S, R3a S, R3b $S \in D$ if $m = 0$,

(ii) R0 S, R2 S, R3a S, R3b S, R3d S, R3e $S \in D$ if $m = 1$,

(iii) R0 S, R1 S, R2 S, R3a S, R3b S, R3d S, R3e S, R4 S, R5 S, R6 S, R7 S, R8 $S \in D$ if $m = 2$,

(iv) $D = \emptyset$, otherwise.

Let us consider S. One can verify that every rule set of S which is 1-ranked is also 0-ranked and every rule set of S which is 2-ranked is also 1-ranked.

Let us consider S. The functor S-rules yields a rule set of S and is defined by:

(Def. 60) $S\text{-rules} = \{\mathrm{R0}\,S, \mathrm{R1}\,S, \mathrm{R2}\,S, \mathrm{R3a}\,S, \mathrm{R3b}\,S, \mathrm{R3d}\,S, \mathrm{R3e}\,S, \mathrm{R4}\,S\} \cup \{\mathrm{R5}\,S, \mathrm{R6}\,S, \mathrm{R7}\,S, \mathrm{R8}\,S\}$.

Let us consider S. Observe that S-rules is 2-ranked.

Let us consider S. Note that there exists a rule set of S which is 2-ranked.

Let us consider S. Observe that there exists a rule set of S which is 1-ranked.

Let us consider S. Note that there exists a rule set of S which is 0-ranked.

Let us consider S, let D be a 1-ranked rule set of S, let X be a D-expanded set, and let us consider a. Observe that X-freeInterpreter a is (X, D)-termEq-respecting.

Let us consider S, let D be a 0-ranked rule set of S, and let X be a D-expanded set. Observe that (X, D)-termEq is total, symmetric, and transitive.

Let us consider S. Observe that there exists a 0-ranked rule set of S which is 1-ranked.

The following proposition is true

(1) If $D_1 \subseteq D_2$ and if D_2 is isotone or D_1 is isotone and if Y is (X, D_1)-derivable, then Y is (X, D_2)-derivable.

Let us consider S, S_6. One can verify that $\{S_6\}$ is S-sequents-like.

Let us consider S, S_{11}, S_5. One can check that $S_{11} \cup S_5$ is S-sequents-like.

Let us consider S and let x, y be S-sequent-like sets. Observe that $\{x, y\}$ is S-sequents-like.

Let us consider S, p_3, p_4. Note that $\langle\{\mathrm{xnot}\,p_3, \mathrm{xnot}\,p_4\}, \langle\mathrm{TheNorSymbOf}\,S\rangle^\frown p_3{}^\frown p_4\rangle$ is $(1, \{\langle\{\mathrm{xnot}\,p_3, \mathrm{xnot}\,p_4\}, \mathrm{xnot}\,p_3\rangle, \langle\{\mathrm{xnot}\,p_3, \mathrm{xnot}\,p_4\}, \mathrm{xnot}\,p_4\rangle\}, \{\mathrm{R6}\,S\})$-derivable.

Let us consider S, p_3, p_4. One can check that $\langle\{p_3, p_4\}, p_4\rangle$ is $(1, \emptyset, \{\mathrm{R0}\,S\})$-derivable.

We now state two propositions:

(2) For every relation R between $2^{S\text{-sequents}}$ and S-sequents such that $\langle S_4, S_6\rangle \in R$ holds $S_6 \in (\mathrm{FuncRule}\,R)(S_4)$.

(3) If $x \in R(X)$, then x is $(1, X, \{R\})$-derivable.

Let us consider S, D, X. Let us observe that X is D-expanded if and only if:

(Def. 61) If x is (X, D)-provable, then $x \in X$.

The following four propositions are true:

(4) If $p_2 \in X$, then p_2 is $(X, \{\mathrm{R0}\,S\})$-provable.

