

# Some Properties of $p$ -Groups and Commutative $p$ -Groups

Xiquan Liang  
Qingdao University of Science  
and Technology  
China

Dailu Li  
Qingdao University of Science  
and Technology  
China

**Summary.** This article describes some properties of  $p$ -groups and some properties of commutative  $p$ -groups.

MML identifier: GROUPE\_1, version: 7.11.07 4.156.1112

The notation and terminology used here have been introduced in the following papers: [7], [4], [8], [6], [10], [9], [11], [5], [1], [3], [2], and [12].

## 1. $p$ -GROUPS

For simplicity, we use the following convention:  $G$  is a group,  $a, b$  are elements of  $G$ ,  $m, n$  are natural numbers, and  $p$  is a prime natural number.

One can prove the following propositions:

- (1) If for every natural number  $r$  holds  $n \neq p^r$ , then there exists an element  $s$  of  $\mathbb{N}$  such that  $s$  is prime and  $s \mid n$  and  $s \neq p$ .
- (2) For all natural numbers  $n, m$  such that  $n \mid p^m$  there exists a natural number  $r$  such that  $n = p^r$  and  $r \leq m$ .
- (3) If  $a^n = \mathbf{1}_G$ , then  $(a^{-1})^n = \mathbf{1}_G$ .
- (4) If  $(a^{-1})^n = \mathbf{1}_G$ , then  $a^n = \mathbf{1}_G$ .
- (5)  $\text{ord}(a^{-1}) = \text{ord}(a)$ .
- (6)  $\text{ord}(a^b) = \text{ord}(a)$ .
- (7) Let  $G$  be a group,  $N$  be a subgroup of  $G$ , and  $a, b$  be elements of  $G$ . Suppose  $N$  is normal and  $b \in N$ . Let given  $n$ . Then there exists an element  $g$  of  $G$  such that  $g \in N$  and  $(a \cdot b)^n = a^n \cdot g$ .

- (8) Let  $G$  be a group,  $N$  be a normal subgroup of  $G$ ,  $a$  be an element of  $G$ , and  $S$  be an element of  $G/N$ . If  $S = a \cdot N$ , then for every  $n$  holds  $S^n = a^n \cdot N$ .
- (9) Let  $G$  be a group,  $H$  be a subgroup of  $G$ , and  $a, b$  be elements of  $G$ . If  $a \cdot H = b \cdot H$ , then there exists an element  $h$  of  $G$  such that  $a = b \cdot h$  and  $h \in H$ .
- (10) Let  $G$  be a finite group and  $N$  be a normal subgroup of  $G$ . If  $N$  is a subgroup of  $Z(G)$  and  $G/N$  is cyclic, then  $G$  is commutative.
- (11) Let  $G$  be a finite group and  $N$  be a normal subgroup of  $G$ . If  $N = Z(G)$  and  $G/N$  is cyclic, then  $G$  is commutative.
- (12) For every finite group  $G$  and for every subgroup  $H$  of  $G$  such that  $\overline{H} \neq \overline{G}$  there exists an element  $a$  of  $G$  such that  $a \notin H$ .

Let  $p$  be a natural number, let  $G$  be a group, and let  $a$  be an element of  $G$ . We say that  $a$  is  $p$ -power if and only if:

(Def. 1) There exists a natural number  $r$  such that  $\text{ord}(a) = p^r$ .

We now state the proposition

- (13)  $\mathbf{1}_G$  is  $m$ -power.

Let us consider  $G, m$ . One can verify that there exists an element of  $G$  which is  $m$ -power.

Let us consider  $p, G$  and let  $a$  be a  $p$ -power element of  $G$ . Observe that  $a^{-1}$  is  $p$ -power.

One can prove the following proposition

- (14) If  $a^b$  is  $p$ -power, then  $a$  is  $p$ -power.

Let us consider  $p, G, b$  and let  $a$  be a  $p$ -power element of  $G$ . One can verify that  $a^b$  is  $p$ -power.

Let us consider  $p$ , let  $G$  be a commutative group, and let  $a, b$  be  $p$ -power elements of  $G$ . Observe that  $a \cdot b$  is  $p$ -power.

Let us consider  $p$  and let  $G$  be a finite  $p$ -group group. One can verify that every element of  $G$  is  $p$ -power.

The following proposition is true

- (15) Let  $G$  be a finite group,  $H$  be a subgroup of  $G$ , and  $a$  be an element of  $G$ . If  $H$  is  $p$ -group and  $a \in H$ , then  $a$  is  $p$ -power.

Let us consider  $p$  and let  $G$  be a finite  $p$ -group group. One can verify that every subgroup of  $G$  is  $p$ -group.

We now state the proposition

- (16)  $\{\mathbf{1}\}_G$  is  $p$ -group.

