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Brouwer Fixed Point Theorem in the General Case

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Summary. In this article we prove the Brouwer fixed point theorem for an arbitrary convex compact subset of $\mathcal{E}^n$ with a non empty interior. This article is based on [15].

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The notation and terminology used here have been introduced in the following papers: [17], [12], [1], [4], [7], [16], [6], [13], [10], [2], [3], [14], [9], [20], [18], [8], [19], [11], [21], and [5].

1. Preliminaries

For simplicity, we adopt the following convention: n is a natural number, p, q, u, w are points of $\mathcal{E}^n_{\mathrm{T}}$, S is a subset of $\mathcal{E}^n_{\mathrm{T}}$, A, B are convex subsets of $\mathcal{E}^n_{\mathrm{T}}$, and r is a real number.

Next we state several propositions:

(1) $(1 - r) \cdot p + r \cdot q = p + r \cdot (q - p)$.

(2) If u, $w \in \mathrm{halfline}(p, q)$ and $|u - p| = |w - p|$, then $u = w$.

(3) Let given S. Suppose $p \in S$ and $p \neq q$ and $S \cap \mathrm{halfline}(p, q)$ is Bounded. Then there exists w such that

(i) $w \in \mathrm{Fr}\, S \cap \mathrm{halfline}(p, q)$,

(ii) for every u such that $u \in S \cap \mathrm{halfline}(p, q)$ holds $|p - u| \leq |p - w|$, and

(iii) for every r such that $r > 0$ there exists u such that $u \in S \cap \mathrm{halfline}(p, q)$ and $|w - u| < r$.

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(4) For every A such that A is closed and $p \in \operatorname{Int} A$ and $p \neq q$ and $A \cap \operatorname{halfline}(p, q)$ is Bounded there exists u such that $\operatorname{Fr} A \cap \operatorname{halfline}(p, q) = \{u\}$.

(5) If $r > 0$, then $\operatorname{Fr} \overline{\operatorname{Ball}}(p, r) = \operatorname{Sphere}(p, r)$.

Let n be an element of $\mathbb{N}$, let A be a Bounded subset of $\mathcal{E}^n_{\mathrm{T}}$, and let p be a point of $\mathcal{E}^n_{\mathrm{T}}$. One can verify that $p + A$ is Bounded.

2. Main Theorems

Next we state four propositions:

(6) Let n be an element of $\mathbb{N}$ and A be a convex subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose A is compact and non boundary. Then there exists a function h from $\mathcal{E}^n_{\mathrm{T}} \restriction A$ into $\operatorname{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ such that h is homeomorphism and $h^\circ \operatorname{Fr} A = \operatorname{Sphere}((0_{\mathcal{E}^n_{\mathrm{T}}}), 1)$.

(7) Let given A, B. Suppose A is compact and non boundary and B is compact and non boundary. Then there exists a function h from $\mathcal{E}^n_{\mathrm{T}} \restriction A$ into $\mathcal{E}^n_{\mathrm{T}} \restriction B$ such that h is homeomorphism and $h^\circ \operatorname{Fr} A = \operatorname{Fr} B$.

(8)[1] For every A such that A is compact and non boundary holds every continuous function from $\mathcal{E}^n_{\mathrm{T}} \restriction A$ into $\mathcal{E}^n_{\mathrm{T}} \restriction A$ has a fixpoint.

(9) Let A be a non empty convex subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose A is compact and non boundary. Let F_1 be a non empty subspace of $\mathcal{E}^n_{\mathrm{T}} \restriction A$. If $\Omega_{(F_1)} = \operatorname{Fr} A$, then F_1 is not a retract of $\mathcal{E}^n_{\mathrm{T}} \restriction A$.

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