(5) Suppose that

(i) $D_1 \cup D_2$ is isotone,

(ii) $D_1 \cup D_2 \cup D_3$ is isotone,
(iii) x is (m, S_{11}, D_1)-derivable,
(iv) y is (m, S_5, D_2)-derivable, and
(v) z is $(n, \{x, y\}, D_3)$-derivable.
Then z is $(m + n, S_{11} \cup S_5, D_1 \cup D_2 \cup D_3)$-derivable.

(6) Suppose D_1 is isotone and $D_1 \cup D_2$ is isotone and y is (m, X, D_1)-derivable and z is $(n, \{y\}, D_2)$-derivable. Then z is $(m + n, X, D_1 \cup D_2)$-derivable.

(7) If x is (m, X, D)-derivable, then $\{x\}$ is (X, D)-derivable.

Let us consider S. Observe that R6 S is isotone.

One can prove the following propositions:

(8) If $D_1 \subseteq D_2$ and if D_1 is isotone or D_2 is isotone and if x is (X, D_1)-provable, then x is (X, D_2)-provable.

(9) If $X \subseteq Y$ and x is (X, D)-provable, then x is (Y, D)-provable.

Let us consider S. Note that R8 S is isotone.

Let us consider S. Observe that R1 S is isotone.

Next we state the proposition

(10) If $\{y\}$ is (S_4, D)-derivable, then there exists m_1 such that y is (m_1, S_4, D)-derivable.

Let us consider S, D, X. Observe that every set which is (X, D)-derivable is also S-sequents-like.

Let us consider S, D, X, x. Let us observe that x is (X, D)-provable if and only if:

(Def. 62) There exists a set H and there exists m such that $H \subseteq X$ and $\langle H, x \rangle$ is $(m, \emptyset, D)$-derivable.

The following proposition is true

(11) If $D_1 \subseteq D_2$ and if D_2 is isotone or D_1 is isotone and if x is (m, X, D_1)-derivable, then x is (m, X, D_2)-derivable.

Let us consider S. Observe that R7 S is isotone.

Next we state the proposition

(12) If x is (X, D)-provable, then x is a w.f.f. string of S.

In the sequel F denotes a rule set of S.

Let us consider S, D_1 and let X be a D_1-expanded set. One can verify that (S, X)-freeInterpreter is (X, D_1) -termEq-respecting.

Let us consider S, let D be a 0-ranked rule set of S, and let X be a D-expanded set. The functor D Henkin X yielding a function is defined by:

(Def. 63) D Henkin $X = (S, X)$-freeInterpreter quotient(X, D) -termEq .

Let us consider S, let D be a 0-ranked rule set of S, and let X be a D-expanded set. One can check that D Henkin X is OwnSymbolsOf S-defined.

Let us consider S, D_1 and let X be a D_1-expanded set. Observe that D_1 Henkin X is $(S, \mathrm{Classes}(X, D_1)$ -termEq)-interpreter-like.

Let us consider S, D_1 and let X be a D_1-expanded set. Then D_1 Henkin X is an element of Classes((X, D_1) -termEq)-InterpretersOf S.

Let us consider S, p_3, p_4. One can verify that $\langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4$ is $(\{\mathrm{xnot}\, p_3, \mathrm{xnot}\, p_4\}, \{\mathrm{R0}\, S\} \cup \{\mathrm{R6}\, S\})$-provable.

Let us consider S. Note that every 0-ranked rule set of S is non empty.

Let us consider S, x. We say that x is S-premises-like if and only if:

(Def. 64) $x \subseteq \mathrm{AllFormulasOf}\, S$ and x is finite.

Let us consider S. One can verify that every set which is S-premises-like is also finite.

Let us consider S, p_2. Note that $\{p_2\}$ is S-premises-like.

Let us consider S and let e be an empty set. One can check that e null S is S-premises-like.

Let us consider X, S. Observe that there exists a subset of X which is S-premises-like.

Let us consider S. Observe that there exists a set which is S-premises-like.

Let us consider S and let X be an S-premises-like set. Observe that every subset of X is S-premises-like.

In the sequel H_3 denotes an S-premises-like set.

Let us consider S, H_2, H_1. Then H_1 null H_2 is a subset of AllFormulasOf S.