Let us consider  $p$  and let  $G$  be a group. Note that there exists a subgroup of  $G$  which is  $p$ -group.

Let us consider  $p$ , let  $G$  be a finite group, let  $G_1$  be a  $p$ -group subgroup of  $G$ , and let  $G_2$  be a subgroup of  $G$ . One can verify that  $G_1 \cap G_2$  is  $p$ -group and  $G_2 \cap G_1$  is  $p$ -group.

Next we state the proposition

- (17) For every finite group  $G$  such that every element of  $G$  is  $p$ -power holds  $G$  is  $p$ -group.

Let us consider  $p$ , let  $G$  be a finite  $p$ -group group, and let  $N$  be a normal subgroup of  $G$ . Note that  $G/N$  is  $p$ -group.

The following four propositions are true:

- (18) Let  $G$  be a finite group and  $N$  be a normal subgroup of  $G$ . If  $N$  is  $p$ -group and  $G/N$  is  $p$ -group, then  $G$  is  $p$ -group.
- (19) Let  $G$  be a finite commutative group and  $H, H_1, H_2$  be subgroups of  $G$ . Suppose  $H_1$  is  $p$ -group and  $H_2$  is  $p$ -group and the carrier of  $H = H_1 \cdot H_2$ . Then  $H$  is  $p$ -group.
- (20) Let  $G$  be a finite group and  $H, N$  be subgroups of  $G$ . Suppose  $N$  is a normal subgroup of  $G$  and  $H$  is  $p$ -group and  $N$  is  $p$ -group. Then there exists a strict subgroup  $P$  of  $G$  such that the carrier of  $P = H \cdot N$  and  $P$  is  $p$ -group.
- (21) Let  $G$  be a finite group and  $N_1, N_2$  be normal subgroups of  $G$ . Suppose  $N_1$  is  $p$ -group and  $N_2$  is  $p$ -group. Then there exists a strict normal subgroup  $N$  of  $G$  such that the carrier of  $N = N_1 \cdot N_2$  and  $N$  is  $p$ -group.

Let us consider  $p$ , let  $G$  be a  $p$ -group finite group, let  $H$  be a finite group, and let  $g$  be a homomorphism from  $G$  to  $H$ . Observe that  $\text{Im } g$  is  $p$ -group.

The following proposition is true

- (22) For all strict groups  $G, H$  such that  $G$  and  $H$  are isomorphic and  $G$  is  $p$ -group holds  $H$  is  $p$ -group.

Let  $p$  be a prime natural number and let  $G$  be a group. Let us assume that  $G$  is  $p$ -group. The functor  $\text{expon}(G, p)$  yields a natural number and is defined by:

$$\text{(Def. 2)} \quad \overline{G} = p^{\text{expon}(G, p)}.$$

Let  $p$  be a prime natural number and let  $G$  be a group. Then  $\text{expon}(G, p)$  is an element of  $\mathbb{N}$ .

Next we state four propositions:

- (23) For every finite group  $G$  and for every subgroup  $H$  of  $G$  such that  $G$  is  $p$ -group holds  $\text{expon}(H, p) \leq \text{expon}(G, p)$ .
- (24) For every strict finite group  $G$  such that  $G$  is  $p$ -group and  $\text{expon}(G, p) = 0$  holds  $G = \{1\}_G$ .
- (25) For every strict finite group  $G$  such that  $G$  is  $p$ -group and  $\text{expon}(G, p) = 1$  holds  $G$  is cyclic.

- (26) Let  $G$  be a finite group,  $p$  be a prime natural number, and  $a$  be an element of  $G$ . If  $G$  is  $p$ -group and  $\text{expon}(G, p) = 2$  and  $\text{ord}(a) = p^2$ , then  $G$  is commutative.

## 2. COMMUTATIVE $p$ -GROUPS

Let  $p$  be a natural number and let  $G$  be a group. We say that  $G$  is  $p$ -commutative group-like if and only if:

- (Def. 3) For all elements  $a, b$  of  $G$  holds  $(a \cdot b)^p = a^p \cdot b^p$ .

Let  $p$  be a natural number and let  $G$  be a group. We say that  $G$  is  $p$ -commutative group if and only if:

- (Def. 4)  $G$  is  $p$ -group and  $p$ -commutative group-like.

Let  $p$  be a natural number. Observe that every group which is  $p$ -commutative group is also  $p$ -group and  $p$ -commutative group-like and every group which is  $p$ -group and  $p$ -commutative group-like is also  $p$ -commutative group.

The following proposition is true

- (27)  $\{1\}_G$  is  $p$ -commutative group-like.

Let us consider  $p$ . Note that there exists a group which is  $p$ -commutative group, finite, cyclic, and commutative.

