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Differentiable Functions into Real Normed Spaces

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Noboru Endou
Nagano National College of Technology
Nagano, Japan

Keiko Narita
Hirosaki-city
Aomori, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we formalize the differentiability of functions from the set of real numbers into a normed vector space [14].

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The notation and terminology used here have been introduced in the following papers: [12], [2], [3], [7], [9], [11], [1], [4], [10], [13], [6], [17], [18], [15], [8], [16], [19], and [5].

For simplicity, we adopt the following rules: F denotes a non trivial real normed space, G denotes a real normed space, X denotes a set, x, x_0, r, p denote real numbers, n, k denote elements of $\mathbb{N}$, Y denotes a subset of $\mathbb{R}$, Z denotes an open subset of $\mathbb{R}$, s_1 denotes a sequence of real numbers, s_2 denotes a sequence of G, f, f_1, f_2 denote partial functions from $\mathbb{R}$ to the carrier of F, h denotes a convergent to 0 sequence of real numbers, and c denotes a constant sequence of real numbers.

We now state two propositions:

(1) If for every n holds $\|s_2(n)\| \leq s_1(n)$ and s_1 is convergent and $\lim s_1 = 0$, then s_2 is convergent and $\lim s_2 = 0_G$.

(2) $(s_1 \uparrow k)\,(s_2 \uparrow k) = (s_1\, s_2) \uparrow k$.

Let us consider F and let I_1 be a partial function from $\mathbb{R}$ to the carrier of F. We say that I_1 is rest-like if and only if:

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(Def. 1) I_1 is total and for every h holds $h^{-1}\,(I_{1*}h)$ is convergent and $\lim(h^{-1}\,(I_{1*}h)) = 0_F$.

Let us consider F. One can check that there exists a partial function from $\mathbb{R}$ to the carrier of F which is rest-like. Let us consider F. A rest of F is a rest-like partial function from $\mathbb{R}$ to the carrier of F. Let us consider F and let I_1 be a function from $\mathbb{R}$ into the carrier of F. We say that I_1 is linear if and only if:

(Def. 2) There exists a point r of F such that for every real number p holds $I_1(p) = p \cdot r$.

Let us consider F. Note that there exists a function from $\mathbb{R}$ into the carrier of F which is linear. Let us consider F. A linear of F is a linear function from $\mathbb{R}$ into the carrier of F.

We use the following convention: R, R_1, R_2 denote rests of F and L, L_1, L_2 denote linears of F.

The following propositions are true:

(3) $L_1 + L_2$ is a linear of F and $L_1 - L_2$ is a linear of F.

(4) $r\,L$ is a linear of F.

(5) Let h_1, h_2 be partial functions from $\mathbb{R}$ to the carrier of F and s_2 be a sequence of real numbers. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1 + h_2)_* s_2 = (h_{1*}s_2) + (h_{2*}s_2)$ and $(h_1 - h_2)_* s_2 = (h_{1*}s_2) - (h_{2*}s_2)$.

(6) Let h_1, h_2 be partial functions from $\mathbb{R}$ to the carrier of F and s_2 be a sequence of real numbers. If h_1 is total and h_2 is total, then $(h_1 + h_2)_* s_2 = (h_{1*}s_2) + (h_{2*}s_2)$ and $(h_1 - h_2)_* s_2 = (h_{1*}s_2) - (h_{2*}s_2)$.

(7) $R_1 + R_2$ is a rest of F and $R_1 - R_2$ is a rest of F.

(8) $r\,R$ is a rest of F.

Let us consider F, f and let x_0 be a real number. We say that f is differentiable in x_0 if and only if:

(Def. 3) There exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and there exist L, R such that for every x such that $x \in N$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x - x_0}$.

Let us consider F, f and let x_0 be a real number. Let us assume that f is differentiable in x_0. The functor $f'(x_0)$ yielding a point of F is defined by the condition (Def. 4).

(Def. 4) There exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and there exist L, R such that $f'(x_0) = L(1)$ and for every x such that $x \in N$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x - x_0}$.