Let us consider S, H, x. Note that H null x is S-premises-like.

Let us consider S, H_1, H_2. Note that $H_1 \cup H_2$ is S-premises-like.

Let us consider S, H, p_2. Observe that $\langle H, p_2\rangle$ is S-sequent-like.

Let us consider S, H_1, H_2, p_2. One can verify that $\langle H_1 \cup H_2, p_2\rangle$ is $(1, \{\langle H_1, p_2\rangle\}, \{\mathrm{R1}\, S\})$-derivable.

Let us consider S, H, p_2, p_3, p_4. One can check that $\langle H \,\mathrm{null}\, p_3 \frown p_4, \mathrm{xnot}\, p_2\rangle$ is $(1, \{\langle H \cup \{p_2\}, p_3\rangle, \langle H \cup \{p_2\}, \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4\rangle\}, \{\mathrm{R8}\, S\})$-derivable.

Let us consider S. One can verify that $\emptyset$ null S is S-sequents-like.

Let us consider S, H, p_2. Observe that $\langle H \cup \{p_2\}, p_2\rangle$ is $(1, \emptyset, \{\mathrm{R0}\, S\})$-derivable. Let us consider p_3, p_4. Note that $\langle H \,\mathrm{null}\, p_4, \mathrm{xnot}\, p_3\rangle$ is

$(2, \{\langle H, \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4\rangle\}, \{\mathrm{R0}\, S\} \cup \{\mathrm{R1}\, S\} \cup \{\mathrm{R8}\, S\})$-derivable.

Let us consider S, H, p_3, p_4. Note that $\langle H, \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_4 \frown p_3\rangle$ is $(1, \{\langle H, \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4\rangle\}, \{\mathrm{R7}\, S\})$-derivable.

Let us consider S, H, p_3, p_4. Observe that $\langle H \,\mathrm{null}\, p_3, \mathrm{xnot}\, p_4\rangle$ is $(3, \{\langle H, \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4\rangle\}, \{\mathrm{R0}\, S\} \cup \{\mathrm{R1}\, S\} \cup \{\mathrm{R8}\, S\} \cup \{\mathrm{R7}\, S\})$-derivable.

Let us consider S, S_6. Observe that $(S_6)_{\mathbf{1}}$ is S-premises-like.

Let us consider S, X, D. Then D null X is a rule set of S.

Let us consider S, p_3, p_4, l_1, H and let l_2 be an $H \cup \{p_3\} \cup \{p_4\}$-absent literal element of S.

Note that $\langle (H \cup \{\langle l_1 \rangle ^\frown p_3\}) \operatorname{null} l_2,\ p_4 \rangle$ is $(1, \{\langle H \cup \{(l_1, l_2)\text{-SymbolSubstIn}\, p_3\}, p_4\rangle\}, \{\mathrm{R5}\, S\})$-derivable.

Let us consider S, D, X. We say that X is D-inconsistent if and only if:

(Def. 65) There exist p_3, p_4 such that p_3 is (X, D)-provable and $\langle \text{TheNorSymbOf}\, S \rangle ^\frown p_3 ^\frown p_4$ is (X, D)-provable.

Let us consider m_2, S, H_1, H_2, p_2. Note that $\langle (H_1 \cup H_2) \operatorname{null} m_2,\ p_2 \rangle$ is $(m_2, \{\langle H_1, p_2\rangle\}, \{\mathrm{R1}\, S\})$-derivable.

Let us consider S. Observe that there exists an isotone rule set of S which is non empty.

We now state the proposition

(13) If X is D-inconsistent and D is isotone and $\mathrm{R1}\, S, \mathrm{R8}\, S \in D$, then $\operatorname{xnot} p_2$ is (X, D)-provable.

Let us consider S. Observe that $\mathrm{R5}\, S$ is isotone.

Let us consider S, l, t, p_2. Observe that $\langle \{(l, t) \operatorname{SubstIn} p_2\},\ \langle l \rangle ^\frown p_2 \rangle$ is $(1, \emptyset, \{\mathrm{R4}\, S\})$-derivable.