Let us consider  $p$  and let  $G$  be a  $p$ -commutative group-like finite group. Note that every subgroup of  $G$  is  $p$ -commutative group-like.

Let us consider  $p$ . Note that every group which is  $p$ -group, finite, and commutative is also  $p$ -commutative group.

We now state the proposition

- (28) For every strict finite group  $G$  such that  $\overline{G} = p$  holds  $G$  is  $p$ -commutative group.

Let us consider  $p, G$ . One can check that there exists a subgroup of  $G$  which is  $p$ -commutative group and finite.

Let us consider  $p$ , let  $G$  be a finite group, let  $H_1$  be a  $p$ -commutative group-like subgroup of  $G$ , and let  $H_2$  be a subgroup of  $G$ . One can check that  $H_1 \cap H_2$  is  $p$ -commutative group-like and  $H_2 \cap H_1$  is  $p$ -commutative group-like.

Let us consider  $p$ , let  $G$  be a finite  $p$ -commutative group-like group, and let  $N$  be a normal subgroup of  $G$ . One can verify that  $G/N$  is  $p$ -commutative group-like.

One can prove the following propositions:

- (29) Let  $G$  be a finite group and  $a, b$  be elements of  $G$ . Suppose  $G$  is  $p$ -commutative group-like. Let given  $n$ . Then  $(a \cdot b)^{p^n} = a^{p^n} \cdot b^{p^n}$ .
- (30) Let  $G$  be a finite commutative group and  $H, H_1, H_2$  be subgroups of  $G$ . Suppose  $H_1$  is  $p$ -commutative group and  $H_2$  is  $p$ -commutative group and the carrier of  $H = H_1 \cdot H_2$ . Then  $H$  is  $p$ -commutative group.

- (31) Let  $G$  be a finite group,  $H$  be a subgroup of  $G$ , and  $N$  be a strict normal subgroup of  $G$ . Suppose  $N$  is a subgroup of  $Z(G)$  and  $H$  is  $p$ -commutative group and  $N$  is  $p$ -commutative group. Then there exists a strict subgroup  $P$  of  $G$  such that the carrier of  $P = H \cdot N$  and  $P$  is  $p$ -commutative group.
- (32) Let  $G$  be a finite group and  $N_1, N_2$  be normal subgroups of  $G$ . Suppose  $N_2$  is a subgroup of  $Z(G)$  and  $N_1$  is  $p$ -commutative group and  $N_2$  is  $p$ -commutative group. Then there exists a strict normal subgroup  $N$  of  $G$  such that the carrier of  $N = N_1 \cdot N_2$  and  $N$  is  $p$ -commutative group.
- (33) Let  $G, H$  be groups. Suppose  $G$  and  $H$  are isomorphic and  $G$  is  $p$ -commutative group-like. Then  $H$  is  $p$ -commutative group-like.
- (34) Let  $G, H$  be strict groups. Suppose  $G$  and  $H$  are isomorphic and  $G$  is  $p$ -commutative group. Then  $H$  is  $p$ -commutative group.

Let us consider  $p$ , let  $G$  be a  $p$ -commutative group-like finite group, let  $H$  be a finite group, and let  $g$  be a homomorphism from  $G$  to  $H$ . Observe that  $\text{Im } g$  is  $p$ -commutative group-like.

The following propositions are true:

- (35) For every strict finite group  $G$  such that  $G$  is  $p$ -group and  $\text{expon}(G, p) = 0$  holds  $G$  is  $p$ -commutative group.
- (36) For every strict finite group  $G$  such that  $G$  is  $p$ -group and  $\text{expon}(G, p) = 1$  holds  $G$  is  $p$ -commutative group.

#### REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [3] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(5):887–890, 1990.
- [4] Marco Riccardi. The Sylow theorems. *Formalized Mathematics*, 15(3):159–165, 2007, doi:10.2478/v10037-007-0018-3.
- [5] Dariusz Surowik. Cyclic groups and some of their properties – part I. *Formalized Mathematics*, 2(5):623–627, 1991.
- [6] Wojciech A. Trybulec. Classes of conjugation. Normal subgroups. *Formalized Mathematics*, 1(5):955–962, 1990.
- [7] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(5):821–827, 1990.
- [8] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Formalized Mathematics*, 1(5):855–864, 1990.
- [9] Wojciech A. Trybulec. Commutator and center of a group. *Formalized Mathematics*, 2(4):461–466, 1991.
- [10] Wojciech A. Trybulec. Lattice of subgroups of a group. Frattini subgroup. *Formalized Mathematics*, 2(1):41–47, 1991.
- [11] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(4):573–578, 1991.
- [12] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.

Received April 29, 2010

---