Let us consider F, f, X. We say that f is differentiable on X if and only if:

(Def. 5) $X \subseteq \operatorname{dom} f$ and for every x such that $x \in X$ holds $f \restriction X$ is differentiable in x.

The following propositions are true:

(9) If f is differentiable on X, then X is a subset of $\mathbb{R}$.

(10) f is differentiable on Z iff $Z \subseteq \operatorname{dom} f$ and for every x such that $x \in Z$ holds f is differentiable in x.

(11) If f is differentiable on Y, then Y is open.

Let us consider F, f, X. Let us assume that f is differentiable on X. The functor $f'_{\restriction X}$ yields a partial function from $\mathbb{R}$ to the carrier of F and is defined by:

(Def. 6) $\operatorname{dom}(f'_{\restriction X}) = X$ and for every x such that $x \in X$ holds $f'_{\restriction X}(x) = f'(x)$.

Next we state a number of propositions:

(12) Suppose $Z \subseteq \operatorname{dom} f$ and there exists a point r of F such that $\operatorname{rng} f = \{r\}$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})_x = 0_F$.

(13) Let x_0 be a real number and N be a neighbourhood of x_0. Suppose f is differentiable in x_0 and $N \subseteq \operatorname{dom} f$. Let given h, c. Suppose $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}\left((f_*(h + c)) - (f_*c)\right)$ is convergent and $f'(x_0) = \lim(h^{-1}\left((f_*(h + c)) - (f_*c)\right))$.

(14) If f_1 is differentiable in x_0 and f_2 is differentiable in x_0, then $f_1 + f_2$ is differentiable in x_0 and $(f_1 + f_2)'(x_0) = {f_1}'(x_0) + {f_2}'(x_0)$.

(15) If f_1 is differentiable in x_0 and f_2 is differentiable in x_0, then $f_1 - f_2$ is differentiable in x_0 and $(f_1 - f_2)'(x_0) = {f_1}'(x_0) - {f_2}'(x_0)$.

(16) For every real number r such that f is differentiable in x_0 holds $r\,f$ is differentiable in x_0 and $(r\,f)'(x_0) = r \cdot f'(x_0)$.

(17) Suppose $Z \subseteq \operatorname{dom}(f_1 + f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 + f_2)'_{\restriction Z}(x) = {f_1}'(x) + {f_2}'(x)$.

(18) Suppose $Z \subseteq \operatorname{dom}(f_1 - f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 - f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 - f_2)'_{\restriction Z}(x) = {f_1}'(x) - {f_2}'(x)$.

(19) Suppose $Z \subseteq \operatorname{dom}(r\,f)$ and f is differentiable on Z. Then $r\,f$ is differentiable on Z and for every x such that $x \in Z$ holds $(r\,f)'_{\restriction Z}(x) = r \cdot f'(x)$.

(20) If $Z \subseteq \operatorname{dom} f$ and $f\restriction Z$ is constant, then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\restriction Z}(x) = 0_F$.

(21) Let r, p be points of F and given Z, f. Suppose $Z \subseteq \operatorname{dom} f$ and for every x such that $x \in Z$ holds $f_x = x \cdot r + p$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\restriction Z}(x) = r$.

(22) For every real number x_0 such that f is differentiable in x_0 holds f is continuous in x_0.

(23) If f is differentiable on X, then $f\restriction X$ is continuous.

(24) If f is differentiable on X and $Z \subseteq X$, then f is differentiable on Z.

(25) There exists a rest R of F such that $R_0 = 0_F$ and R is continuous in 0.

Let us consider F and let f be a partial function from $\mathbb{R}$ to the carrier of F. We say that f is differentiable if and only if:

(Def. 7) f is differentiable on $\mathrm{dom}\, f$.

Let us consider F. One can check that there exists a function from $\mathbb{R}$ into the carrier of F which is differentiable. We now state the proposition

(26) Let f be a differentiable partial function from $\mathbb{R}$ to the carrier of F. If $Z \subseteq \mathrm{dom}\, f$, then f is differentiable on Z.

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