Let us consider S. One can verify that $\mathrm{R4}\, S$ is isotone.

Let us consider S, X. We say that X is S-witnessed if and only if:

(Def. 66) For all l_1, p_3 such that $\langle l_1 \rangle ^\frown p_3 \in X$ there exists l_2 such that $(l_1, l_2)\text{-SymbolSubstIn}\, p_3 \in X$ and $l_2 \notin \operatorname{rng} p_3$.

We now state the proposition

(14)[3] Let X be a D_1-expanded set. Suppose $\mathrm{R1}\, S$, $\mathrm{R4}\, S$, $\mathrm{R6}\, S$, $\mathrm{R7}\, S$, $\mathrm{R8}\, S \in D_1$ and X is S-mincover and S-witnessed. Then $(D_1 \operatorname{Henkin} X)\text{-TruthEval}\, p_1 = 1$ if and only if $p_1 \in X$.

Let us consider S, D, X. We introduce X is D-consistent as an antonym of X is D-inconsistent.

We now state the proposition

(15) For every subset X of Y such that X is D-inconsistent holds Y is D-inconsistent.

Let us consider S, D, let X be a functional set, and let p_2 be an element of ExFormulasOf S. The functor $(D, p_2) \operatorname{AddAsWitnessTo} X$ is defined by:

(Def. 67)
$$(D, p_2) \operatorname{AddAsWitnessTo} X = \begin{cases} X \cup \{(S\text{-firstChar}(p_2),\ \text{the element of LettersOf } S \setminus \text{SymbolsOf} (((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \cap (X \cup \{\operatorname{head} p_2\})))\text{-SymbolSubstIn} \operatorname{head} p_2\}, \\ \quad \text{if } X \cup \{p_2\} \text{ is } D\text{-consistent and LettersOf } S \setminus \text{SymbolsOf}(((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \cap (X \cup \{\operatorname{head} p_2\})) \neq \emptyset, \\ X, \text{ otherwise.} \end{cases}$$

[3] Henkin's Theorem

Let us consider S, D, let X be a functional set, and let p_2 be an element of ExFormulasOf S. One can check that $X \setminus ((D, p_2)\ \mathrm{AddAsWitnessTo}\ X)$ is empty.

Let us consider S, D, let X be a functional set, and let p_2 be an element of ExFormulasOf S. One can check that $((D, p_2)\ \mathrm{AddAsWitnessTo}\ X) \setminus X$ is trivial.

Let us consider S, D, let X be a functional set, and let p_2 be an element of ExFormulasOf S. Then $(D, p_2)\ \mathrm{AddAsWitnessTo}\ X$ is a subset of $X \cup \mathrm{AllFormulasOf}\ S$.

Let us consider S, D. We say that D is correct if and only if the condition (Def. 68) is satisfied.

(Def. 68) Let given p_2, X. Suppose p_2 is (X, D)-provable. Let given U and I be an element of U-InterpretersOf S. If X is I-satisfied, then $I\text{-TruthEval}\ p_2 = 1$.

Let us consider S, t_1, t_2. One can check that $\mathrm{SubTerms}(\langle \mathrm{TheEqSymbOf}\ S \rangle ^\frown t_1 {}^\frown t_2) \dot{-} \langle t_1, t_2 \rangle$ is empty.

Let us consider S and let R be a rule of S. We say that R is correct if and only if:

(Def. 69) If X is S-correct, then $R(X)$ is S-correct.

Let us consider S. Observe that every set which is S-sequent-like is also S-null.

Let us consider S. Note that $\mathrm{R0}\ S$ is correct.

Let us consider S. Note that there exists a rule of S which is correct.

Let us consider S. One can check that $\mathrm{R1}\ S$ is correct.

Let us consider S. Note that $\mathrm{R2}\ S$ is correct.

Let us consider S. One can check that $\mathrm{R3a}\ S$ is correct.

Let us consider S. Observe that $\mathrm{R3b}\ S$ is correct.

Let us consider S. Observe that $\mathrm{R3d}\ S$ is correct.

Let us consider S. Note that $\mathrm{R3e}\ S$ is correct.

Let us consider S. One can check that $\mathrm{R4}\ S$ is correct.

Let us consider S. One can check that $\mathrm{R5}\ S$ is correct.

Let us consider S. One can verify that $\mathrm{R6}\ S$ is correct.

Let us consider S. Observe that $\mathrm{R7}\ S$ is correct.

Let us consider S. Observe that $\mathrm{R8}\ S$ is correct.

Next we state the proposition

(16) If for every rule R of S such that $R \in D$ holds R is correct, then D is correct.

Let us consider S and let R be a correct rule of S. Note that $\{R\}$ is correct. Observe that S-rules is correct. One can check that $\mathrm{R9}\ S$ is isotone. Let us consider H, p_2. Observe that $\langle H, p_2 \rangle\ \mathrm{null}\ 1$ is $(1, \{\langle H, \mathrm{xnot}\ \mathrm{xnot}\ p_2 \rangle\}, \{\mathrm{R9}\ S\})$-derivable.

Let us consider X, S. Observe that there exists an 0-w.f.f. string of S which is X-implied.

Let us consider X, S. Observe that there exists a w.f.f. string of S which is X-implied.

Let us consider S, X and let p_2 be an X-implied w.f.f. string of S. Observe that xnot xnot p_2 is X-implied.

Let us consider X, S, p_2. We say that p_2 is X-provable if and only if:

(Def. 70) p_2 is $(X, \{\mathrm{R9}\, S\} \cup S\text{-rules})$-provable.

2. Constructions for Countable Languages: Witness Adjoining

Let X be a functional set, let us consider S, D, and let n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. The functor (D, n_1) AddWitnessesTo X yields a function from $\mathbb{N}$ into $2^{X \cup \mathrm{AllFormulasOf}\, S}$ and is defined by:

(Def. 71) $((D, n_1)\,\mathrm{AddWitnessesTo}\, X)(0) = X$ and for every m_1 holds $((D, n_1)\,\mathrm{AddWitnessesTo}\, X)(m_1 + 1) = (D, n_1(m_1))\,\mathrm{AddAsWitnessTo}((D, n_1)\,\mathrm{AddWitnessesTo}\, X)(m_1)$.

Let X be a functional set, let us consider S, D, and let n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. We introduce (D, n_1) addw X as a synonym of (D, n_1) AddWitnessesTo X.

We now state the proposition

(17) Let X be a functional set and n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. Suppose D is isotone and R1 S, R8 S, R2 S, R5 $S \in D$ and LettersOf $S \setminus$ SymbolsOf$(X \cap ((\mathrm{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}))$ is infinite and X is D-consistent. Then $((D, n_1)\,\mathrm{addw}\, X)(k) \subseteq ((D, n_1)\,\mathrm{addw}\, X)(k + m)$ and LettersOf $S \setminus$ SymbolsOf$(((D, n_1)\,\mathrm{addw}\, X)(m) \cap ((\mathrm{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}))$ is infinite and $((D, n_1)\,\mathrm{addw}\, X)(m)$ is D-consistent.

Let X be a functional set, let us consider S, D, and let n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. The functor X WithWitnessesFrom(D, n_1) yielding a subset of $X \cup$ AllFormulasOf S is defined by:

(Def. 72) $X\,\mathrm{WithWitnessesFrom}(D, n_1) = \bigcup \mathrm{rng}((D, n_1)\,\mathrm{AddWitnessesTo}\, X)$.

Let X be a functional set, let us consider S, D, and let n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. We introduce $X\,\mathrm{addw}(D, n_1)$ as a synonym of X WithWitnessesFrom(D, n_1).

Let X be a functional set, let us consider S, D, and let n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. One can verify that $X \setminus (X\,\mathrm{addw}(D, n_1))$ is empty.

The following proposition is true

(18) Let X be a functional set and n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. Suppose that D is isotone and R1 S, R8 S, R2 S, R5 $S \in D$ and LettersOf $S \setminus$ SymbolsOf$(X \cap ((\mathrm{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}))$ is infinite and $X\,\mathrm{addw}(D, n_1) \subseteq Z$ and Z is D-consistent and $\mathrm{rng}\, n_1 =$ ExFormulasOf S. Then Z is S-witnessed.

3. Constructions for Countable Languages: Consistently Maximizing a Set of Formulas of a Countable Language (Lindenbaum's Lemma)

Let us consider X, S, D and let p_2 be an element of AllFormulasOf S. The functor (D, p_2) AddFormulaTo X is defined by:

(Def. 73) $$(D, p_2)\,\text{AddFormulaTo}\,X = \begin{cases} X \cup \{p_2\}, & \\ \quad \text{if xnot}\, p_2 \text{ is not } (X, D)\text{-provable}, & \\ X \cup \{\text{xnot}\, p_2\}, \ \text{otherwise.} & \end{cases}$$

Let us consider X, S, D and let p_2 be an element of AllFormulasOf S. Then (D, p_2) AddFormulaTo X is a subset of $X \cup$ AllFormulasOf S.

Let us consider X, S, D and let p_2 be an element of AllFormulasOf S. Note that $X \setminus ((D, p_2)\,\text{AddFormulaTo}\,X)$ is empty.

Let us consider X, S, D and let n_1 be a function from $\mathbb{N}$ into AllFormulasOf S. The functor (D, n_1) AddFormulasTo X yields a function from $\mathbb{N}$ into $2^{X \cup \text{AllFormulasOf}\, S}$ and is defined by:

(Def. 74) $((D, n_1)\,\text{AddFormulasTo}\,X)(0) = X$ and for every m holds $((D, n_1)\,\text{AddFormulasTo}\,X)(m + 1) = (D, n_1(m))\,\text{AddFormulaTo}((D, n_1)\,\text{AddFormulasTo}\,X)(m)$.

Let us consider X, S, D and let n_1 be a function from $\mathbb{N}$ into AllFormulasOf S. The functor (D, n_1) CompletionOf X yields a subset of $X \cup$ AllFormulasOf S and is defined as follows:

(Def. 75) $(D, n_1)\,\text{CompletionOf}\,X = \bigcup \text{rng}((D, n_1)\,\text{AddFormulasTo}\,X)$.

Let us consider X, S, D and let n_1 be a function from $\mathbb{N}$ into AllFormulasOf S. One can check that $X \setminus ((D, n_1)\,\text{CompletionOf}\,X)$ is empty.

We now state the proposition

(19) For every relation R between $2^{S\text{-sequents}}$ and S-sequents holds $y \in (\text{FuncRule}\,R)(X)$ iff $y \in S$-sequents and $\langle X, y \rangle \in R$.

In the sequel D_2 is a 2-ranked rule set of S.

Let us consider S and let r_1, r_2 be isotone rules of S. Note that $\{r_1, r_2\}$ is isotone.

Let us consider S and let r_1, r_2, r_3, r_4 be isotone rules of S. Observe that $\{r_1, r_2, r_3, r_4\}$ is isotone.

Let us consider S. Observe that S-rules is isotone.

Let us consider S. Observe that there exists an isotone rule set of S which is correct.

Let us consider S. Observe that there exists a correct isotone rule set of S which is 2-ranked.

Let S be a countable language. Observe that AllFormulasOf S is countable.

We now state the proposition

(20) Let S be a countable language and D be a rule set of S. Suppose D is 2-ranked, isotone, and correct and Z is D-consistent and $Z \subseteq$ AllFormulasOf S. Then there exists a non empty set U and there exists an element I of U-InterpretersOf S such that Z is I-satisfied.

In the sequel C denotes a countable language and p_2 denotes a w.f.f. string of C.

We now state the proposition

(21) If $X \subseteq$ AllFormulasOf C and p_2 is X-implied, then p_2 is X-provable.

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